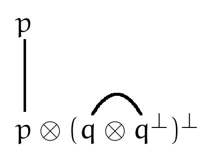
Modelling Linear Logic without Units (Preliminary Results)

Robin Houston, Dominic Hughes and Andrea Schalk

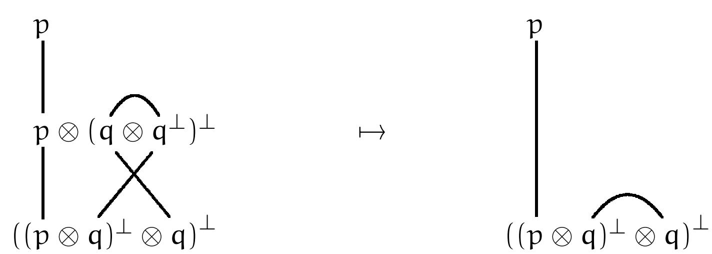
October 10, 2005

1 Introduction

Proof nets for MLL$^-$, unit-free Multiplicative Linear Logic (Girard, 1987), provide elegant, abstract representations of proofs. Cut-free MLL$^-$ proof nets form a category, under path composition[1], in the manner of Eilenberg-Kelly-Mac Lane graphs (Eilenberg and Kelly, 1966; Kelly and Mac Lane, 1971). Objects are formulas of MLL$^-$, and a morphism $A \to B$, a proof net from A to B, is a linking or matching between complementary leaves (occurrences of variables). For example, here is a morphism from p (a variable) to $p \otimes (q \otimes q^\perp)^\perp$,



and here is an example of post-composing this proof net with another from $p \otimes (q \otimes q^\perp)^\perp$ to $((p \otimes q)^\perp \otimes q)^\perp$, giving a proof net from p to $((p \otimes q)^\perp \otimes q)^\perp$:



To obtain the result of composition, one simply traces paths.

The category of proof nets is almost, but not quite, a star-autonomous category (Barr, 1979). The mismatch is the lack of units. This prompts the question of what categorical structure *does* match MLL$^-$. More precisely:

> **Question:**
> *Can one axiomatise a categorical structure, a relaxation of star-autonomy, suitable for modelling MLL$^-$? The category of MLL$^-$ proof nets* (*with explicit negation*[2]) *should be a free such category.*

Here by a *categorical structure* we mean data (functors, natural isomorphisms, and so forth) together with coherence diagrams, akin to the axiomatisation of star-autonomous categories.

[1] Path composition coincides with normalisation by cut elimination (Girard, 1987).

[2] *I.e.*, with formulas generated from variables by $\otimes$ and $(-)^\perp$, rather than from literals by tensor and par. In other words, one drops the quotienting by de Morgan duality which is implicit in the usual definition of MLL$^-$ formulas.

The naive proposal simply drops the units from a standard axiomatisation of star-autonomous category $\mathbb{C}$ (Barr, 1979):

- Tensor. A functor $- \otimes - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.
- Associativity. A natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ natural in objects $A, B, C \in \mathbb{C}$ such that the usual pentagon commutes.
- Symmetry. A natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ natural in objects $A, B \in \mathbb{C}$ such that $\sigma^{-1}_{B,A} = \sigma_{A,B}$ and the usual hexagon commutes.
- Involution. A functor $(-)^{\perp} : \mathbb{C}^{op} \to \mathbb{C}$ together with a natural isomorphism $A \to A^{\perp\perp}$.
- An isomorphism $\mathbb{C}(A \otimes B, C^{\perp}) \to \mathbb{C}(A, (B \otimes C)^{\perp})$ natural in all objects A, B, C.

However, while there is a proof net from p to $p \otimes (q \otimes q^{\perp})^{\perp}$ (the first proof net depicted at the beginning of the Introduction), this axiomatisation fails to provide a corresponding morphism from p to $p \otimes (q \otimes q^{\perp})^{\perp}$ in the free such category generated from the variables p and q. The problem of finding the right axiomatisation is non-trivial.

The solution predates the problem. As so often, the solution long predates the problem. Day (1970) defines a *promonoidal* category[3] as a generalisation of a monoidal category. Rather than having a functor

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

and a unit object $I \in \mathbb{C}$, a promonoidal category has functors

$$P : \mathbb{C}^{op} \times \mathbb{C}^{op} \times \mathbb{C} \to \text{Set},$$
$$J : \mathbb{C} \to \text{Set}.$$

This brings us to our primary definition:

> *A* ***semi-monoidal category*** $\mathbb{C}$ *is a promonoidal category such that* $P(A, B, C) = \mathbb{C}(A \otimes B, C)$ *for some functor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

The motivation behind the choice of terminology *semi* here is twofold. A monoidal category is a promonoidal category satisfying:

(a) $P(A, B, C) = \mathbb{C}(A \otimes B, C)$ for some functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and

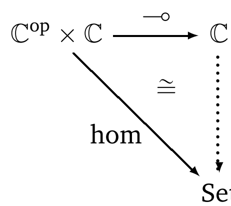
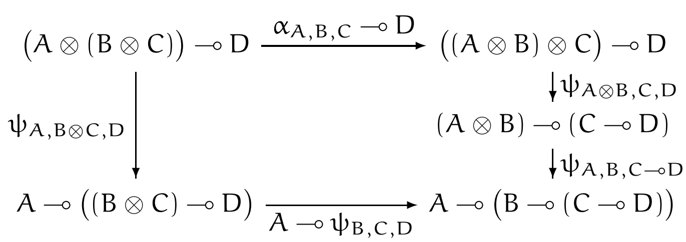
(b) $J(A) = \mathbb{C}(I, A)$ for some object I.

Semi refers to our use of just one of the two properties. Secondly, if one views *semi* as short for *semigroupal*, then one has an analogy with semigroups, which are monoids without unit.

Emulating the usual progression from monoidal category to star-autonomous category via symmetry and closure, we progress to a notion of semi star-autonomous category, our candidate for an answer to the question posed at the beginning of the Introduction. (This being a preliminary

[3] The original paper on the subject (Day, 1970) uses the term 'premonoidal'. To add to the confusion, the word premonoidal is now used to mean something quite different.

report on work in progress, we have yet to complete the proof that the category of proof nets is a free semi star-autonomous category.) A key step in this progression is the following:

> *A* ***symmetric semi-monoidal closed category*** (SSMCC) *is a category* $\mathbb{C}$ *equipped with an associative, symmetric functor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*, a functor* $\multimap : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ *through which hom factors up to isomorphism,*
>
> 
>
>
> *and a natural isomorphism*
>
> $$\psi_{A,B,C} \;:\; (A \otimes B) \multimap C \;\to\; A \multimap (B \multimap C)$$
>
> *such that*
>
> 
>
>
> *commutes.*

See Section 4 for details. Once again, the solution predates the problem: the natural isomorphism ψ, and its commuting diagram are exactly as in the definition of symmetric monoidal closed category in Eilenberg and Kelly (1965).

Related work. Our interest in obtaining an axiomatisation was sparked by the desire to characterise the category of unit-free proof nets for Multiplicative-Additive Linear Logic (Hughes and van Glabbeek, 2005) as a free category.

