Proofs Without Syntax

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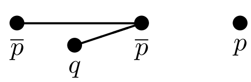
"[M]athematicians care no more for logic than logicians for mathematics."
Augustus de Morgan, 1868

Proofs are traditionally syntactic, inductively generated objects. This paper presents an abstract mathematical formulation of propositional calculus (propositional logic) in which proofs are combinatorial (graph-theoretic), rather than syntactic. It defines a *combinatorial proof* of a proposition ϕ as a graph homomorphism $h : G \rightarrow G(\phi)$, where $G(\phi)$ is a graph associated with ϕ and G is a coloured graph. The main theorem is soundness and completeness: ϕ is true iff there exists a combinatorial proof $h : G \rightarrow G(\phi)$.

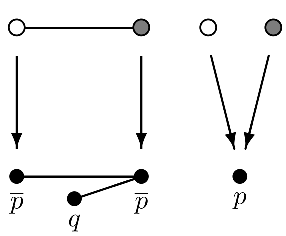
1 Introduction

In 1868, de Morgan lamented the rift between mathematics and logic [deM68]: "*[M]athematicians care no more for logic than logicians for mathematics.*" The dry syntactic manipulations of formal logic can be off-putting to mathematicians accustomed to beautiful symmetries, geometries, and rich layers of structure. Figure 1 shows a syntactic proof in a standard Hilbert system taught to mathematics undergraduates [Hil28, Joh87]. Although the system itself is elegant (*e.g.* just three axiom schemata suffice), the syntactic proofs generated in it need not be. Other syntactic systems include [Fr1879, Gen35].

This paper presents an abstract mathematical formulation of propositional calculus (propositional logic) in which proofs are combinatorial (graph-theoretic), rather than syntactic. It defines a *combinatorial proof* of a proposition ϕ as a graph homomorphism $h : G \rightarrow G(\phi)$, where $G(\phi)$ is a graph associated with ϕ, and G is a coloured graph. For example, if $\phi = ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$ then $G(\phi)$ is:



A combinatorial proof $h : G \rightarrow G(\phi)$ of ϕ is shown below:



The upper graph G has two colours (white and grey), and the arrows define h. The same proposition is proved syntactically in Figure 1.

The main theorem of the paper is soundness and completeness:

A proposition is true iff it has a combinatorial proof.

As with conventional syntactic soundness and completeness, this theorem matches a universal quantification with an existential one: a proposition ϕ is true if it evaluates to 1 *for all* $0/1$ assignments of its variables, and ϕ is provable if *there exists* a proof of ϕ. However, where conventional completeness provides an inductively generated *syntactic* witness (*e.g.* Figure 1), this theorem provides an abstract *mathematical* witness for every true proposition (*e.g.* the homomorphism h drawn above).

Just three conditions suffice for soundness and completeness: a graph homomorphism $h : G \rightarrow G(\phi)$ is a combinatorial proof of ϕ if (1) G is a suitable coloured graph, (2) h is a *skew fibration*, a lax form of graph fibration, and (3) the image of each colour class is labelled appropriately. Each condition can be checked in polynomial time, so combinatorial proofs constitute a formal *proof system* [CR79].

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2 Notation and terminology

Graphs. An ***edge*** on a set V is a two-element subset of V. A ***graph*** (V, E) is a finite set V of ***vertices*** and a set E of edges on V. Write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G, respectively, and vw for $\{v, w\}$. The ***complement*** of (V, E) is the graph (V, E^{c}) with $vw \in E^{\mathsf{c}}$ iff $vw \notin E$. A graph (V, E) is ***coloured*** if V carries an equivalence relation $\sim$ such that $v \sim w$ only if $vw \notin E$; each equivalence class is a ***colour class***. Given a set L, a graph

A syntactic proof of $((p\Rightarrow q)\Rightarrow p)\Rightarrow p$ in a standard Hilbert system — ***Figure 1***

Below is a proof of Peirce's law $((p\Rightarrow q)\Rightarrow p)\Rightarrow p$ in a standard Hilbert formulation of propositional logic, taught to mathematics undergraduates [Joh87], with axiom schemata

$$(a) \quad x\Rightarrow(y\Rightarrow x)$$
$$(b) \quad (x\Rightarrow(y\Rightarrow z))\Rightarrow((x\Rightarrow y)\Rightarrow(x\Rightarrow z))$$
$$(c) \quad ((x\Rightarrow\bot)\Rightarrow\bot)\Rightarrow x$$

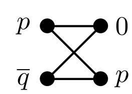
and where (m^i_j) marks *modus ponens* with hypotheses numbered i and j. Hilbert systems tend to emphasise the elegance of the schemata (*e.g.* just (a)–(c) suffice) over the elegance of the proofs generated by the schemata. (Note: there may exist a shorter proof of Peirce's law in this system.)

