# Duality

A structured set is a set, equipped with some structure. Examples of structured sets are:

**posets.** A poset is a pair $(S, \leq)$ with $S$ a set and $\leq \subseteq S \times S$ a partial order. A poset $(S, \leq)$ is *finitary* if for every element $x \in S$ the set $\{y \in S \mid y \leq x\}$ is finite.

**lattices.** A lattice is a triple $(S, \vee, \wedge)$ with $S$ a set and $\vee : S \times S \to S$ and $\wedge : S \times S \to S$ functions satisfying:

$$\begin{array}{rclcrcl} x \vee y &=& y \vee x & \qquad & x \wedge y &=& y \wedge x \\ x \vee (y \vee z) &=& (x \vee y) \vee z & & x \wedge (y \wedge z) &=& (x \wedge y) \wedge z \\ x \vee x &=& x & & x \wedge x &=& x \\ x \vee (y \wedge x) &=& x & & x \wedge (y \vee x) &=& x \end{array}$$

Define the relation $\leq \subseteq E \times E$ by $x \leq y$ iff $x \wedge y = x$. Then the axioms above are sufficient to quarantee that $\leq$ is a partial order, $x \vee y$ denotes the least upperbound of $x$ and $y$ and $x \wedge y$ denotes the greatest lower bound of $x$ and $y$. (And dually, if $(S, \leq)$ is any poset such that for each pair $x, y \in S$ a least upperbound $x \vee y$ and a greatest lower bound $x \wedge y$ exists, then $\vee$ and $\wedge$ satisfy the axioms above.) A lattice is *distributive* if it satisfies

$$x \vee (y \wedge z) \;=\; (x \vee y) \wedge (x \vee z) \quad \text{and} \quad x \wedge (y \vee z) \;=\; (x \wedge y) \vee (x \wedge z).$$

**lattices with top and bottom.** A lattice with top and bottom is a tuple $(S, \vee, \wedge, \bot, \top)$ with $(S, \vee, \wedge)$ a lattice, $\top \in S$ and $\bot \in S$, such that

$$\bot \vee x = x = x \wedge \top \qquad \top \vee x = \top \qquad \bot \wedge x = \bot$$

$\top$ and $\bot$ are the greatest and the least element, respectively, in the partial order $(S, \leq)$.

**boolean algebras.** These are tuples $(S, \vee, \wedge, \bot, \top, \neg)$ with $(S, \vee, \wedge, \bot, \top, \neg)$ a distributive lattice with top and bottom, and $\neg : S \to S$ an operator satisfying:

$$x \vee \neg x = \top \quad \text{and} \quad x \wedge \neg x = \bot.$$

From this it follows that

$$\neg\neg x = x \qquad \neg\top = \bot \qquad \neg(x \vee y) = \neg x \wedge \neg y \qquad \neg(x \wedge y) = \neg x \vee \neg y.$$

**semilattices.** A semilattice is a pair $(S, \wedge)$ with $S$ a set and $\wedge : S \times S \to S$ a function satisfying:

$$x \wedge y \;=\; y \wedge x \qquad x \wedge (y \wedge z) \;=\; (x \wedge y) \wedge z \qquad x \wedge x \;=\; x$$

Define the relation $\leq \subseteq E \times E$ by $x \leq y$ iff $x \wedge y = x$. Then the axioms above are sufficient to quarantee that $\leq$ is a partial order. (And dually, if $(S, \leq)$ is any poset such that for each pair $x, y \in S$ a least upperbound $x \wedge y$ exists, then $\wedge$ satisfies the axioms above.) A semilattice (or a lattice) is *finitary* if the associated partial order is. Two elements $x$ and $y$ in a finitary semilattice either have no upperbound at all, or have a least upperbound. If the least upperbound of $x$ and $y$ exists, it is denoted by $x \vee y$. A semilattice is *distributive* if it satisfies

$$x \vee (y \wedge z) \;=\; (x \vee y) \wedge (x \vee z) \quad \text{and} \quad x \wedge (y \vee z) \;=\; (x \wedge y) \vee (x \wedge z)$$

whenever both sides of these equations are defined.

**topological spaces.** A topological space is a pair $(S, \mathcal{O})$ with $S$ a set and $\mathcal{O}$ a collection of subsets of $S$ (the *open sets*), closed under arbitrary unions and finite intersections, and containing $\emptyset$ and $S$.