Soon after beginning to think about the problem, we came across an interesting and informative proposal and discussion in a draft of Lamarche and Straßburger (2005). In this draft the authors define what they call (*unitless*) *autonomous categories*, motivated (like us) by the desire to model unitless fragments of MLL.

Our definition of SSMCC is apparently stronger, in that every SSMCC is a (unitless) autonomous category in the sense of (*op. cit.*), while the converse appears to be false. In fact, certain properties are desired of categories in Lamarche and Straßburger (2005) which do not appear to be derivable from the given conditions. Indeed, in the presence of a tensor unit object I, the axioms do not seem to imply symmetric monoidal closure. One of the motivations behind producing this preprint is to suggest a solution to the problem of finding a definition with the desired properties.[4] (Section 7.1 discusses some other apparent divergences from the

[4] In correspondence Lamarche and Straßburger have indicated that they may change

desired properties.)

In our initial exploration of candidates for semi star-autonomous category, we (independently) considered essentially the same definition as in Došen and Petrić (2005): a unitless linearly distributive category with a suitable duality on objects. Ultimately we chose the approach more analogous to the standard progression from monoidal category to star-autonomous category, via symmetry and closure: it ties directly into the pioneering work of Eilenberg and Kelly (1965), with the (ψ) diagram, and also Day's promonoidal categories (Day, 1970).

Structure of paper. Section 2 gives two different (but equivalent) elementary definitions of SSMCC. Most of the remainder of the note is devoted to showing that the first of these is equivalent to the conceptual definition (Definition 5.1).

Section 3 describes the background needed to understand the subsequent development, in what is intended to be a clear and gentle (but not rigorous) way. In particular, the definitions of *coend* and *promonoidal category* are explained. Section 4 describes symmetric semi-monoidal categories, using Appendix A (which gives a simple but non-standard axiomatisation of symmetric monoidal categories). Section 5 shows that the conditions of §2.1 are necessary and sufficient. Section 6 defines semi star-autonomous categories.

Finally section 7 discusses the relationship between our definitions and the recent proposals of Lamarche and Straßburger (2005) and Došen and Petrić (2005).

Acknowledgements Many thanks to Richard Garner and Martin Hyland for their penetrating advice about an earlier version of this work. We are also grateful to Lutz Straßburger for useful discussion by email. We acknowledge use of Paul Taylor's diagrams package.

2 Summary of Results

The definition of SSMCC (in terms of promonoidal categories) is given in Def. 5.1. The main technical contribution of this note is to show that this conceptual definition can be recast in more elementary terms.

2.1 First Description

Proposition 2.1. *A symmetric semi-monoidal closed category can be described by the following data:*

- *A category* $\mathbb{C}$*,*
- *Functors* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *and* $\multimap : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$*,*
- *Natural isomorphisms* α*,* σ *and* ψ *with components*

$$\begin{aligned} \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C), \\ \sigma_{A,B} &: A \otimes B \to B \otimes A, \\ \psi_{A,B,C} &: (A \otimes B) \multimap C \to A \multimap (B \multimap C), \end{aligned}$$

the definition in the final version of their paper, as a result.

such that

$$\sigma_{A,B} = \sigma^{-1}_{B,A} \qquad (\boldsymbol{\sigma})$$

and the following coherence diagrams commute for all $A, B, C, D \in \mathbb{C}$*:*

$$\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A\otimes B,C,D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C\otimes D}} & A \otimes (B \otimes (C \otimes D)) \\
{\scriptstyle \alpha_{A,B,C} \otimes D} \searrow & & (\boldsymbol{\alpha}) & & \nearrow {\scriptstyle A \otimes \alpha_{B,C,D}} \\
(A \otimes (B \otimes C)) \otimes D & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & A \otimes ((B \otimes C) \otimes D),
\end{array}$$

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A \\
{\scriptstyle \sigma_{A,B} \otimes C} \downarrow & & (\boldsymbol{\alpha\sigma}) & & \downarrow {\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) & \xrightarrow[B \otimes \sigma_{A,C}]{} & B \otimes (C \otimes A)
\end{array}$$

$$\begin{array}{ccc}
(A \otimes (B \otimes C)) \multimap D & \xrightarrow{\alpha_{A,B,C} \multimap D} & ((A \otimes B) \otimes C) \multimap D \\
 & & \downarrow {\scriptstyle \psi_{A\otimes B,C,D}} \\
{\scriptstyle \psi_{A,B\otimes C,D}} \downarrow & (\boldsymbol{\psi}) & (A \otimes B) \multimap (C \multimap D) \\
 & & \downarrow {\scriptstyle \psi_{A,B,C\multimap D}} \\
A \multimap ((B \otimes C) \multimap D) & \xrightarrow[A \multimap \psi_{B,C,D}]{} & A \multimap (B \multimap (C \multimap D))
\end{array}$$

- *A functor* $J : \mathbb{C} \to \mathrm{Set}$ *and a natural isomorphism* e *with components* $e_{A,B} : J(A \multimap B) \to \mathbb{C}(A, B)$.

This proposition is proved in section 5 below.

2.2 Second Description

In fact there is a canonical choice for J and e; in order to state what it is, we need a few definitions:

Definition 2.2. A *category with tensor* is a category $\mathbb{C}$ equipped with a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and a natural isomorphism α satisfying condition $(\boldsymbol{\alpha})$.

A *category with symmetric tensor* is a category $\mathbb{C}$ with tensor, together with a natural isomorphism σ such that conditions $(\boldsymbol{\sigma})$ and $(\boldsymbol{\alpha\sigma})$ hold.

Definition 2.3. Let $\mathbb{C}$ be a category with tensor. A *linear element* a of the object $A \in \mathbb{C}$ is a natural transformation with components

$$a_X : X \to A \otimes X$$

such that

$$\begin{array}{ccc}
X \otimes Y & \xrightarrow{a_X \otimes Y} & (A \otimes X) \otimes Y \\
 & {\scriptstyle a_{X\otimes Y}} \searrow & \downarrow {\scriptstyle \alpha_{A,X,Y}} \\
 & & A \otimes (X \otimes Y)
\end{array}$$

commutes for all $X, Y \in \mathbb{C}$.

Definition 2.4. Given a category $\mathbb{C}$ with tensor, define a functor

$$\mathrm{Lin}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Set}$$

as follows. For $A \in \mathbb{C}$, let $\mathrm{Lin}_{\mathbb{C}}(A)$ be the set of linear elements of A. For $f : A \to B$ and $a \in \mathrm{Lin}_{\mathbb{C}}(A)$, let $\mathrm{Lin}_{\mathbb{C}}(f)(a)$ be the linear element of B with components

$$X \xrightarrow{a_X} A \otimes X \xrightarrow{f \otimes X} B \otimes X.$$

It turns out that, in the situation of Prop. 2.1, there is a canonical natural isomorphism between J and $\mathrm{Lin}_{\mathbb{C}}$. Furthermore, it happens that this natural isomorphism takes e to a particular natural transformation l, which is defined as follows.

Definition 2.5. Suppose we have $(\mathbb{C}, \otimes, \alpha)$ as above, together with a functor $\multimap : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ and a natural isomorphism

$$\mathrm{curry}_{A,B,C} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C)$$

with counit (i.e. evaluation map) $\varepsilon^A_B : (A \multimap B) \otimes A \to B$.