1	(c)	$((q\Rightarrow\bot)\Rightarrow\bot)\Rightarrow q$
2	(a)	$(((q\Rightarrow\bot)\Rightarrow\bot)\Rightarrow q)\Rightarrow(\bot\Rightarrow(((q\Rightarrow\bot)\Rightarrow\bot)\Rightarrow q))$
3	(m^1_2)	$\bot\Rightarrow(((q\Rightarrow\bot)\Rightarrow\bot)\Rightarrow q)$
4	(b)	$(\bot\Rightarrow(((q\Rightarrow\bot)\Rightarrow\bot)\Rightarrow q))\Rightarrow((\bot\Rightarrow((q\Rightarrow\bot)\Rightarrow\bot))\Rightarrow(\bot\Rightarrow q))$
5	(m^3_4)	$(\bot\Rightarrow((q\Rightarrow\bot)\Rightarrow\bot))\Rightarrow(\bot\Rightarrow q)$
6	(a)	$\bot\Rightarrow((q\Rightarrow\bot)\Rightarrow\bot)$
7	(m^6_5)	$\bot\Rightarrow q$
8	(a)	$(\bot\Rightarrow q)\Rightarrow(p\Rightarrow(\bot\Rightarrow q))$
9	(m^7_8)	$p\Rightarrow(\bot\Rightarrow q)$
10	(b)	$(p\Rightarrow(\bot\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))$
11	(m^9_{10})	$(p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)$
12	(a)	$((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p))$
13	(b)	$((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p))$
14	(a)	$(((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p))))$
15	(m^{13}_{14})	$((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))$
16	(b)	$(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p))))\Rightarrow((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p))))$
17	(m^{15}_{16})	$(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow((p\Rightarrow q)\Rightarrow p)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))$
18	(m^{12}_{17})	$((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p))$
19	(b)	$(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))\Rightarrow((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))$
20	(m^{18}_{19})	$(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))$
21	(a)	$((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))$
22	(a)	$((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))))$
23	(m^{20}_{22})	$((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))$
24	(b)	$(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))))\Rightarrow((((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q)))))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))))$
25	(m^{23}_{24})	$(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))))\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p)))$
26	(m^{21}_{25})	$((p\Rightarrow\bot)\Rightarrow(p\Rightarrow q))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))$
27	(m^{11}_{26})	$((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p)$
28	(a)	$(p\Rightarrow\bot)\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot))\Rightarrow(p\Rightarrow\bot))$
29	(b)	$((p\Rightarrow\bot)\Rightarrow(((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot))\Rightarrow(p\Rightarrow\bot)))\Rightarrow(((p\Rightarrow\bot)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot)))\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot)))$
30	(m^{28}_{29})	$((p\Rightarrow\bot)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot)))\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot))$
31	(a)	$(p\Rightarrow\bot)\Rightarrow((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot))$
32	(m^{31}_{30})	$(p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot)$
33	(b)	$((p\Rightarrow\bot)\Rightarrow(p\Rightarrow\bot))\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow\bot))$
34	(m^{32}_{33})	$((p\Rightarrow\bot)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow\bot)$
35	(c)	$((p\Rightarrow\bot)\Rightarrow\bot)\Rightarrow p$
36	(a)	$(((p\Rightarrow\bot)\Rightarrow\bot)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow\bot)\Rightarrow p))$
37	(m^{35}_{36})	$((p\Rightarrow\bot)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow\bot)\Rightarrow p)$
38	(b)	$(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow\bot)\Rightarrow p))\Rightarrow((((p\Rightarrow\bot)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow\bot))\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p))$
39	(m^{37}_{38})	$(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow\bot))\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p)$
40	(m^{34}_{39})	$((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p$
41	(a)	$(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p)\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p))$
42	(m^{40}_{41})	$((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p)$
43	(b)	$(((p\Rightarrow q)\Rightarrow p)\Rightarrow(((p\Rightarrow\bot)\Rightarrow p)\Rightarrow p))\Rightarrow((((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow p))$
44	(m^{42}_{43})	$(((p\Rightarrow q)\Rightarrow p)\Rightarrow((p\Rightarrow\bot)\Rightarrow p))\Rightarrow(((p\Rightarrow q)\Rightarrow p)\Rightarrow p)$
45	(m^{27}_{44})	$((p\Rightarrow q)\Rightarrow p)\Rightarrow p$

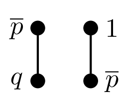
is L-***labelled*** if every vertex has an element of L associated with it, its ***label***. Let $G=(V,E)$ and $G'=(V',E')$ be graphs. A ***homomorphism*** $h:G\to G'$ is a function $h:V\to V'$ such that $vw\in E$ implies $h(v)h(w)\in E'$. If V and V' are disjoint, the ***union*** $G\vee G'$ is $(V\cup V', E\cup E')$ and the ***join*** $G\wedge G'$ is $(V\cup V', E\cup E'\cup\{vv' : v\in V,\ v'\in V'\})$; colourings or labellings of G and G' are inherited. A graph (V,E) is a ***cograph*** (see *e.g.* [BLS99]) if V is non-empty and for any distinct $v,w,x,y\in V$, the restriction of E to edges on $\{v,w,x,y\}$ is not $\{vw,wx,xy\}$. A set $W\subseteq V(G)$ ***induces a matching*** if it is non-empty and for all $w\in W$ there is a unique $w'\in W$ such that $ww'\in E(G)$.

