Understanding Sequentialization

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Abstract

We present a syntactic account of MLL proof net correctness. This yields an extremely succinct proof of the sequentialization theorem, and elicits an intuitive proof-theoretic understanding of correctness.

1 Introduction

What do graph-theoretic correctness criteria for MLL proof nets *mean*? The Danos-Regnier criterion [DR89], Girard's original trip criterion [Gir87] and the contractability criterion [Dan90, Laf95] all demand real combinatorial work to extract sequentializations. Thus the criteria are far enough removed from proof theory that they have an almost mystical quality: why do they work?

The goal of this paper is to deepen our understanding of the meaning of correctness criteria by exhibiting a direct proof-theoretic description of a variant of the contractability criterion. The best overview is obtained by perusing Figures 3–7, which illustrate the idea of constructing a sequentialization by gluing proof fragments, directly mimicking the correctness criterion.

Not only does this yield insight as to what correctness criteria are about, but by capturing the correctness criterion syntactically, the proof of the sequentialization theorem becomes trivial. Indeed, once the idea of gluing proof fragments with broken par rules is presented (Section 3), the proof of the theorem is reduced to four lines, plus a lemma with a five-line proof.

2 Proof nets

We work with Lafont's definition of proof structures [Laf95] (see Figure 2). Principal ports of nodes are depicted as black semicircles; a ***concluding port*** is a principal port with no incident edge. The two ***upper edges*** of a par are the edges immediately above.

We introduce a variant of the contractibility correctness criterion [Dan90, Laf95]. To ***break*** a par of a proof structure is to delete its two upper edges. A broken par p is ***gluable*** if its deleted upper edges go up to the same connected component, distinct from the component of p. A ***gluing*** is the replacement of the deleted upper edges of a gluable par.

Definition (Proof net). *A proof structure Θ is a **proof net** if breaking every par of Θ leaves a forest, from which Θ can be restored by a sequence of gluings.*

See Figure 2 for an example.

Theorem (Sequentialization). *A proof structure Θ is sequentializable iff Θ is a proof net.*

The proof is in Section 4. Figures 3–7 illustrate the simple idea: construct a sequentialization by mimicking the gluing sequence on proof fragments.

3 Gluing proof fragments

A ***broken proof structure*** is any subgraph of a proof structure obtained by breaking zero or more pars; a connected component of a broken proof structure is called a ***proof structure fragment***. A ***proof fragment*** is any proof of the extension of MLL with the ***broken par rule***:

$$\frac{}{\vdash\ A \parr B}\,(\parr) \qquad\qquad (A \text{ and } B \text{ are arbitrary MLL formulas})$$

Just as every MLL proof defines a proof structure by simple recursion [Gir87], every MLL proof fragment defines a proof structure fragment: add the base case that a broken par rule defines a broken par. A proof structure fragment ϕ is ***sequentializable*** if it is defined by a proof fragment; the proof fragment is a ***sequentialization*** of ϕ. Assume that sequents are multisets, so the permutation rule is omitted. Then the rules of a sequentialization π of ϕ are in bijection with the nodes of ϕ, and the concluding formulas of π are in bijection with the concluding ports of ϕ.

An acyclic proof structure fragment in which every par is broken is called a ***tensor tree***. By induction on the number of nodes, a tensor tree ϕ is trivially sequentializable: the base case of a broken par is sequentialized by a broken par rule; otherwise ϕ comprises two smaller tensor trees joined at a tensor. Figure 3 shows five tensor trees and their sequentializations.

Given a broken par rule ρ with conclusion $A \parr B$ in a proof fragment π, and a proof fragment π' with conclusion Γ, A, B (see Figure 1), the ***gluing*** of ρ to π' at A, B consists in (i) placing Γ, A, B as the hypothesis of ρ, (ii) replacing every sequent Δ below Γ, A, B by Γ, Δ.