Define the natural transformation $l_{A,B} : \mathrm{Lin}_{\mathbb{C}}(A \multimap B) \to \mathbb{C}(A, B)$ as follows: for each $x \in \mathrm{Lin}_{\mathbb{C}}(A \multimap B)$, let $l_{A,B}(x)$ be the composite

$$A \xrightarrow{x_A} (A \multimap B) \otimes A \xrightarrow{\varepsilon^A_B} B.$$

Our first description (Prop. 2.1) is equivalent to the following.

Proposition 2.6. *An* SSMCC *can be described by:*

- *A category $\mathbb{C}$ with symmetric tensor,*
- *A functor $\multimap : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ with a natural isomorphism*

$$\mathrm{curry}_{A,B,C} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C)$$

such that the natural transformation l of Def. 2.5 is invertible.

The proof of Prop. 2.6 is still in draft form, and is not included in this preliminary note.

3 Technical Background

This section gives an informal introduction to coends and promonoidal categories.

3.1 Coends

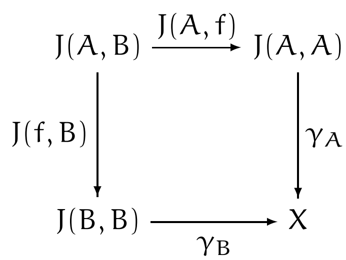
The definition of promonoidal category involves coends, so we need to understand them to some extent. Fortunately they are quite simple: a coend is just a slightly more general version of a colimit.

Recall that a colimit of a functor $J : \mathbb{D} \to \mathbb{C}$ is a universal natural transformation from J to some object $X \in \mathbb{C}$. Coends just extend this idea to mixed-variance functors $J : \mathbb{D} \times \mathbb{D}^{op} \to \mathbb{C}$: a coend of J is a universal *dinatural*[5] transformation from J to an object $X \in \mathbb{C}$.

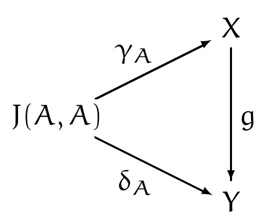
A dinatural transformation $\gamma : J \Rightarrow X$ is a family of arrows

$$\gamma_A : J(A, A) \to X$$

indexed by the objects $A \in \mathbb{D}$, such that for every $f : A \to B$ in $\mathbb{D}$ the diagram



commutes. Such a dinatural transformation is universal if, for every object Y and dinatural transformation $\delta : J \Rightarrow Y$, there is a unique fill-in morphism $g : X \to Y$ such that



commutes for every $A \in \mathbb{D}$.

It is conventional, and very handy, to write coends using integral notation. The coend of J is written $\int^{A \in \mathbb{D}} J(A, A)$; we usually omit the '$\in \mathbb{D}$' part when it is obvious from the context.

In the rest of this note, we only need to use coends in Set or in functor categories of the form $[\mathbb{C}, \text{Set}]$ for some $\mathbb{C}$. These categories have all (small) coends, so we shall never have to worry about whether or not a particular coend exists.[6]

Here are some important properties of coends. We use them heavily in the sequel, often without remark.

- Left adjoints preserve coends. In particular, in a cartesian closed category $\mathbb{C}$ we have

$$A \times \int^{X \in \mathbb{D}} J(X, X) \cong \int^{X \in \mathbb{D}} A \times J(X, X)$$

[5] The dinatural transformations we need to consider are of the special kind called *extraordinary natural transformations*. The distinction is important in enriched category theory, where extraordinary natural transformations can be defined but dinatural transformations can not in general.

[6] We are glossing over some size issues here, which can be dealt with in the usual way. The foundationally conservative reader may read the word 'category', where it appears in a definition, as 'small category'.

for every $A \in \mathbb{C}$;

- The 'Fubini theorem': if $\int^{Y \in \mathbb{D}_2} J(X, X', Y, Y)$ exists for all X, $X' \in \mathbb{D}_1$ then

$$\int^{X \in \mathbb{D}_1} \int^{Y \in \mathbb{D}_2} J(X, X, Y, Y) \cong \int^{(X,Y) \in \mathbb{D}_1 \times \mathbb{D}_2} J(X, X, Y, Y)$$

for $J : \mathbb{D}_1 \times \mathbb{D}_1^{op} \times \mathbb{D}_2 \times \mathbb{D}_2^{op} \to \mathbb{C}$;

- For every $F : \mathbb{C} \to \mathrm{Set}$ and $Y \in \mathbb{C}$,

$$FY \cong \int^{X \in \mathbb{C}} FX \times \mathbb{C}(X, Y)$$

and the coend on the right exists;

- The dual of the above: for every $F : \mathbb{C}^{op} \to \mathrm{Set}$ and $X \in \mathbb{C}$,

$$FX \cong \int^{Y \in \mathbb{C}} FY \times \mathbb{C}(X, Y).$$

The latter two isomorphisms can be regarded as versions of the Yoneda lemma. For proofs of all these facts, and a generally very nice tutorial introduction to coends, see the lecture notes by Càccamo et al. (2002).

3.2 Promonoidal Categories

As mentioned in the introduction, a promonoidal category is a category $\mathbb{C}$ together with functors

$$P : \mathbb{C}^{op} \times \mathbb{C}^{op} \times \mathbb{C} \to \mathrm{Set},$$
$$J : \mathbb{C} \to \mathrm{Set}.$$

and natural isomorphisms α, λ and ρ satisfying conditions analogous[7] to those in the definition of a monoidal category.

In order to understand what these conditions are, we develop an informal procedure for translating the language of monoidal categories into the language of promonoidal categories. In a monoidal category $\mathbb{C}$ we can form various functors $\mathbb{C}^n \to \mathbb{C}$ for some natural number n, using the tensor product and unit object. For example we have the functor $F : \mathbb{C}^2 \to \mathbb{C}$ defined as

$$F(A_1, A_2) = (A_1 \otimes I) \otimes (A_2 \otimes A_1).$$

In general, such a functor is described by an expression formed from variables $A_1, \ldots, A_n$ and the constant I using the binary operation $\otimes$. These expressions can be thought of as denoting the objects of the free monoidal category on a countably infinite set of generators.

Given such an expression S and promonoidal category $\mathbb{C}$, we can define a corresponding expression $\ulcorner S \urcorner$, representing a functor $(\mathbb{C}^{op})^n \to [\mathbb{C}, \mathrm{Set}]$, recursively as follows:

$$\begin{aligned} \ulcorner I \urcorner &= J, \\ \ulcorner A_i \urcorner &= \mathbb{C}(A_i, -), \\ \ulcorner S \otimes T \urcorner &= \int^{X,Y} \ulcorner S \urcorner(X) \times \ulcorner T \urcorner(Y) \times P(X, Y, -). \end{aligned}$$

[7] In fact this is no mere analogy: monoidal and promonoidal categories are both instances of the general notion of *pseudomonoid* in a monoidal bicategory (Day and Street, 1997). A promonoidal category is a pseudomonoid in the bicategory whose objects are categories and whose 1-cells are modules (aka profunctors or distributors).

(We assume that bound variables are renamed where necessary.)