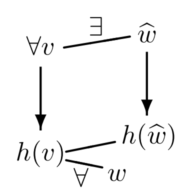
Propositions. Fix a set $\mathcal{V}$ of ***variables***. A ***proposition*** is any expression generated freely from variables by the binary operations ***and*** $\wedge$, ***or*** $\vee$, and ***implies*** $\Rightarrow$, the unary operation ***not*** $\neg$, and the constants (nullary operations) ***true*** 1 and ***false*** 0. A ***valuation*** is a function $f:\mathcal{V}\to\{0,1\}$. Write $\hat{f}$ for the extension of a valuation f to propositions defined by $\hat{f}(0)=0$, $\hat{f}(1)=1$, $\hat{f}(\neg\phi)=1-\hat{f}(\phi)$, $\hat{f}(\phi\wedge\rho)=\min\{\hat{f}(\phi),\hat{f}(\rho)\}$, $\hat{f}(\phi\vee\rho)=\max\{\hat{f}(\phi),\hat{f}(\rho)\}$, $\hat{f}(\phi\Rightarrow\rho)=\hat{f}((\neg\phi)\vee\rho)$. A proposition ϕ is ***true*** if $\hat{f}(\phi)=1$ for all valuations f. Variables $p\in\mathcal{V}$ and their negations $\overline{p}=\neg p$ are ***literals***; p and $\overline{p}$ are ***dual***, as are 0 and 1. An ***atom*** is a literal or constant, and $\mathcal{A}$ denotes the set of atoms.

3 Combinatorial proofs



Given an $\mathcal{A}$-labelled graph G, define $\neg G$ as the result of complementing G and every label of G. For example, if G is the graph shown right, then $\neg G$ is the graph below left. Define $G\Rightarrow G' = (\neg G)\vee G'$. Identify each atom a with a single vertex labelled a; thus, having defined operations $\neg$, $\vee$, $\wedge$ and $\Rightarrow$ on $\mathcal{A}$-labelled graphs, every proposition ϕ determines an $\mathcal{A}$-labelled graph, denoted $G(\phi)$. For example, $G\big((p\vee\neg q)\wedge(0\vee p)\big)$ is above right, $G\big((q\wedge\neg p)\vee(1\wedge\neg p)\big)$ is left, and $G\big(((p\Rightarrow q)\Rightarrow p)\Rightarrow p\big)$ is in the Introduction.





A colouring is ***nice*** if every colour class has at most two vertices and no union of two-vertex colour classes induces a matching.[1] A graph homomorphism $h:G\to G'$ is a ***skew fibration*** (see figure right) if for all $v\in V(G)$ and $h(v)w\in E(G')$ there exists $v\widehat{w}\in E(G)$ with $h(\widehat{w})w\notin E(G')$. Given a graph homomorphism $h:G\to G'$ with G' an $\mathcal{A}$-labelled graph, a vertex $v\in V(G)$ is ***axiomatic*** if $h(v)$ is labelled 1, and a pair $\{v,w\}\subseteq V(G)$ is axiomatic if $h(v)$ and $h(w)$ are labelled by dual literals.

[1] *I.e.*, every colour class has one or two vertices, and if $c_1,\ldots,c_n$ are two-vertex colour classes then $c_1\cup\cdots\cup c_n$ does not induce a matching.

DEFINITION 1 *A **combinatorial proof** of a proposition ϕ is a skew fibration $h : G \to G(\phi)$ from a nicely coloured cograph G to the graph $G(\phi)$ of ϕ, such that every colour class of G is axiomatic.*

A combinatorial proof of $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$ is shown in the Introduction. The reader may find it instructive to consider why $p \wedge \neg p$ has no combinatorial proof.

THEOREM 1 (SOUNDNESS AND COMPLETENESS) *A proposition is true iff it has a combinatorial proof.*

Section 4 reformulates this theorem in terms of combinatorial (non-syntactic, non-inductive) notions of *proposition* and *truth*. Section 5 proves the reformulated theorem.

Notes. The translation $\phi \mapsto G(\phi)$ is based on a well understood translation of a boolean formula into a graph (see *e.g.* [CLS81]), and (up to standard graph isomorphism[2]) represents propositions modulo associativity and commutativity of $\wedge$ and $\vee$, double negation $\neg\neg\phi = \phi$, de Morgan duality $\neg(\phi \wedge \rho) = (\neg\phi) \vee (\neg\rho)$ and $\neg(\phi \vee \rho) = (\neg\phi) \wedge (\neg\rho)$, and $\phi \Rightarrow \rho = (\neg\phi) \vee \rho$. Perhaps the earliest graphical representation of propositions is due to Peirce [Pei58, vol. 4:2], dating from the late 1800s.