4 Proof of the sequentialization theorem

Lemma (Gluing preserves sequentializability). *Let ϕ, ϕ' be fragments of a broken proof structure, let p be a broken par of ϕ with deleted edges e_1, e_2 up to principal ports x_1, x_2 in ϕ', and let θ be the fragment resulting from the gluing of p (thus $\theta = \phi \cup \{e_1, e_2\} \cup \phi'$). If ϕ and ϕ' are sequentializable, then θ is sequentializable.*

Proof. Let π, π' be sequentializations of ϕ, ϕ' respectively, let ρ be the broken par rule of π corresponding to p, and let A, B be the conclusion formulas of π' corresponding to x_1, x_2 respectively. Without loss of generality, the conclusion C of ρ is $A \parr B$ (if necessary, rename C and its descendents). Let $\widehat{\pi}$ be the result of gluing ρ to π' at A, B. By induction on the number of rules below ρ, $\widehat{\pi}$ is a sequentialization of θ. □

Theorem (Sequentialization). *A proof structure Θ is sequentializable iff Θ is a proof net.*

Proof. (*Only if.*) A simple and uninteresting induction on the number of nodes. (*If.*) By hypothesis, breaking all pars of Θ leaves a forest $\mathcal{F}$, and a sequence s of gluings restores $\mathcal{F}$ to Θ. Each fragment of $\mathcal{F}$ is a tensor tree, hence sequentializable. By Lemma 4, every consecutive gluing of s preserves sequentializability, hence Θ is sequentializable. □

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References

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[DR89] V. Danos and L. Regnier. The structure of multiplicatives. *Archive for Mathematical Logic*, 28:181–203, 1989.

[Gir87] Jean-Yves Girard. Linear Logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[Laf95] Yves Lafont. From proof-nets to interaction nets. *London Mathematical Society Lecture Note Series*, 222:225–247, 1995. Advances in Linear Logic.

$$\dfrac{\dfrac{}{P^\perp \parr P}\,(\parr) \qquad \dfrac{\dfrac{}{Q^\perp, Q}\,(\mathsf{ax}) \quad \dfrac{}{R^\perp, R}\,(\mathsf{ax})}{Q^\perp \otimes R^\perp, R, Q}\,(\otimes)}{(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp), R, Q}\,(\otimes)$$

(*Top*)

$$\dfrac{\dfrac{}{R \parr Q}\,(\parr) \qquad \dfrac{\dfrac{}{S^\perp, S}\,(\mathsf{ax})}{S^\perp \parr S}\,(\parr)}{(R \parr Q) \otimes (S^\perp \parr S)}\,(\otimes)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P^\perp \parr P}\,(\parr) \qquad \dfrac{\dfrac{}{Q^\perp, Q}\,(\mathsf{ax}) \quad \dfrac{}{R^\perp, R}\,(\mathsf{ax})}{Q^\perp \otimes R^\perp, R, Q}\,(\otimes)}{(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp), R, Q}\,(\otimes)}{R \parr Q}\,(\parr) \qquad \dfrac{\dfrac{}{S^\perp, S}\,(\mathsf{ax})}{S^\perp \parr S}\,(\parr)}{(R \parr Q) \otimes (S^\perp \parr S)}\,(\otimes)$$

(*Middle*)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P^\perp \parr P}\,(\parr) \qquad \dfrac{\dfrac{}{Q^\perp, Q}\,(\mathsf{ax}) \quad \dfrac{}{R^\perp, R}\,(\mathsf{ax})}{Q^\perp \otimes R^\perp, R, Q}\,(\otimes)}{(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp), R, Q}\,(\otimes)}{(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp),\ R \parr Q}\,(\parr) \qquad \dfrac{\dfrac{}{S^\perp, S}\,(\mathsf{ax})}{S^\perp \parr S}\,(\parr)}{(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp),\ (R \parr Q) \otimes (S^\perp \parr S)}\,(\otimes)$$

(*Bottom*)

Figure 1: Gluing a broken par rule. This example is an instance of the definition on page 2 with $A = R$, $B = Q$, and $\Gamma = (P^\perp \otimes P) \otimes (Q^\perp \otimes R^\perp)$. To save space, sequent turnstiles are omitted. (*Top*) The proof fragments π' (above) and π (below). (*Middle*) Step (i) of the definition: placing Γ, R, Q as the hypothesis of the broken par rule. (*Bottom*) Step (ii) of the definition: replacing every sequent Δ below Γ, R, Q by Γ, Δ.