Between structured sets one defines structure preserving functions or *homomorphisms*. A structure preserving function between two topological spaces $(S, \mathcal{O})$ and $(S', \mathcal{O}')$, also called a *continuous function*, is a function $f : S \to S'$ such that if $X' \in \mathcal{O}'$ then $f^{-1}(X') \stackrel{\text{def}}{=} \{x \in S \mid f(x) \in X'\} \in \mathcal{O}$, i.e. the inverse image of an open set is open. A homomorphism $h$ between two posets is a function between their underlying sets, satisfying $d \leq e \Rightarrow h(d) \leq h(e)$. A lattice homomorphism satisfies $h(x \vee y) = h(x) \vee h(y)$ and $h(x \wedge y) = h(x) \wedge h(y)$. If the lattice has a top and bottom one furthermore requires $h(\top) = \top$ and $h(\bot) = \bot$, and in the case of a boolean algebra one adds $h(\neg x) = \neg h(x)$. A semilattice homomorphism satisfies just $h(x \wedge y) = h(x) \wedge h(y)$, and a distributive semilattice homomorphism additionally satisfies $h(x \vee y) = h(x) \vee h(y)$, whenever the left-hand side is defined.

If $S$ and $T$ are sets, $T^S$ denotes the set of functions from $S$ to $T$. This terminology fits with the classic number theoretic notion of exponentation, because $|T^S| = |T|^{|S|}$, i.e. if $S$ has $n$ elements and $T$ has $m$ elements then $T^S$ has $m^n$ elements. By analogy, for structured sets $A$ and $B$, $B^A$ denotes the set of structure preserving functions from $A$ to $B$. The more structure there is in $A$ and $B$, the less elements there are in $B^A$. The set $B^A$ usually has some natural structure as well, dependent on (but not necessary of the same type as) the structure of $A$ and $B$.

2 is the name of a particular structured set. It has two elements, called 0 and 1, or $\bot$ and $\top$. It is a poset, under the natural ordering $\bot < \top$, and even a boolean algebra.

If $S$ is a set, then $2^S$ denotes the *powerset* of $S$, the set of all its subsets. Again this terminology fits with arithmetic, as $|2^S| = 2^{|S|}$. It is also consistent with the point of view that $2^S$ denotes the set of functions from $S$ to 2; each function $f : S \to 2$ describes a subset of $S$, namely the set $\{x \in S \mid f(x) = 1\}$. The powerset $2^S$ is naturally ordered by inclusion, i.e. for $f, g \in 2^S$ write $f \leq g$ iff $f \subseteq g$, i.e. iff $f(x) \leq g(x)$ for all $x \in S$. Moreover, union $\cup$ and intersection $\cap$ are well-defined operations on $2^S$, as well wel as complement. In fact, with these operations $2^S$ is a boolean algebra. The top is $S$ and the bottom $\emptyset$.

If $P = (S, \leq)$ is a poset, than $2^P$ is the set of upwards-closed subsets of $P$. Namely, a set $X$ is of the form $\{x \in S \mid h(x) = 1\}$ with $h : P \to 2$ an order preserves function, iff it satisfyies $f \geq e \in X \Rightarrow f \in X$. (Likewise, the sets of the form $\{x \in S \mid h(x) = 0\}$ are the downwards-closed subsets of $S$.) As before, $2^P$ is ordered by inclusion, and contains a top and bottom. Moreover, the union or intersection of two upwards-closed subsets of $S$ is again an upwards-closed subset. Thus on $2^P$ one can define $\vee$ as $\cup$ and $\wedge$ as $\cap$. However, there is no appropriate notion of $\neg$: the complement of an upwards-closed subset is self in general not upwards-closed. Thus $2^P$ cannot be regarded as a boolean algebra. It is a lattice with top and bottom however. By checking the equations one find that it is even a distributive lattice with $\top$ and $\bot$.

Now let $D = (S, \vee, \wedge, \bot, \top)$ be a distributive lattice with top and bottom. Then one can check that $2^D$ consists of the upwards-closed subsets of $D$ satisfying

$$x, y \in D \Rightarrow x \wedge y \in D \qquad x, y \notin D \Rightarrow x \vee y \notin D \qquad \top \in D \quad \text{and} \quad \bot \notin D.$$

What structure can one find on $2^D$? Compared with the previous case, we have lost top and bottom. The candidate for top is the function sending all elements of $S$ to top, i.e. the subset $S$. However, this subset is not in $2^D$, and neither is $\emptyset$, the function sending all of $S$ to $\bot$. One can also prove that if $X$ and $Y$ are two incomparable subsets of $S$ (i.e. we have neither $X \subseteq Y$ nor $Y \subseteq X$), then $X \cup Y$ and $X \cap Y$ can not be in $2^D$. Thus $\vee$ and $\wedge$ are not defined on $2^D$. The only thing left is the inclusion order on subsets. Thus $2^D$ is just a poset.