A skew fibration is a lax notion of graph fibration. A graph homomorphism $h : G \to G'$ is a ***graph fibration*** (see *e.g.* [BV02]) if for all $v \in V(G)$ and $h(v)w \in E(G')$ there is a unique $v\widehat{w} \in E(G)$ with $h(\widehat{w}) = w$.[3] The definition of skew fibration drops uniqueness and relaxes $h(\widehat{w}) = w$ to 'skewness' $h(\widehat{w})w \notin E(G')$.

In the example of a combinatorial proof drawn in the Introduction, observe that the image of the colour class $\circ\ \circ$ under h is $\overset{\bullet}{\overline{p}}\ \ \overset{\bullet}{p}$. Think of the colour class as actively pairing an occurrence of a variable p with an occurrence of its dual $\overline{p}$. The idea of pairing dual variable occurrences has arisen in the study of various forms of syntax, such as closed categories [KM71], contraction-free predicate calculus [KW84] and linear logic [Gir87]. By pairing dual occurrences, combinatorial proofs relate (superficially) to the connection/matrix method [Dav71, Bib74, And81].

A partially combinatorial notion of proof for classical logic, called a *proof net*, was presented in [Gir91]. Proof nets are rather syntactic: a proof net of a proposition ϕ has an underlying syntax tree containing not only $\wedge$'s and $\vee$'s from ϕ, but also auxiliary syntactic connectives which are not even boolean operations (*contraction* and *weakening*).

Nicely coloured cographs with two vertices in every colour class correspond to *unlabelled chorded R&B-cographs* [Ret03]. When labelled, the latter represent proof nets of mixed multiplicative linear logic [Gir87].

Combinatorial proofs constitute a formal *proof system* [CR79] since correctness can be checked in polynomial time. The skew fibration and axiomatic conditions are clearly polynomial. Checking that a graph G is a cograph is polynomial by constructing its modular decomposition tree $T(G)$ [BLS99], and checking that G is nicely coloured is a simple breadth-first search on $T(G)$.

There is a polynomial-time computable function taking a propositional sequent calculus proof of ϕ with $n \geq 0$ cuts [Gen35] to a combinatorial proof of ϕ with n cuts, where by the latter we mean a combinatorial proof of $\phi \vee (\theta_1 \wedge \neg\theta_1) \vee \ldots \vee (\theta_n \wedge \neg\theta_n)$ for propositions θ_i.

4 Combinatorial propositions and truth

A set $W \subseteq V(G)$ is ***stable*** if $vw \notin E(G)$ for all $v, w \in W$. A ***clause*** is a maximal stable set. A clause of an $\mathcal{A}$-labelled graph is ***true*** if it contains a 1-labelled vertex or two vertices labelled by dual literals; an $\mathcal{A}$-labelled graph is ***true*** if its clauses are true. For example, $\overset{\bullet}{\overline{p}}\ \ \overset{\bullet}{p}\!\!-\!\!\overset{\bullet}{1}$ $(= G(p \Rightarrow (p \wedge 1)))$ is true, with true clauses $\overset{\bullet}{\overline{p}}\ \ \overset{\bullet}{p}$ and $\overset{\bullet}{\overline{p}}\ \ \ \ \overset{\bullet}{1}$.

LEMMA 1 *A proposition ϕ is true iff its graph $G(\phi)$ is true.*

Proof. Exhaustively apply distributivity $\theta \vee (\psi_1 \wedge \psi_2) \to (\theta \vee \psi_1) \wedge (\theta \vee \psi_2)$ to ϕ modulo associativity and commutativity of $\wedge$ and $\vee$, yielding a conjunction ϕ' of syntactic clauses (disjunctions of atoms). The lemma is immediate for ϕ' since $G(\phi')$ is a join of clauses, and $G\big(\theta \vee (\psi_1 \wedge \psi_2)\big)$ is true iff $G\big((\theta \vee \psi_1) \wedge (\theta \vee \psi_2)\big)$ is true since for non-empty graphs G_1 and G_2, a clause of $G_1 \vee G_2$ (resp. $G_1 \wedge G_2$) is a clause of G_1 and (resp. or) a clause of G_2. □

A ***combinatorial proposition*** is an $\mathcal{A}$-labelled cograph. Since a graph is a cograph iff it is derivable from individual vertices by union, join and complement [BLS99, §11.3], the graph $G(\phi)$ of any syntactic proposition ϕ is a combinatorial proposition; conversely every combinatorial proposition is (isomorphic[2] to) $G(\phi)$ for some ϕ.