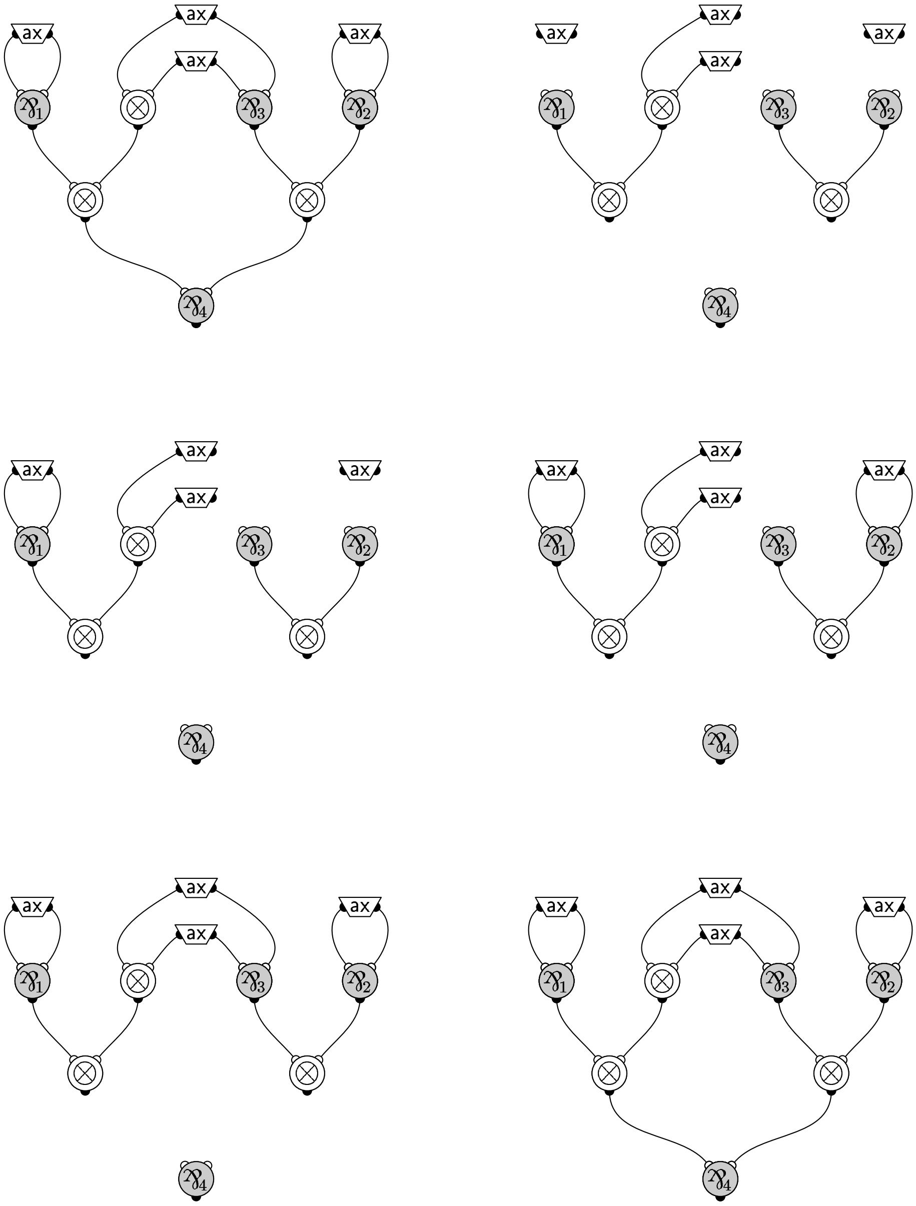


Figure 2: Verifying the correctness criterion for the proof net shown top-left. Breaking all pars leaves a forest (top-right), from which the proof structure can be restored by a sequence of gluings (mid-left, mid-right, bottom-left, bottom-right). Note that prior to gluing par 3, par 4 is not gluable, since its parents are in separate connected components.