For example, we have

$$\begin{aligned}
\ulcorner A_1 \otimes A_2 \urcorner &= \int^{X,Y} \ulcorner A_1 \urcorner(X) \times \ulcorner A_2 \urcorner(Y) \times P(X,Y,-) \\
&= \int^{X,Y} \mathbb{C}(A_1,X) \times \mathbb{C}(A_2,Y) \times P(X,Y,-) \\
&\cong \int^{X} (\mathbb{C}(A_1,X) \times \int^{Y} \mathbb{C}(A_2,Y) \times P(X,Y,-)) \\
&\cong \int^{X} \mathbb{C}(A_1,X) \times P(X,A_2,-) \\
&\cong P(A_1,A_2,-)
\end{aligned}$$

and

$$\begin{aligned}
\ulcorner A_1 \otimes I \urcorner &= \int^{X,Y} \ulcorner A_1 \urcorner(X) \times \ulcorner I \urcorner(Y) \times P(X,Y,-) \\
&= \int^{X,Y} \mathbb{C}(A_1,X) \times J(Y) \times P(X,Y,-) \\
&\cong \int^{Y} J(Y) \times P(A_1,Y,-).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\ulcorner I \otimes A_1 \urcorner &\cong \int^{X} J(X) \times P(X,A_1,-), \\
\ulcorner (A_1 \otimes A_2) \otimes A_3 \urcorner &\cong \int^{X} P(A_1,A_2,X) \times P(X,A_3,-), \\
\ulcorner A_1 \otimes (A_2 \otimes A_3) \urcorner &\cong \int^{X} P(A_1,X,-) \times P(A_2,A_3,X).
\end{aligned}$$

Now we can say what the types of α, λ and ρ should be. They should have components

$$\begin{aligned}
\alpha_{A,B,C} &: \int^{X} P(A,B,X) \times P(X,C,-) \to \int^{X} P(A,X,-) \times P(B,C,X) \\
\lambda_{A} &: \int^{X} J(X) \times P(X,A,-) \to \mathbb{C}(A,-) \\
\rho_{A} &: \int^{Y} J(Y) \times P(A,Y,-) \to \mathbb{C}(A,-)
\end{aligned}$$

Each component here is *itself* a natural transformation, so we can add an extra variable and ask for natural isomorphisms

$$\begin{aligned}
\alpha_{A,B,C,Z} &: \int^{X} P(A,B,X) \times P(X,C,Z) \to \int^{X} P(A,X,Z) \times P(B,C,X) \\
\lambda_{A,Z} &: \int^{X} J(X) \times P(X,A,Z) \to \mathbb{C}(A,Z) \\
\rho_{A,Z} &: \int^{Y} J(Y) \times P(A,Y,Z) \to \mathbb{C}(A,Z)
\end{aligned}$$

between functors to Set.

We impose the usual coherence conditions, as described for monoidal categories in the appendix and elaborated below in the semi-monoidal case.

3.3 Notation for Diagrams

In a diagram containing several cells which are known to commute, we often label each such cell with the reason that it commutes, removing the need for separate explanations that must be cross-referenced with the diagram. The symbol $\natural$ is used to indicate that a cell commutes by naturality of some natural transformation.

4 Symmetric Semi-monoidal Categories

We consider the special case of a *semi-monoidal category*, i.e. a promonoidal category $\mathbb{C}$ whose P is represented by a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Therefore suppose we have such a functor $\otimes$, and that $P(A,B,X) = \mathbb{C}(A \otimes B, X)$. Now we have

$$\begin{aligned} \ulcorner(A_1 \otimes A_2) \otimes A_3\urcorner &\cong \int^X \ulcorner A_1 \otimes A_2\urcorner(X) \times \mathbb{C}(X \otimes A_3, -) \\ &\cong \int^X \mathbb{C}(A_1 \otimes A_2, X) \times \mathbb{C}(X \otimes A_3, -) \\ &\cong \mathbb{C}((A_1 \otimes A_2) \otimes A_3, -) \end{aligned}$$

and similarly

$$\ulcorner A_1 \otimes (A_2 \otimes A_3)\urcorner \cong \mathbb{C}(A_1 \otimes (A_2 \otimes A_3), -).$$

Thus, by Yoneda, the associativity isomorphism may be represented by a natural isomorphism α with components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

just as in an ordinary monoidal category, subject to the usual pentagon condition ($\boldsymbol{\alpha}$).

We are really interested in *symmetric* semi-monoidal categories, so suppose that there is also a symmetry σ with components $\sigma_{A,B} : A \otimes B \to B \otimes A$ such that $\sigma_{A,B} = \sigma^{-1}_{B,A}$ for all $A,B \in \mathbb{C}$, and satisfying the hexagon condition ($\boldsymbol{\alpha\sigma}$).

By the argument in the appendix – more precisely, by reinterpreting that argument in the promonoidal setting – we have a symmetric semi-monoidal category just when there is a natural isomorphism λ with components

$$\lambda_{A,Z} : \int^X J(X) \times \mathbb{C}(X \otimes A, Z) \to \mathbb{C}(A, Z)$$

such that the diagram

$$\begin{array}{ccc} \int^X J(X) \times \mathbb{C}((X \otimes B) \otimes C, Z) & \xrightarrow{\int^X J(X) \times \mathbb{C}(\alpha_{X,B,C}, Z)} & \int^X J(X) \times \mathbb{C}(X \otimes (B \otimes C), Z) \\ \cong\downarrow & & \\ \int^{X,Y} J(X) \times \mathbb{C}(X \otimes B, Y) \times \mathbb{C}(Y \otimes C, Z) & (1) & \downarrow\lambda_{B\otimes C,Z} \\ \int^Y \lambda_{B,Y} \times \mathbb{C}(Y \otimes C, Z)\downarrow & & \\ \int^Y \mathbb{C}(B, Y) \times \mathbb{C}(Y \otimes C, Z) & \xrightarrow[\cong]{} & \mathbb{C}(B \otimes C, Z) \end{array}$$

commutes for all $B,C,Z \in \mathbb{C}$.

5 Symmetric Semi-monoidal Closed Categories

Generally speaking, a promonoidal category $\mathbb{C}$ is *left closed* if it has a functor $\multimap : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ with a natural isomorphism

$$P(A, B, C) \cong \mathbb{C}(A, B \multimap C).$$

In the case of present interest, we have:

Definition 5.1. A *symmetric semi-monoidal closed category* (SSMCC) is a symmetric semi-monoidal category $\mathbb{C}$ together with a functor

$$\multimap : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$$

and a natural isomorphism with components

$$\mathrm{curry}_{A,B,C} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C).$$

Recall the characterisation claimed in Prop. 2.1. This differs from Def. 5.1 in the following ways: instead of λ we have e; instead of curry we have ψ; and instead of (1) we have (ψ). (Note that no conditions are imposed explicitly on J or e.) The rest of the section is devoted to proving Prop. 2.1.