DEFINITION 2 *A **combinatorial proof** of a combinatorial proposition P is a skew fibration $h : G \to P$ from a nicely coloured cograph G whose colour classes are axiomatic.*

Thus a combinatorial proof of a syntactic proposition ϕ (Def. 1) is a combinatorial proof of $G(\phi)$ (Def. 2). By Lemma 1, the following is equivalent to Theorem 1.

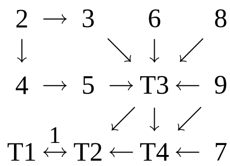
THEOREM 2 (COMBINATORIAL SOUNDNESS AND COMPLETENESS) *A combinatorial proposition is true iff it has a combinatorial proof.*

[2] Graphs (V, E) and (V', E') are isomorphic if there exists a bijection $h : V \to V'$ with $vw \in E$ iff $h(v)h(w) \in E'$.

[3] This is simply a convenient restatement of the familiar notions of fibration in topology [Whi78] and category theory [Gro59, Gra66]: a graph homomorphism is a graph fibration *iff* it satisfies the homotopy lifting property (when viewed as a continuous map by identifying each edge with a copy of the unit interval) *iff* it has all requisite cartesian liftings (when viewed as a functor by identifying each graph with its path category).

5 Proof of Theorem 2

The diagram right shows the dependency between the Lemmas (1–9) and Theorems (T1–T4) in this paper.



Given a graph homomorphism $h : G \to G'$, an edge $v\widehat{w} \in E(G)$ is a ***skew lifting of*** $h(v)w \in E(G')$ ***at*** v if $h(\widehat{w})w \notin E(G')$. Thus h is a skew fibration iff every edge $h(v)w \in E(G')$ has a skew lifting at v.

A graph G is a ***subgraph*** of G', denoted $G \subseteq G'$, if $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$. The subgraph $G[W]$ of G ***induced by*** $W \subseteq V(G)$ is $(W, \{ vw \in E(G) : v,w \in W \})$. Let $h : G \to H$ be a graph homomorphism and let G' and H' be induced subgraphs of G and H, respectively. Write $h(G')$ for the induced subgraph $H[h(V(G'))]$ and $h^{-1}(H')$ for the induced subgraph $G[h^{-1}(V(H'))]$. Define the ***restriction*** $h_{\restriction H'} : h^{-1}(H') \to H'$ by $h_{\restriction H'}(v) = h(v)$.

LEMMA 2 *Let* $\diamond \in \{\wedge, \vee\}$. *If* $h : G \to H_1 \diamond H_2$ *is a skew fibration then both restrictions* $h_{\restriction H_i}$ *are skew fibrations.*

Proof. We prove that if $v\widehat{w}$ is a skew lifting of $h_{\restriction H_i}(v)w = h(v)w \in E(H_i)$ at v with respect to h, then $h(\widehat{w}) \in H_i$; hence $v\widehat{w}$ is a well defined skew lifting with respect to $h_{\restriction H_i}$. Suppose $h(\widehat{w}) \in H_j$ and $j \neq i$. If $\diamond = \vee$, since h is a homomorphism, $h(v)h(\widehat{w})$ is an edge between H_1 and H_2 in $H_1 \vee H_2$, a contradiction; if $\diamond = \wedge$, since $H_1 \wedge H_2$ has all edges between H_1 and H_2, $h(\widehat{w})w$ is an edge, contradicting $v\widehat{w}$ being a skew lifting with respect to h. □

LEMMA 3 *Let* $h : (G_1 \wedge G_2) \vee (H_1 \vee H_2) \to (K_1 \wedge K_2) \vee L$ *be a skew fibration with* $h(G_i) \subseteq K_i$ *and* $h(H_i) \subseteq L$. *Then* $h_i : G_i \vee H_i \to K_i \vee L$ *defined by* $h_i(v) = h(v)$ *is a skew fibration.*

Proof. Since a graph union $X_1 \vee X_2$ has no edges between X_1 and X_2, (a) if $k : X_1 \vee X_2 \to Y$ is a skew fibration, so also is $k^{\restriction X_i} : X_i \to Y$ defined by $k^{\restriction X_i}(x) = k(x)$, and (b) if $k_i : Z_i \to X_i$ is a skew fibration for $i = 1, 2$, so also is $k_1 \vee k_2 : Z_1 \vee Z_2 \to X_1 \vee X_2$ defined by $(k_1 \vee k_2)(z) = k_i(z)$ iff $z \in V(Z_i)$. Since $h_i = h_{\restriction K_i} \vee (h_{\restriction L})^{\restriction H_i}$, h_i is a skew fibration by (a), (b) and Lemma 2. □

LEMMA 4 *If* $h : G \to K$ *is a skew fibration into a cograph* K, *then every clause of* K *contains a clause of* $h(G)$.