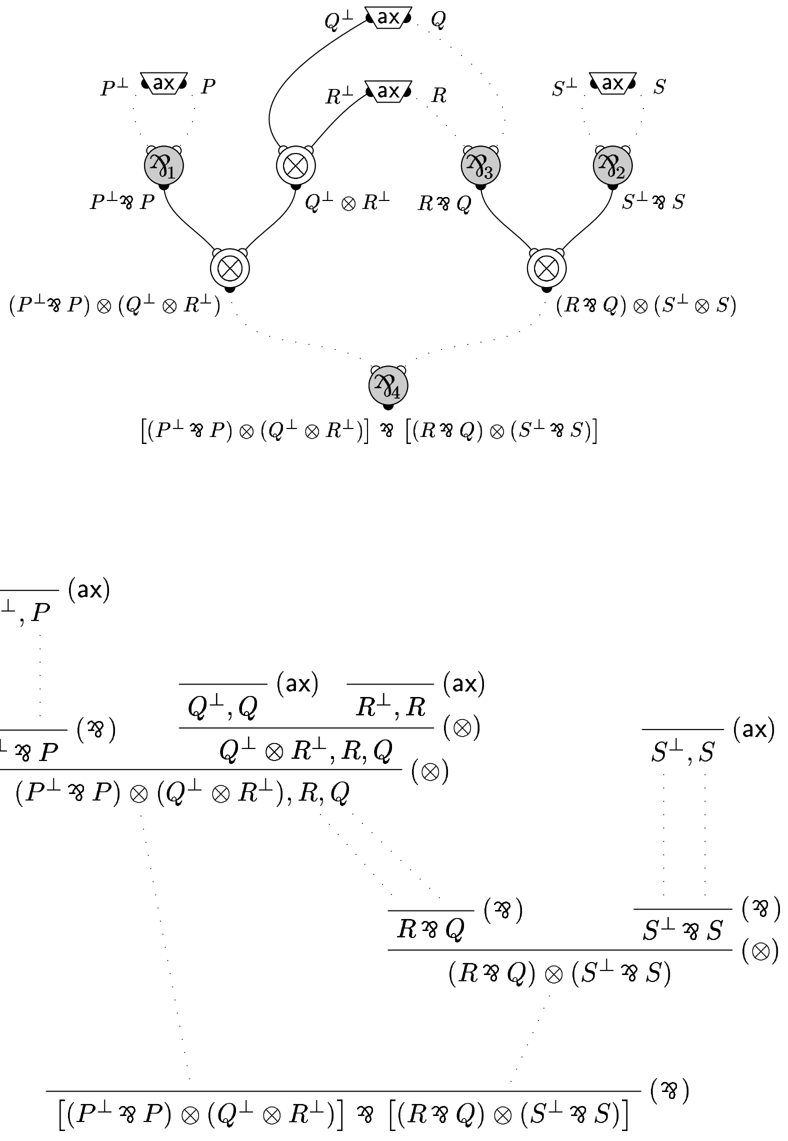


Figure 3: (*Above*) The forest $\mathcal{F}$ of five tensor trees resulting from breaking every par of the proof net Θ. (*Below*) Sequentializations of the five tensor trees, obtained trivially since every par of a tensor tree is broken. To save space, sequent turnstiles are omitted. To aid pattern-matching, the principal ports of Θ have been labelled. Dotted lines are an informal addition.