Lemma 5.2. *There is an isomorphism* SSMCC*,* $\ulcorner I \otimes A \urcorner \cong J(A \multimap -)$*, natural in* A*.*

Proof. We have the chain of natural isomorphisms

$$\begin{aligned} \ulcorner I \otimes A \urcorner &= \textstyle\int^X J(X) \times \mathbb{C}(X \otimes A, -) \\ &\cong \textstyle\int^X J(X) \times \mathbb{C}(X, A \multimap -) \\ &\cong J(A \multimap -); \end{aligned}$$

□

Lemma 5.3. *To give a natural isomorphism* $\lambda_A : \ulcorner I \otimes A \urcorner \Rightarrow \ulcorner A \urcorner$ *such that* (*1*) *commutes is to give a natural isomorphism*

$$e_{A,Z} : J(A \multimap Z) \to \mathbb{C}(A, Z).$$

such that the diagram

$$\begin{array}{ccc}
\int^X JX \times \mathbb{C}(X \otimes (B \otimes C), Z) & \xrightarrow{\int^X JX \times \mathbb{C}(\alpha_{X,B,C}, Z)} & \int^X JX \times \mathbb{C}((X \otimes B) \otimes C, Z) \\
 & & \Big\downarrow {\scriptstyle \int^X JX \times \mathrm{curry}_{X \otimes B, C, Z}} \\
{\scriptstyle \int^X JX \times \mathrm{curry}_{X, B \otimes C, Z}} \Big\downarrow & & \int^X JX \times \mathbb{C}(X \otimes B, C \multimap Z) \\
 & & \Big\downarrow {\scriptstyle \int^X JX \times \mathrm{curry}_{X, B, C \multimap Z}} \\
\int^X JX \times \mathbb{C}(X, (B \otimes C) \multimap Z) & (2) & \int^X JX \times \mathbb{C}(X, B \multimap (C \multimap Z)) \\
{\scriptstyle \cong} \Big\downarrow & & \Big\downarrow {\scriptstyle \cong} \\
J((B \otimes C) \multimap Z) & & J(B \multimap (C \multimap Z)) \\
{\scriptstyle e_{B \otimes C, Z}} \Big\downarrow & & \Big\downarrow {\scriptstyle e_{B, C \multimap Z}} \\
\mathbb{C}(B \otimes C, Z) & \xrightarrow[\mathrm{curry}_{B,C,Z}]{} & \mathbb{C}(B, C \multimap Z)
\end{array}$$

commutes for all $B, C, Z \in \mathbb{C}$*.*

Proof. By Lemma 5.2 we can derive e from λ and vice versa, and the translations are mutually inverse. So it makes no difference whether we are given e or λ, since each can be derived from the other in a canonical way.

It remains to show that (1) is equivalent to (2). Consider the diagram in Fig. 1. The left-hand cell commutes by the relationship between e and λ. The remaining cells commute by naturality, or functoriality of the coend. The left edge of this diagram is equal to the left and lower edge of (1). Therefore condition (1) is equivalent to the condition in the statement.

□

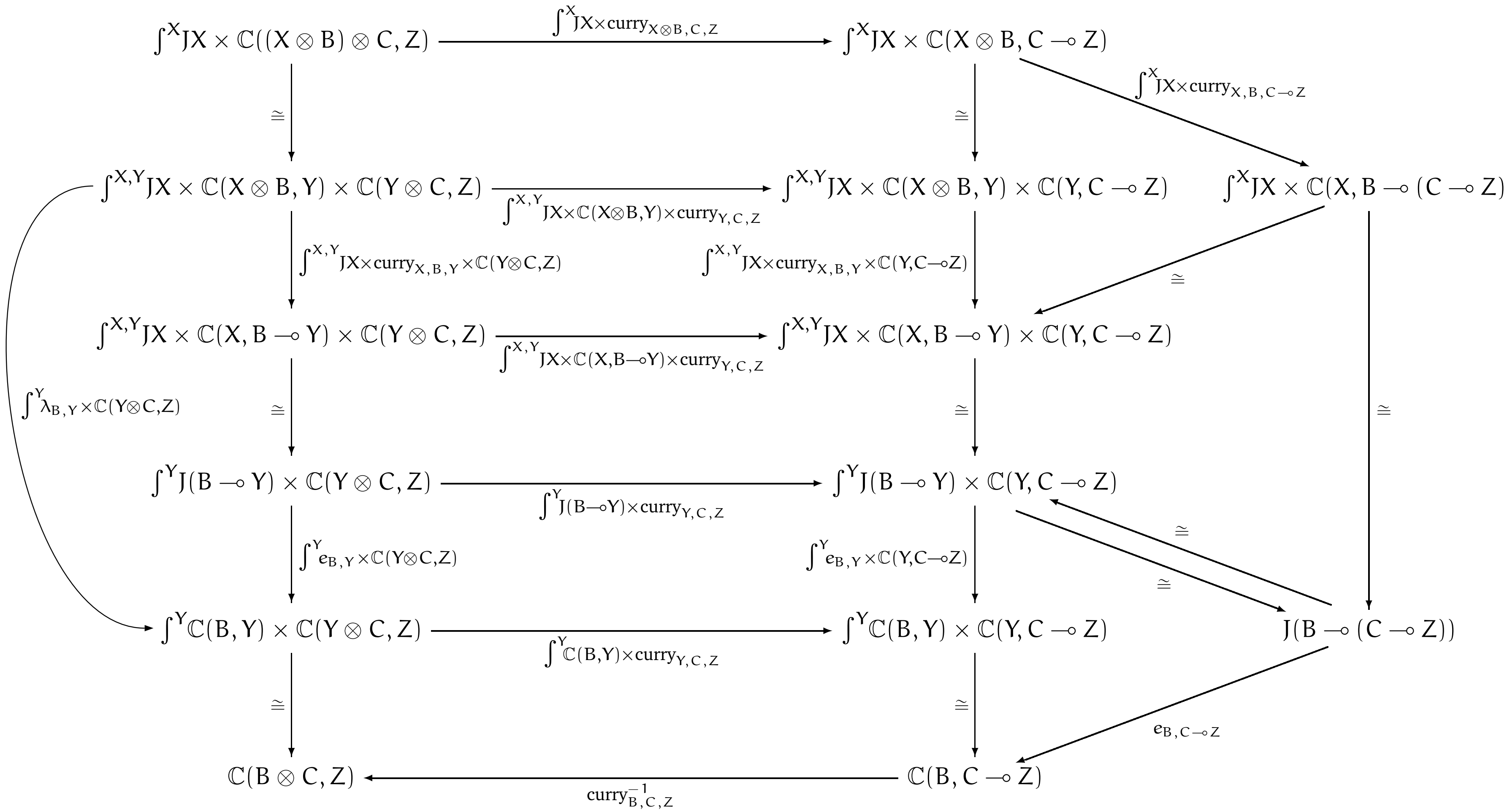


Figure 1: Diagram used in the proof of Lemma 5.3

We are still working in an SSMCC as originally defined, so we have a curry isomorphism. We construct a natural isomorphism ψ' as follows.

Definition 5.4. Given a natural isomorphism

$$\mathrm{curry}_{A,B,C} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C)$$

we define the natural isomorphism

$$\psi'_{A,B,C} : (A \otimes B) \multimap C \quad \longrightarrow \quad A \multimap (B \multimap C)$$

to be the unique such natural transformation for which

$$\begin{array}{ccc}
\mathbb{C}(A, (X \otimes Y) \multimap Z) & \xrightarrow{\mathbb{C}(A, \psi')} & \mathbb{C}(A, X \multimap (Y \multimap Z)) \\
\mathrm{curry}^{-1}_{A,X\otimes Y,Z} \downarrow & & \\
\mathbb{C}(A \otimes (X \otimes Y), Z) & (3) & \uparrow \mathrm{curry}_{A,X,Y\multimap Z} \\
\mathbb{C}(\alpha_{A,X,Y}, Z) \downarrow & & \\
\mathbb{C}((A \otimes X) \otimes Y, Z) & \xrightarrow[\mathrm{curry}_{A\otimes X,Y,Z}]{} & \mathbb{C}(A \otimes X, Y \multimap Z)
\end{array}$$

commutes for all $A,X,Y,Z \in \mathbb{C}$. (Uniqueness is a consequence of the Yoneda lemma.)