Proof. By induction on the number of vertices in K. The base case with K a single vertex is immediate. Otherwise $K = K_1 \diamond K_2$ for $\diamond \in \{\wedge, \vee\}$ and cographs K_i. Let $G_i = h^{-1}(K_i)$ and $h_i = h_{\restriction K_i} : G_i \to K_i$, a skew fibration by Lemma 2. Let C be a clause of K. If $\diamond = \wedge$ then C is a clause of K_j for $j = 1$ or 2; by induction C contains a clause C' of $h_j(G_j)$, also a clause of $h_1(G_1) \wedge h_2(G_2) = h(G)$. If $\diamond = \vee$ then $C = C_1 \cup C_2$ for clauses C_i of K_i; by induction C_i contains a clause C'_i of $h_i(G_i)$, so C contains the clause $C'_1 \cup C'_2$ of $h_1(G_1) \vee h_2(G_2) = h(G)$. □

LEMMA 5 *Let* $h : G \to P$ *be a skew fibration into a combinatorial proposition* P. *If* $h(G)$ *is true then* P *is true.*

Proof. Lemma 4 and the definition of *true*. □

The ***empty*** graph is the graph with no vertices. A graph is ***disconnected*** if it is a union of non-empty graphs, and ***connected*** otherwise. A ***component*** is a maximal non-empty connected subgraph. A graph homomorphism $h : G \to H$ is ***shallow*** if $h^{-1}(K)$ has at most one component for every component K of H.

LEMMA 6 *For any combinatorial proof* $h : G \to P$ *there exists a shallow combinatorial proof* $h' : G \to P'$ *such that* P *is true iff* P' *is true.*

Proof. Let $G_1, \ldots, G_n$ be the components of G, and let P' be the union of n copies of P defined by $V(P') = V(P) \times \{1, \ldots, n\}$ and $\langle v, i\rangle\langle w, j\rangle \in E(P')$ iff $vw \in E(P)$ and $i = j$, and the label of $\langle v, i\rangle$ in P' equal to the label of v in P. Define $h' : G \to P'$ on $v \in V(G_i)$ by $h'(v) = \langle h(v), i\rangle$, Since P' is a union of copies of P, it is true iff P is true (every clause of P' contains a clause of P; conversely the union of n copies of a clause of P is a clause of P'), and h' is a combinatorial proof (skew liftings copied from h). □

A subgraph G' of G is a ***portion*** of G if $G = G' \vee G''$ for some G''. A ***fusion*** of graphs G and H is any graph obtained from $G \vee H$ by selecting portions G' of G and H' of H and adding edges between every vertex of G' and every vertex of H'. Union and join are extremal cases of fusion: union with G', H' empty; join with $G' = G$, $H' = H$. On coloured graphs, fusion does not reduce to union and join: the coloured cograph ○ ○—● ● is a fusion of ○ ○ and ● ●, but is not a union or join of coloured graphs (since Section 2 defined a colouring as an equivalence relation). Henceforth abbreviate *nicely coloured* to *nice*.

LEMMA 7 *A fusion of nice cographs is a nice cograph.*

Proof. Let G be the fusion of nice cographs G_1 and G_2 obtained by joining portions G'_i of G_i. Suppose U is a union of two-vertex colour classes in G inducing a matching. Let $U_i = U \cap V(G_i)$ and $U'_i = U \cap V(G'_i)$. By definition of fusion, the only edges in G between U_1 and U_2 are between U'_1 and U'_2, and there are edges between all vertices of U'_1 and all vertices of U'_2; thus ($\star$) *there is at most one edge between* U_1 *and* U_2, or else two edges of G on U would intersect. Since U is a union of two-vertex colour classes, each either in U_1 or U_2, each U_i contains an even number of vertices. Therefore, since U induces a matching, (†) *there must be an even number of edges between* U_1 *and* U_2. Together ($\star$) and (†) imply there is no edge between U_1 and U_2, hence, for whichever U_i is non-empty (perhaps both), U_i is a union of two-vertex colour classes inducing a matching in G_i, contradicting G_i being nice. □

LEMMA 8 *Every nice cograph with more than one colour class is a fusion of nice cographs.*

Proof. Let G be a nice cograph. Since G is a cograph, its underlying (uncoloured) graph has the form $(G_1 \wedge G_2) \vee (G_3 \wedge G_4) \vee \ldots \vee (G_{n-1} \wedge G_n) \vee H$ for cographs G_i and H with no edges. Assume $n \neq 0$, otherwise the result is trivial. Let $\widehat{G}$ be the graph whose vertices are the G_i, with $G_iG_j \in E(\widehat{G})$ iff there is an edge or colour class $\{v, w\}$ in G with $v \in V(G_i)$ and $w \in V(G_j)$ (*cf.* the proof of Theorem 4 in [Ret03]). A *perfect matching* is a set of pairwise disjoint edges whose union contains all vertices. Since G is nice, $M = \{G_1G_2, G_3G_4, \ldots, G_{n-1}G_n\}$ is the only perfect matching of $\widehat{G}$. For if M' is another perfect matching, then $M' \setminus M$ determines a set of two-vertex colour classes in G whose union induces a matching in G: for each $G_iG_j \in M' \setminus M$ pick a colour class $\{v, w\}$ with $v \in V(G_i)$ and $w \in V(G_j)$. Since $\widehat{G}$ has a unique perfect matching, some $G_kG_{k+1} \in M$ is a bridge [Kot59, LP86], *i.e.*, $(\, V(\widehat{G}),\, E(\widehat{G}) \setminus G_kG_{k+1}\,) = X \vee Y$ with $G_k \in V(X)$ and $G_{k+1} \in V(Y)$. Let W be the union of all colour classes of G coincident with any G_i in X, and let $W' = V(G) \setminus W$. Then $G[W]$ and $G[W']$ are nice (since W and W' are unions of colour classes), and G is the fusion of $G[W]$ and $G[W']$ joining portions G_k of $G[W]$ and G_{k+1} of $G[W']$. □