Next look at boolean algebras. If $B = (S, \vee, \wedge, \bot, \top, \neg)$ is a boolean algebra, then $2^B$ consists of the subsets $X$ of $S$ defined as above, that also satisfy $x \in X \Leftrightarrow \neg x \notin X$. These subsets are all incomparable under inclusion; no one includes another. In this case $2^B$ is a plain set.

**Theorem 1** If $S$ is a finite set, then $2^S$ is a finite boolean algebra and $2^{2^S} \cong S$.
If $B$ is a finite boolean algebra, then $2^B$ is a finite set and $2^{2^B} \cong B$.
If $P$ is a finite poset, then $2^P$ is a finite distributive lattice with $\top$ and $\bot$ and $2^{2^P} \cong P$.
If $D$ is a finite distributive lattice with $\top$ and $\bot$, then $2^D$ is a finite poset and $2^{2^D} \cong D$.

This phenonenon is called *duality*. Finite boolean algebras are the duals of finite sets, and finite distributive lattices are the duals of finite posets. This duality generalizes smoothly to infinite structures, but in that case the definitions need to be sharpened. Infinite sets for instance dualize into *complete atomic* boolean algebras, two properties that could be skipped in Theorem 1, as finite boolean algebras are always complete and atomic.

On the spectrum

$$\text{sets} \longrightarrow \text{posets} \longrightarrow \text{distributive lattices with } \top \text{ and } \bot \longrightarrow \text{boolean algebras}$$

the added structure increases. Plain sets have the least structure, and boolean algebras the most. We see that if a structured set has more structure, its dual has less.

### Homework

6. Give examples of
   (a) a poset for which no semilattice operator $\wedge$ exists,
   (b) a distributive semilattice that is not a lattice,
   (c) a lattice that is not distributive,
   (d) a distributive lattice that is not a boolean algebra.

7. Prove that the lattice axioms are indeed sufficient to quarantee that $\leq$ is a partial order, $x \vee y$ denotes the least upperbound, and $x \wedge y$ denotes the greatest lower bound of $x$ and $y$.

8. Prove that in the presence of the lattice axioms only one of the two distributive laws is needed; the other then follows.

9. Show that in a finitary semilattice any nonempty set of elements (possibly infinite) has a greatest lowerbound, and any set with an upperbound has a least upperbound.

10. What are the duals of
    (a) finite posets with top and bottom?
    (b) finite distributive lattices (without top and bottom)?
    (c) finite semilattices with bottom?

    Explain your answer (same level of detail as preceeding text).

## The dual of prime event structures with binary conflict

It would be great if prime event structures with binary conflict could be dualized in the same way as sets and posets above. In fact, the definition of a configuration goes already a long way in that direction, as they are just subsets satisfying certain closure conditions. There are two problems however: first of all the posets from the previous section give rise to duals consisting of upwards-closed subsets, whereas configurations are required to be downward-closed; secondly the conflict relation doesn't fit so well in the setup from the previous section. The first problem is solved by dualizing the *converse* of an event structure, obtained by reversing the order. The second problem is obtained by encoding the conflict relation in terms of a partial $\vee$-operator. For simplicity I restrict myself to finite event structures here.

**Definition 1** An *encoded event structure* is a triple $(E, \top, \vee)$ with $\top \in E$ and $\vee : E \times E \to E$ a partial function, satisfying

$$\text{for all } d, e \in E : (d \vee e = d) \text{ or } (d \vee e = e) \text{ or } (d \vee e = \top) \text{ or } (d \vee e) \text{ is undefined}$$

$$x \vee y \;=\; y \vee x \qquad x \vee (y \vee z) \;=\; (x \vee y) \vee z \qquad x \vee x \;=\; x$$

and each side of such an equation is defined iff the other is.

If $(E, <, \#)$ is an event structure, a structure $(E, \top, \vee)$ is obtained by adding a fresh element $\top$, writing $d \vee e = e$ iff $d \leq e$, and writing $d \vee e = \top$ iff $d \# e$. In case $d \smile e$, $d \vee e$ is undefined. It is trivial to see that in this encoding information is neither lost nor added, i.e. each form of event structure can be fully recovered from the other form.

Now for E an event structure, let $\mathrm{E}^{\smallsmile}$ be its converse. In encoded form $\mathrm{E}^{\smallsmile}$ is obtained by writing $\vee$ and $\top$ as $\wedge$ and $\perp$. Now the elements of $2^{\mathrm{E}^{\smallsmile}}$ are exactly the configurations of E.

### Homework

11. Why is $2^{\mathrm{E}^{\smallsmile}} = C(\mathrm{E})$? What is the structure on $2^{\mathrm{E}^{\smallsmile}}$? What would be the duals of prime event structures with binary conflict?