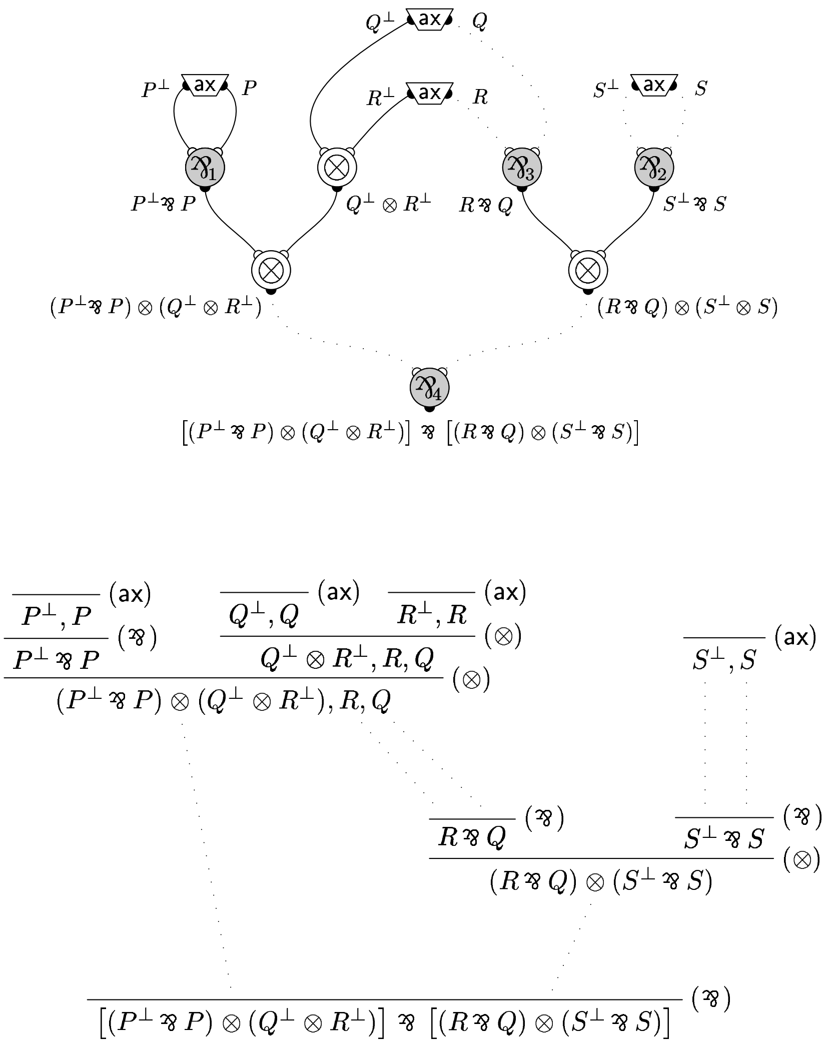


Figure 4: (*Above*) The gluing of par 1 in the forest $\mathcal{F}$. (*Below*) The corresponding gluing of tensor trees.

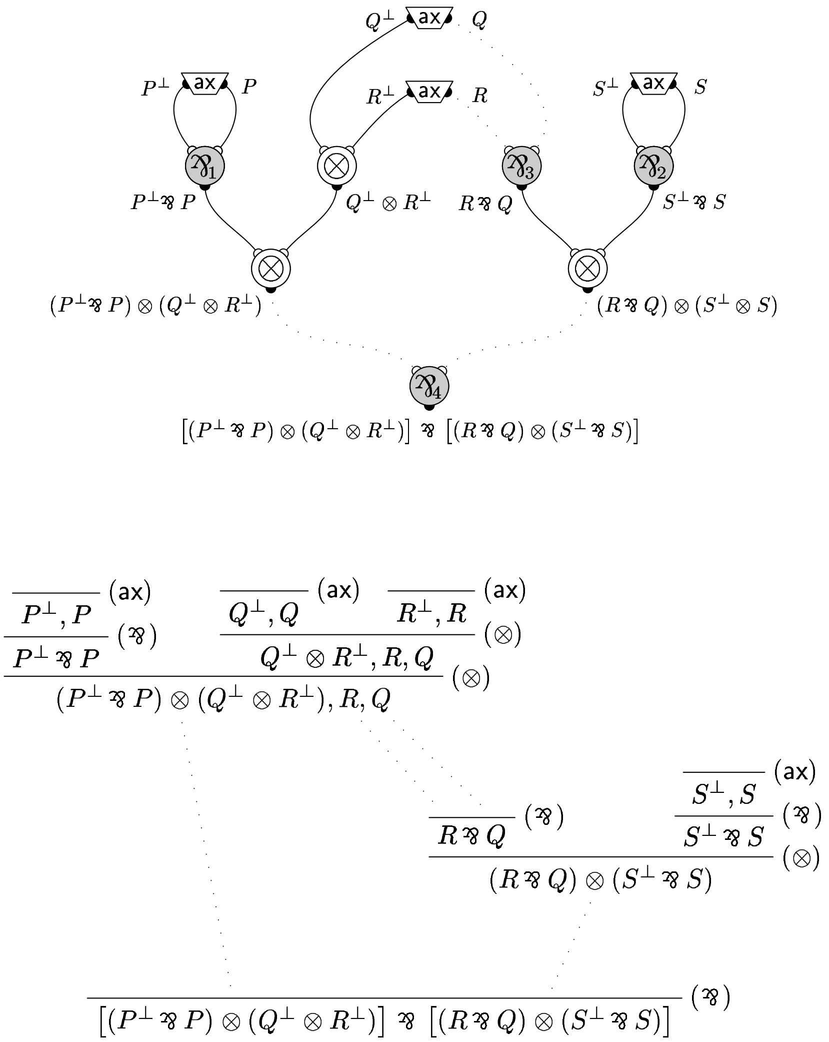
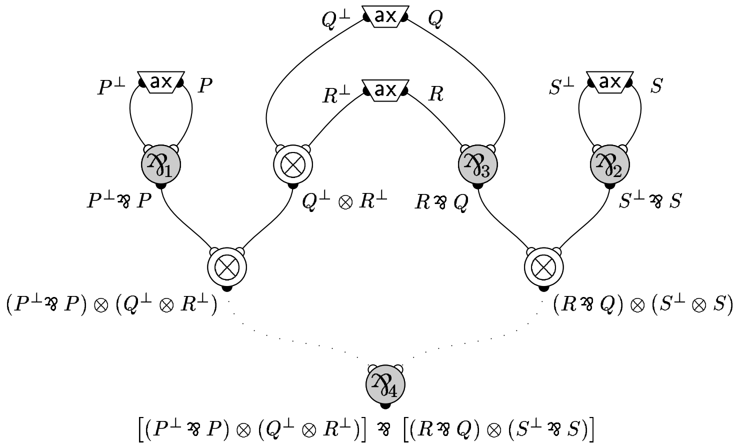