LEMMA 9 *Let P_1 and P_2 be combinatorial propositions and Q a combinatorial proposition or the empty graph. Then $(P_1 \wedge P_2) \vee Q$ is true iff $P_1 \vee Q$ and $P_2 \vee Q$ are true.*

Proof. A clause of $(P_1 \wedge P_2) \vee Q$ is a clause of $P_1 \vee Q$ or $P_2 \vee Q$, and vice versa. □

THEOREM 3 (COMBINATORIAL SOUNDNESS) *If a combinatorial proposition has a combinatorial proof, it is true.*

Proof. Let $h : G \to P$ be a combinatorial proof. We prove P is true by induction on the number of colour classes in G. In the base case, $V(G)$ is a colour class. If $v \in V(G)$ then $h(v)$ is in no edge of P (for if $h(v)w \in E(P)$ then its skew lifting at v is an edge in G, a contradiction), hence is in every clause C of P. Since $V(G)$ is axiomatic, C is true.

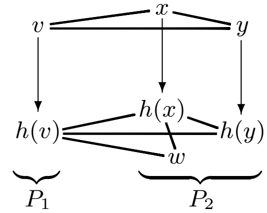
Induction step. By Lemmas 5 and 6, assume h is shallow and surjective. By Lemma 8, G is a fusion of nice cographs G_1 and G_2 obtained from $G_1 \vee G_2$ by joining portions G'_i of G_i. If $G = G_1 \vee G_2$ then $h' : G_1 \to P$ defined by $h'(v) = h(v)$ is a combinatorial proof, and P is true by induction hypothesis. Otherwise each G'_i is non-empty. Let $P_i = h(G'_i)$. Since $G'_1 \wedge G'_2$ is a component of G and h is a shallow surjection, $P_1 \wedge P_2$ is a component of P, say $P = (P_1 \wedge P_2) \vee Q$. Define $h_i : G_i \to P_i \vee Q$ by $h_i(v) = h(v)$, a combinatorial proof: G_i is a nice cograph, the axiomatic colour class property is inherited from h, and h_i is a skew fibration by Lemma 3 (applied after forgetting colourings). By induction hypothesis $P_i \vee Q$ is true, hence P is true by Lemma 9. □

THEOREM 4 (COMBINATORIAL COMPLETENESS) *Every true combinatorial proposition has a combinatorial proof.*

Proof. Let P be a true combinatorial proposition. We construct a combinatorial proof of P by induction on the number of edges in P. In the base case $V(P)$ is a true clause, so there exists $W \subseteq V(P)$ comprising a 1-labelled vertex or a pair of vertices labelled with dual literals. Inclusion $W \to P$ is a combinatorial proof (viewing W as a graph with no edge and a single colour class, and forgetting its labels).

Induction step. Since P is a cograph with an edge, $P = (P_1 \wedge P_2) \vee Q$ for combinatorial propositions P_i and Q a combinatorial proposition or the empty graph. Assume Q is empty or not true; otherwise by induction there is a combinatorial proof $G \to Q$ composable with inclusion $Q \to P$ for a combinatorial proof of P, and we are done. By Lemma 9, $P_i \vee Q$ is true, so by induction has a combinatorial proof $h_i : G_i \to P_i \vee Q$. Let G be the fusion of G_1 and G_2 obtained by joining the portions $h_i^{-1}(P_i)$ of G_i. By Lemma 7, G is nice. Define $h : G \to P$ by $h(v) = h_i(v)$ iff $v \in V(G_i)$. Then h is a graph homomorphism: let $vw \in E(G)$ with $v \in V(G_i)$ and $w \in V(G_j)$; if $i = j$ then $h(v)h(w) \in E(P)$ since h_i is a homomorphism; if $i \neq j$ then vw arose from fusion, so $h(v) \in P_i$ and $h(w) \in P_j$, hence $h(v)h(w) \in E(P)$ since $P_1 \wedge P_2 \subseteq P$ has all edges between P_1 and P_2.