Using this, we can recast condition (1) very simply.

Lemma 5.5. *In an SSMCC, condition (1) holds iff*

$$\begin{array}{ccc}
J((A \otimes B) \multimap C) & \xrightarrow{J(\psi'_{A,B,C})} & J(A \multimap (B \multimap C)) \\
e_{A\otimes B,C} \downarrow & (4) & \downarrow e_{A,B\multimap C} \\
\mathbb{C}(A \otimes B, C) & \xrightarrow[\mathrm{curry}_{A,B,C}]{} & \mathbb{C}(A, B \multimap C)
\end{array}$$

commutes for all $A, B, C \in \mathbb{C}$.

Proof. By Lemma 5.3 we know that (1) is equivalent to (2). Now we have

$$\begin{array}{ccc}
\int^X JX \times \mathbb{C}(X \otimes (B \otimes C), Z) & \xrightarrow{\int^X JX\times\mathbb{C}(\alpha_{X,B,C},Z)} & \int^X JX \times \mathbb{C}((X \otimes B) \otimes C, Z) \\
 & & \downarrow \int^X JX\times\mathrm{curry}_{X\otimes B,C,Z} \\
\int^X JX\times\mathrm{curry}_{X,B\otimes C,Z} \downarrow & (3) & \int^X JX \times \mathbb{C}(X \otimes B, C \multimap Z) \\
 & & \downarrow \int^X JX\times\mathrm{curry}_{X,B,C\multimap Z} \\
\int^X JX \times \mathbb{C}(X, (B \otimes C) \multimap Z) & \xrightarrow{\int^X JX\times\mathbb{C}(X,\psi'_{B,C,Z})} & \int^X JX \times \mathbb{C}(X, B \multimap (C \multimap Z)) \\
\cong \downarrow & \natural & \downarrow \cong \\
J((B \otimes C) \multimap Z) & \xrightarrow[J(\psi'_{B,C,Z})]{} & J(B \multimap (C \multimap Z)) \\
e_{B\otimes C,Z} \downarrow & & \downarrow e_{B,C\multimap Z} \\
\mathbb{C}(B \otimes C, Z) & \xrightarrow[\mathrm{curry}_{B,C,Z}]{} & \mathbb{C}(B, C \multimap Z)
\end{array}$$

The upper two regions commute for the reasons marked, and all the arrows are invertible, therefore the outside (2) commutes iff the lower cell (4) does. □

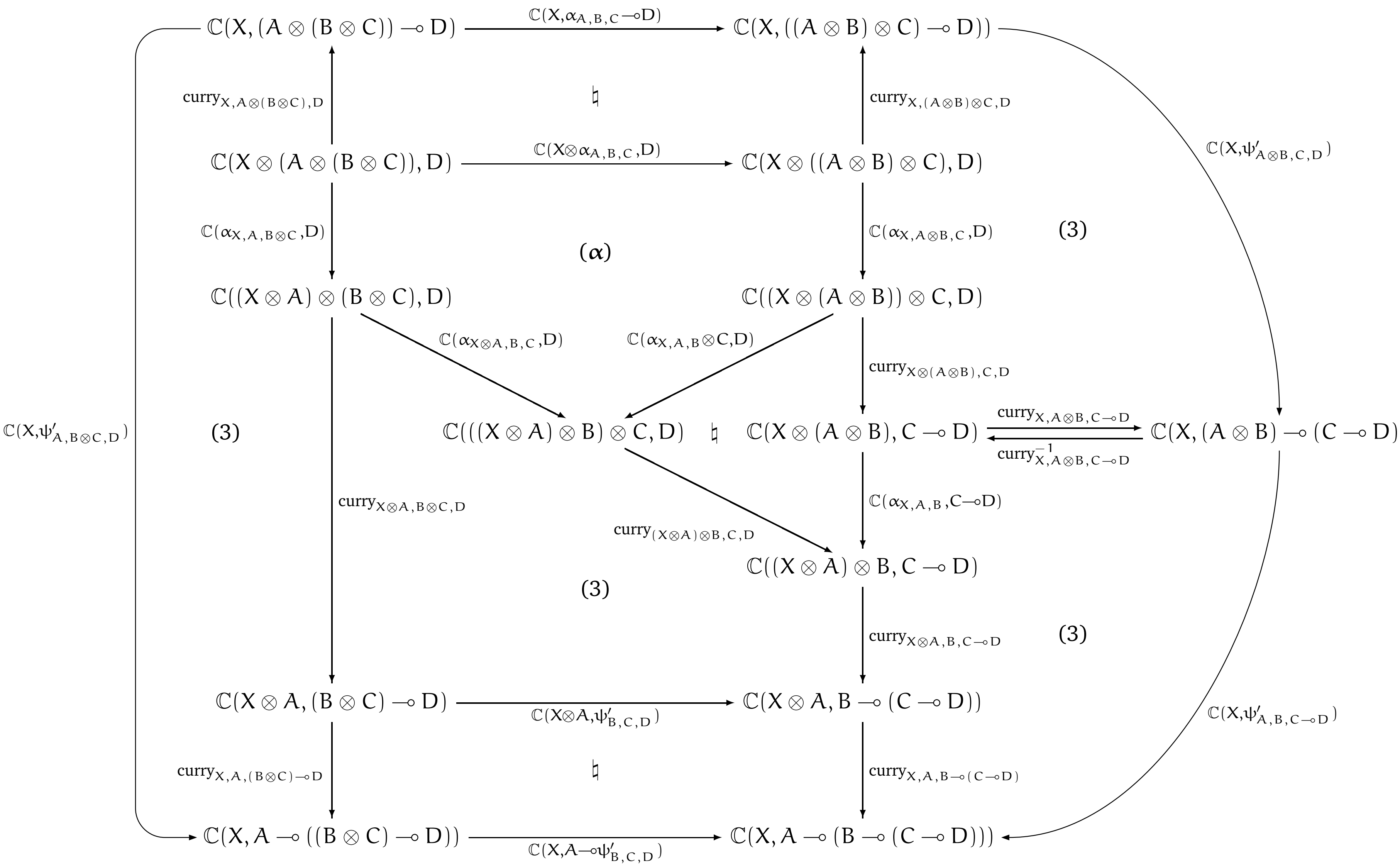


Figure 2: Diagram used in the proof of Lemma 5.6

Lemma 5.6. *In every* SSMCC*, diagram* (ψ) *commutes for* ψ' *in place of* ψ.

