Chu Spaces
Course notes for CS352

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1 Chu Spaces

1.1 Motivation

A Chu space is a two-set structure that is equally useful for representing concurrent computational processes and transformable mathematical objects.

With regard to computation, the two sets of a Chu space consist respectively of events and states. This organization is strongly reminiscent of that of a Petri net, whose transitions correspond more or less with Chu events and whose places, or at least markings thereof, correspond to Chu states.

With regard to mathematics, despite the simplicity of its structure Chu spaces are well-suited to the representation of sets, posets, groups, rings, fields, vector spaces, lattices, Boolean algebras, and topological spaces, to list just some of the basic transformable objects of mathematics.

Chu spaces combine very naturally under the operations of Girard's linear logic suitably interpreted. Girard's axiomatization of linear logic is sound with respect to this interpretation, but it remains an important and seemingly difficult quesion as to whether it is fully complete in the sense that every proof derivable in linear logic has a semantic counterpart in this interpretation.

Chu spaces were defined as a categorical construction in the 1970's by Mike Barr, a category theorist at McGill, and developed in considerable generality by his student Peter Chu in a master's thesis that became an appendix to Barr's 1979 book on *-autonomous categories. In the late 1980's the then unnamed construction began to be referred to as the Chu construction by some computer scientists interested in its apparent connections with linear logic and parallel computation.

In 1991 my student Vineet Gupta and I developed a notion of partial distributive lattice as a generalization of event structures that made clearer the duality of event structures as schedules and families of configurations as automata, and that made a strong connection between a process algebra I had developed having sum and product among its core operations and Girard's language of linear logic.

However our notion was very clumsy, and in 1992 we found it possible to greatly simplify the concept by suitably adapting the Barr-Chu construction as applied to the category of sets, easily the most familiar of all categories to which the construction could be applied and hence the one giving the most straightforward and accessible notion of a Chu space. I suggested to Barr at that time that the objects produced by the construction could be thought of as spaces, and he proposed the term "Chu space" for those objects.[1]

Subsequently I found that Chu spaces were considerably more general than most of the standard structures of mathematics such as algebraic, relational, and topological structures, and the traditional blends thereof. The one exception was categories, and I showed (at least for small categories) that Chu spaces and categories were equally powerful.

1.2 Definition

A Chu space is a rectangular matrix whose rows and columns are understood as representing respectively points and dual points or states. More formally, a Chu space over a set K is a function $R : A \times X \times K$ where A is the set of row indices, X is the set of column indices, and K is the set of possible matrix entries.

Following the tradition of collecting these ingredients as a tuple, a Chu space over K can be represented formally as a triple (A, R, X).

No restrictions of any kind are placed on any of the sets K, A, or X. In particular they may be empty, finite, countable, or uncountable.

1.3 Interpretation

The sets A and X are understood as consisting of points $a, b, \ldots$ and states $x, y, \ldots$ respectively. The relation R is understood as a K-valued binary relation between A and X expressing the pairwise interaction of points and states; it is the primary repository of structure in a Chu space.

Structure of A and X separately may be inferred from R, but only secondarily as a derived notion. That is, structure is primarily heterogeneous (point-state) and only secondarily homogeneous (point-point or state-state).

There are two general kinds of derived homogeneous structure, uniform and ad hoc. Uniform structure is realized via a general notion of property. Ad hoc structure is obtained when K is equipped with structure of some kind and that structure is then lifted to A and X where possible.

The *Cartesian interpretation* of a Chu space draws what we could think of loosely as a "mind-body" distinction by interpreting points and states as respectively *extensional* and *intensional*. Points are understood as coexisting, forming a conjunctive set or *universe* of individuals. States are understood as existing only one at a time, each state of the Chu space determining the states (as elements of K) of the points. The Cartesian interpretation therefore interprets the elements of K as the possible states of each point, with $K = 2$ making points *bits* of memory. This interpretation does not distinguish space from time, but regards points as being distributed in "space-time," leaving such distinctions to ad hoc structure.

[1]As a categorical footnote, in order to relate this to the more general Barr-Chu construction I proposed calling the general objects Chu spaces enriched in a symmetric monoidal category V, by strong analogy with the notion of a category enriched in such a V. Just as an ordinary category is a category enriched in **Set**, so is an ordinary Chu space, the only kind studied here, one that is enriched in **Set**.

1.4 Isomorphism and Equivalence

Two spaces over a common K differing only in the choice of individual row and column indices are called isomorphic. More formally, (A, R, X) and (B, S, Y) are *isomorphic* when there exist bijections (functions that are both injective and surjective) $f : A \to B$, $g : X \to Y$ such that for all $a \in A$ and $x \in X$, $R(a, x) = S(f(a), g(x))$. That is, if the rows and columns of S are permuted according to f and g then the resulting matrix is exactly R.

In addition to whole Chu spaces being isomorphic, there is also a notion of isomorphism for individual rows and columns. When a row (resp. column) appears twice we regard the two occurrences as two distinct but isomorphic points (resp. states). Formally, points $a, b \in A$ are *isomorphic* when for all $x \in X$, $R(a, x) = R(b, x)$.

A Chu space is *extensional* when it contains no repeated columns, more formally, when no two distinct points are isomorphic. It is T_0 or *separable* when it contains no repeated rows, and skeletal or *biextensional* when both extensional and separable.

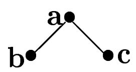
A *skeleton* of a Chu space is the result of deleting repetitions of rows and columns to yield a biextensional space. Formally, (A, R, X) is a *skeleton* of (B, S, Y) when it is biextensional, $A \subseteq B$, $X \subseteq Y$, and for all $a \in A$ and $x \in X$, $R(a, x) = S(a, x)$.

It is easily seen that all skeletons of a Chu space are isomorphic. Two Chu spaces are called *equivalent* when they differ only in the number of repetitions of rows and columns, more formally, when they have isomorphic skeletons. It follows that isomorphic Chu spaces are always equivalent, and equivalent Chu spaces are isomorphic just when they have the same number of repetitions of any given row or column.

Our main interest lies in the structure and interrelationships of Chu spaces, for which it suffices to regard Chu spaces as being defined only up to equivalence. Under this assumption it suffices to treat only biextensional Chu spaces, no repeated rows or columns, which we henceforth assume unless explicitly stated to the contrary.

1.5 Examples

We now give two examples of objects represented by Chu spaces. We shall interpret the Hasse diagram



The wedge

as representing first a poset (the left matrix below) and then a semilattice (the right matrix).

$$\begin{array}{cc} a & 01111 \\ b & 00101 \\ c & 00011 \end{array} \qquad\qquad \begin{array}{cc} a & 0111 \\ b & 0101 \\ c & 0011 \end{array}$$

The wedge as a poset — The wedge as a semilattice

We define a *functional* to be a function to the two-element poset $2 = \{0, 1\}$ with $0 \leq 1$, preserving the relevant structure: the order in the case of the poset, the semilattice operation in the case of the semilattice. A functional on a poset is a monotone function to 2, which in this case a function satisfying $f(b) \leq f(a)$ and $f(c) \leq f(a)$. A functional on a semilattice is a semilattice homomorphism to 2 construed as a semilattice, which in this case is a function satisfying $f(b \vee c) = f(a)$, the identity $b \vee c = a$ being a full description of the semilattice.

Homework. Prove this last remark.

In both cases the columns represent the possible functionals on the wedge, under their respective interpretations. A functional on the wedge as a poset is a cut or upward-closed subset.