Figure 5: (*Above*) The gluing of par 2. (*Below*) The corresponding gluing of proof fragments.

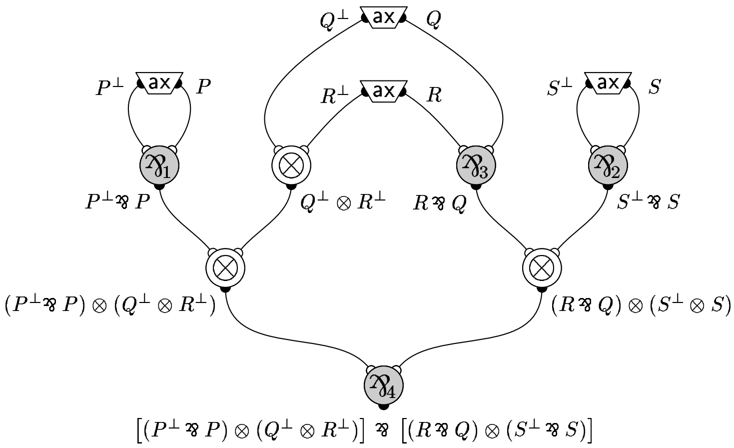


$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P^\perp, P}\,(\mathsf{ax})}{P^\perp ⅋ P}\,(⅋) \qquad \dfrac{\dfrac{}{Q^\perp, Q}\,(\mathsf{ax}) \quad \dfrac{}{R^\perp, R}\,(\mathsf{ax})}{Q^\perp \otimes R^\perp, R, Q}\,(\otimes)}{(P^\perp ⅋ P) \otimes (Q^\perp \otimes R^\perp), R, Q}\,(\otimes)}{(P^\perp ⅋ P) \otimes (Q^\perp \otimes R^\perp),\ R ⅋ Q}\,(⅋) \qquad \dfrac{\dfrac{}{S^\perp, S}\,(\mathsf{ax})}{S^\perp ⅋ S}\,(⅋)}{(P^\perp ⅋ P) \otimes (Q^\perp \otimes R^\perp),\ (R ⅋ Q) \otimes (S^\perp ⅋ S)}\,(\otimes)$$

$$\vdots$$

$$\dfrac{}{\big[(P^\perp ⅋ P) \otimes (Q^\perp \otimes R^\perp)\big] ⅋ \big[(R ⅋ Q) \otimes (S^\perp ⅋ S)\big]}\,(⅋)$$

Figure 6: (*Above*) The gluing of par 3. (*Below*) The corresponding gluing of proof fragments.



$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$P^\perp, P$}
\RightLabel{($\parr$)}
\UnaryInfC{$P^\perp \parr P$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$Q^\perp, Q$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$R^\perp, R$}
\RightLabel{($\otimes$)}
\BinaryInfC{$Q^\perp \otimes R^\perp, R, Q$}
\RightLabel{($\otimes$)}
\BinaryInfC{$(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp), R, Q$}
\RightLabel{($\parr$)}
\UnaryInfC{$(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp),\ R \parr Q$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$S^\perp, S$}
\RightLabel{($\parr$)}
\UnaryInfC{$S^\perp \parr S$}
\RightLabel{($\otimes$)}
\BinaryInfC{$(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp),\ (R \parr Q) \otimes (S^\perp \parr S)$}
\RightLabel{($\parr$)}
\UnaryInfC{$\big[(P^\perp \parr P) \otimes (Q^\perp \otimes R^\perp)\big] \parr \big[(R \parr Q) \otimes (S^\perp \parr S)\big]$}
\end{prooftree}
$$

Figure 7: (*Above*) The gluing of the final par, par 4, restoring the proof net Θ. (*Below*) The corresponding gluing of proof fragments, yielding a sequentialization of Θ.