Proof. Consider the diagram in Fig. 2. All the regions commute for the reasons marked, thus the outside commutes. By Yoneda, it follows that (ψ) commutes as required. □

Lemma 5.7. *Suppose we have a natural isomorphism* ψ *with components*

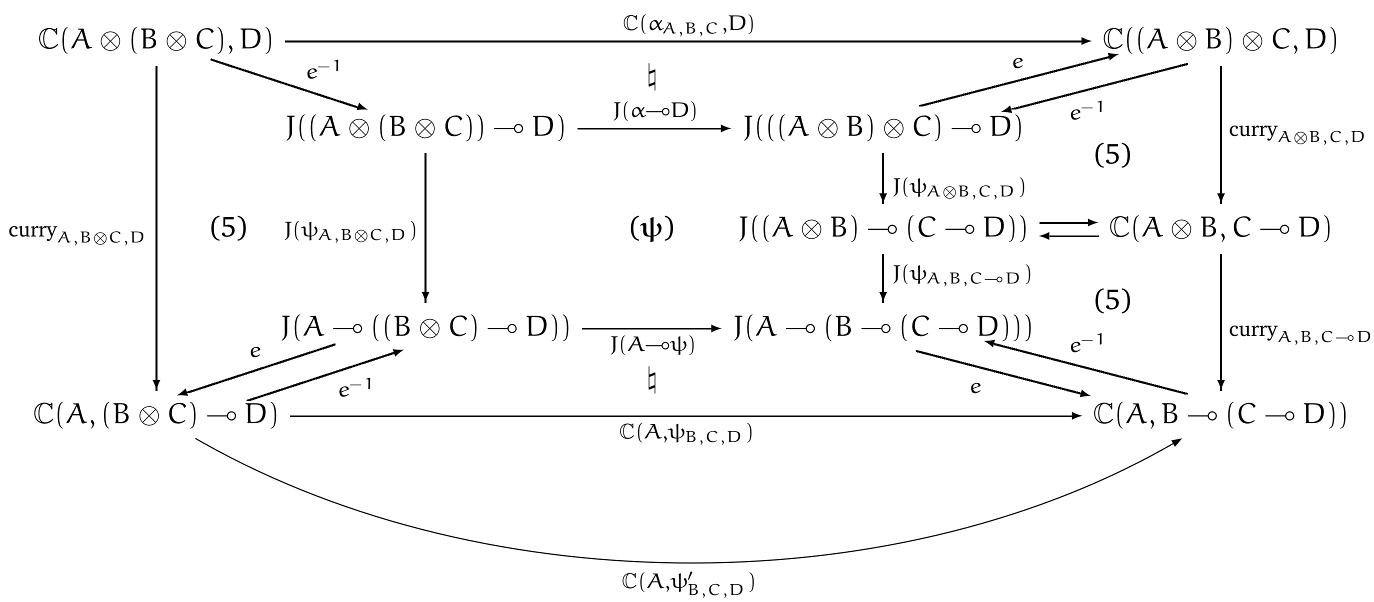
$$\psi_{A,B,C} : (A \otimes B) \multimap C \longrightarrow A \multimap (B \multimap C)$$

such that

$$\begin{array}{ccc} J((A \otimes B) \multimap C) & \xrightarrow{J(\psi_{A,B,C})} & J(A \multimap (B \multimap C)) \\ \downarrow e_{A\otimes B,C} & (5) & \downarrow e_{A,B\multimap C} \\ \mathbb{C}(A \otimes B, C) & \xrightarrow[\mathrm{curry}_{A,B,C}]{} & \mathbb{C}(A, B \multimap C) \end{array}$$

commutes for all $A, B, C \in \mathbb{C}$*. If* ψ *satisfies condition* (ψ) *then* $\psi = \psi'$.

Proof. Suppose ψ satisfies condition (ψ). Then we have



The marked cells commute for the reasons indicated, and the outer edge by assumption. Since all arrows are invertible, it follows that the lower cell also commutes; hence $\psi = \psi'$ by Yoneda. □

In other words, if we are given a ψ satisfying (ψ), we can use (5) to construct a curry isomorphism such that (2) holds. (We have already proved that, given a curry isomorphism satisfying (2), we can use (3) to construct a natural isomorphism ψ' such that (ψ) holds.) This completes the proof of Prop. 2.1.

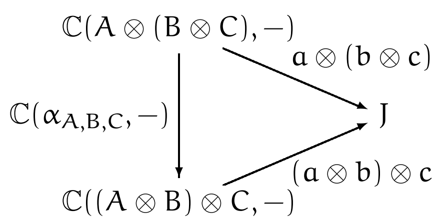
5.1 Tensor of Elements

By Yoneda's lemma, an element of JA corresponds to a natural transformation $\mathbb{C}(A, -) \Rightarrow J$. If we have elements $a \in JA$ and $b \in JB$ then we may define a natural transformation

$$\begin{aligned} \mathbb{C}(A \otimes B, -) &\xrightarrow{\cong} \mathbb{C}(A, B \multimap -) \\ &\xrightarrow{a} J(B \multimap -) \\ &\xrightarrow{\cong} \mathbb{C}(B, -) \\ &\xrightarrow{b} J, \end{aligned}$$

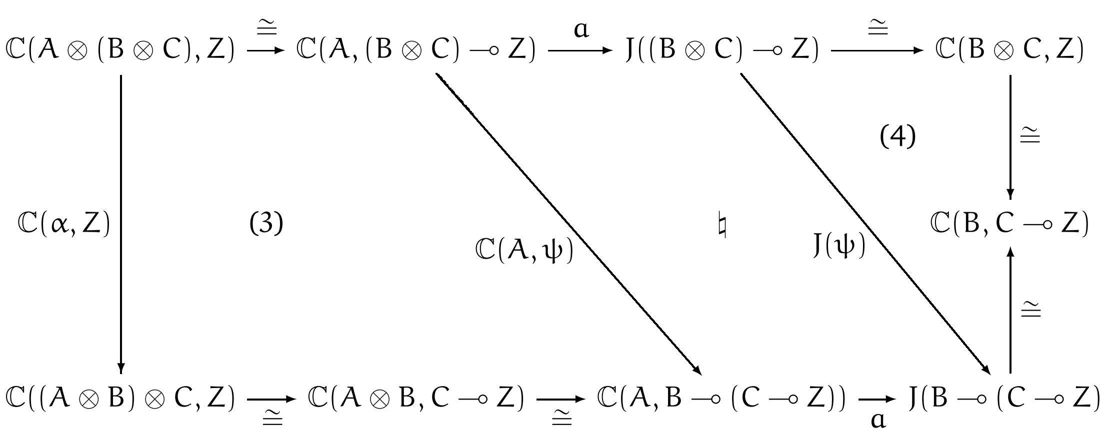
corresponding to an element of $J(A \otimes B)$. We denote this element $a \otimes b$. It is easy to check that this operation defines a natural transformation with components $m_{A,B} : JA \times JB \to J(A \otimes B)$.

Proposition 5.8. *This natural transformation agrees with the associativity, in the sense that the diagram*



commutes for every $A, B, C \in \mathbb{C}$ *with* $a \in JA$, $b \in JB$ *and* $c \in JC$.