The axiomatic colour class property for h is inherited from the h_i, so it remains to show that h is a skew fibration. Let $v \in V(G)$ and $h(v)w \in E(P)$. By symmetry, assume $v \in V(G_1)$. Assume $h(v) \in V(P_1)$ and $w \in V(P_2)$, otherwise we immediately obtain a skew lifting of $h(v)w$ since h_1 is a skew fibration. There is a vertex x in $h_2^{-1}(P_2)$: if Q is empty, this is immediate; otherwise Q is not true and $h_{2\upharpoonright Q} : G_2 \to Q$ would be a combinatorial proof, contradicting soundness. Since fusion joined the $h_i^{-1}(P_i)$, we have $vx \in E(G)$. If $h(x)w \notin E(P_2)$ we are done; otherwise since h_2 is a skew fibration and $h(x)w \in E(P_2)$ there exists $xy \in E(G_2)$ with $h(y)w \notin E(P_2)$. Since $vy \in E(G)$ (again by fusion), we have the desired skew lifting of $h(v)w$ at v. (See figure below. Note $h(y) = w$ is possible.)



□

References

[And81] Andrews, P. B. *Theorem Proving via General Matings. J. ACM* 28 1981 193–214.

[Bib74] Bibel, W. *An approach to a systematic theorem proving procedure in first-order logic. Computing* 12 1974 43–55.

[BV02] Boldi, P., & S. Vigna. *Fibrations of graphs. Discrete Math.* 243 2002 21–66.

[BLS99] Brandstädt, A., V. B. Le & J. P. Spinrad. *Graph Classes: A Survey. SIAM monographs on Discr. Math. & Applic.* 1999.

[CR79] Cook, S. A. & R. A. Reckhow. *The relative efficiency of propositional proof systems. J. Symb. Logic* 44(1) 1979 36–50.

[CLS81] Corneil, D. G., H. Lerchs & L. Stewart-Burlingham. *Complement reducible graphs. Discr. Appl. Math.* 3 1981 163–174.

[Dav71] Davydov, G. V. *The synthesis of the resolution method and the inverse method. Zapiski Nauchnykh Seminarov Lomi* 20 1971 24–35. Translation in J. Sov. Math. 1(1) 1973 12–18.

[Fr1879] Frege, G. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* 1879. Translation [vHe67, 5–82].

[Gen35] Gentzen, G. *Untersuchungen über das logische Schließen I, II. Mathematische Zeitschrift* 39 1935 176–210 405–431. Translation [Gen69, 68–131].

[Gen69] Gentzen, G. *The Collected Papers of Gerhard Gentzen.* Ed. M. E. Szabo. North-Holland 1969.

[Gir87] Girard, J.-Y. *Linear logic. Theor. Comp. Sci.* 50 1987 1–102.

[Gir91] Girard, J.-Y. *A new constructive logic: Classical logic. Math. Struc. Comp. Sci.* 1(3) 1991 255-296.

[Gra66] Gray, J. W. *Fibred and cofibred categories.* In *Proc. Conf. on Categorical Algebra, La Jolla 1965* 21–83. Springer 1966.

[Gro59] Grothendieck, A. *Technique de descente et théorèmes d'existence en géométrie algébrique. I: Généralités. Descente par morphismes fidèlement plats. Séminaire Bourbaki* 190 1959–60.

[vHe67] van Heijenoort, J. (ed.). *From Frege to Gödel. A Source Book in Mathematical Logic 1879–1931.* Harvard Univ. Press 1967.

[Hil28] Hilbert, D. *Die Grundlagen der Mathematik, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität* 6 1928 65-85. Translation [vHe67, 465–479].

[Joh87] Johnstone, P. T. *Notes on Logic and Set Theory.* Cambridge University Press 1987.

[KM71] Kelly, G. M. & S. Mac Lane. *Coherence in closed categories. J. Pure Appl. Algebra* 1 1971 97–140; erratum, *ibid.* no. 2, 219.

[KW84] Ketonen, J. & R. Weyhrauch. *A Decidable Fragment of Predicate Calculus. Theor. Comp. Sci.* 32 1984 297–307.

[Kot59] Kotzig, A. *On the theory of finite graphs with a linear factor. Mat.-Fyz. Casopis Slovensk. Akad. Vied* 9 & 10 1959.

[LP86] Lovász, L. & M. D. Plummer. *Matching Theory.* North-Holland 1986.

[deM68] de Morgan, A. *Review of J. M. Wilson's 'Elementary Geometry'. The Athenaeum* 2 1868 71–73.

[Pei58] Peirce, C. S. *Collected Papers.* Ed. Hartshorne, Weiss & Burks. Harvard University Press 1931–5 & 1958.

[Ret03] Retoré, C. *Handsome proof-nets: perfect matchings and cographs. Theor. Comp. Sci.* 294 2003 473–488.

[Whi78] Whitehead, G. W. *Elements of Homotopy Theory.* Springer 1978.