Proof. Consider the diagram



Since the internal cells commute, so does the outside. Now observe that, by definition, the natural transformation $a \otimes (b \otimes c)$ is equal to the upper path followed by the composite

$$\mathbb{C}(B, C \multimap Z) \xrightarrow{b} J(C \multimap Z) \cong \mathbb{C}(C, Z) \xrightarrow{c} JZ,$$

while $(a \otimes b) \otimes c$ is equal to the lower path followed by this composite. Thus the claim follows. $\square$

6 The Star-autonomous Case

There is a general notion of promonoidal star-autonomous category (Day and Street, 2004, §7). A symmetric promonoidal star-autonomous category is a symmetric promonoidal category $\mathbb{C}$ equipped with a full and faithful functor $-^{\perp} : \mathbb{C} \to \mathbb{C}^{op}$ and a natural isomorphism

$$P(A, B, C^{\perp}) \cong P(A, C, B^{\perp}).$$

This specialises in the obvious way:

Definition 6.1. A semi star-autonomous category is a symmetric semi-monoidal category $\mathbb{C}$ with a full and faithful functor $-^{\perp} : \mathbb{C} \to \mathbb{C}^{op}$ and a natural isomorphism

$$\mathbb{C}(A \otimes B, C^{\perp}) \cong \mathbb{C}(A \otimes C, B^{\perp}). \tag{6}$$

Note that, since $-^{\perp}$ is full and faithful, there is a natural isomorphism $\mathbb{C}(A, B) \cong \mathbb{C}^{op}(A^{\perp}, B^{\perp}) = \mathbb{C}(B^{\perp}, A^{\perp})$.

Lemma 6.2. *There is a natural isomorphism* $B \cong B^{\perp\perp}$.

Proof. There is a sequence of natural isomorphisms

$$\begin{aligned}
\mathbb{C}(A, B) &\cong \mathbb{C}(B^{\perp}, A^{\perp}) && {}^{\perp} \text{ is full and faithful} \\
&\cong \textstyle\int^{X} JX \times \mathbb{C}(X \otimes B^{\perp}, A^{\perp}) && \text{using } \lambda^{-1} \\
&\cong \textstyle\int^{X} JX \times \mathbb{C}(X \otimes A, B^{\perp\perp}) && \text{by (6)} \\
&\cong \mathbb{C}(A, B^{\perp\perp}) && \text{using } \lambda.
\end{aligned}$$

Therefore, by Yoneda's lemma, it follows that B is naturally isomorphic to $B^{\perp\perp}$, as required. □

Proposition 6.3. *Every semi star-autonomous category is a* SSMCC, *with* $A \multimap B$ *defined as* $(A \otimes B^{\perp})^{\perp}$.

Proof. Clearly $\multimap$ is a functor of the correct type, so it remains only to establish the existence of a natural isomorphism $\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, B \multimap C)$. We have the following sequence of isomorphisms:

$$\begin{aligned}
\mathbb{C}(A \otimes B, C) &\cong \mathbb{C}(A \otimes B, C^{\perp\perp}) && \text{by Lemma 6.2} \\
&\cong \mathbb{C}(B \otimes A, C^{\perp\perp}) && \text{by symmetry} \\
&\cong \mathbb{C}(B \otimes C^{\perp}, A^{\perp}) && \text{by (6)} \\
&\cong \mathbb{C}(A^{\perp\perp}, (B \otimes C^{\perp})^{\perp}) && \text{since } {}^{\perp} \text{ is full and faithful} \\
&\cong \mathbb{C}(A, (B \otimes C^{\perp})^{\perp}) && \text{by Lemma 6.2} \\
&= \mathbb{C}(A, B \multimap C) && \text{by definition of } \multimap.
\end{aligned}$$

□

7 Related Work

7.1 The Lamarche-Straßburger Definition

Not long after starting work on this we came across the draft of Lamarche and Straßburger (2005). The authors define what they call (*unitless*) *autonomous categories*, motivated (like us) by the desire to model unitless fragments of MLL.

Using our notation, an 'autonomous category' in the sense of Lamarche and Straßburger (2005) consists of:

- a category $\mathbb{C}$,
- functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ with a natural isomorphism $\mathrm{curry}_{A,B,C} : \mathbb{C}(A \otimes B, C) \xrightarrow{\cong} \mathbb{C}(A, B \multimap C)$,
- natural isomorphisms α and σ such that $\sigma_{A,B} = \sigma^{-1}_{B,A}$, satisfying conditions $(\boldsymbol{\alpha})$ and $(\boldsymbol{\alpha\sigma})$,
- a functor $J : \mathbb{C} \to \mathrm{Set}$, with a natural isomorphism

$$e_{A,B} : J(A \multimap B) \xrightarrow{\cong} \mathbb{C}(A, B),$$

  subject to the condition obtained by translating (A.4) into the promonoidal setting.

(For the original definition in the draft (Lamarche and Straßburger, 2005), in terms of *virtual objects*, see Appendix B.) It is clear that every SSMCC is a Lamarche-Straßburger category; however the converse appears to be false. In fact, certain properties are desired of these categories which do not appear to be derivable from the given conditions. In the case where the functor J is representable, the axioms do not seem to imply that the category is symmetric monoidal closed. Also it is claimed (p. 3 of op. cit.) that the canonical natural transformation $JA \times JB \to J(A \otimes B)$ 'agrees well with associativity'.[8] In fact, one of the motivations behind producing this preprint is to point out that, at least, we can provide a solution to the problem of finding a definition with the desired properties. In correspondence Lamarche and Straßburger have indicated that they might change this definition in the final version of their paper.

For the moment, we should like to point out that their current definition of 'autonomous functor' (their Def. 2.1.4) also does not quite do what one might expect. On the one hand it does not demand (nor imply) that the functor preserves the curry isomorphism, on the other it demands a natural *isomorphism* $J \cong JF$. (This latter condition is not satisfied, for example, by the unique functor $\mathrm{Set} \to \mathbf{1}$.)

7.2 The Došen-Petrić Definition

Došen and Petrić (2005) define what they call a *proof-net category* to be a unitless linearly distributive category in which each object has a dual in a suitable sense. This is a reasonable approach, and we conjecture that the resulting definition is equivalent to ours.

Early in our exploration of candidates for semi star-autonomous category, we considered essentially the same definition. Ultimately we chose the approach more analogous to the standard progression from monoidal category to star-autonomous category, via symmetry and closure: it ties directly into the pioneering work of Eilenberg and Kelly (1965), with the $(\boldsymbol{\psi})$ diagram, and also Day's promonoidal categories (1970). Along the way it gives us a resonable notion of model for the intuitionistic fragment of MLL^{-}.

[8] Our definition does have this property, as shown in §5.1.

7.3 Eilenberg and Kelly

The pioneering work of Eilenberg and Kelly (1965) – in which closed categories are defined for the first time – also deserves to be mentioned here. The authors define 'symmetric monoidal closed category' using a large number of axioms with a great deal of redundancy. (Our diagram (ψ) is one of them.) It seems reasonable to conjecture that, if one were to delete the unit and the axioms involving it from this original definition, the structures satisfying the remaining axioms would be just the SSMCCs.

8 Ongoing Work

There are general definitions of *lax promonoidal functor* (Day, 1977), *strong promonoidal functor* (Day and Street, 1995), and *promonoidal star-autonomous functor* (Day and Street, 2004). These definitions need to be specialised to SSMCCs and semi star-autonomous categories in a sensible way.

We are also working on refining the definition of semi star-autonomous category, to give an elementary description that does not rely on the definition of SSMCC. We conjecture that our definition is equivalent to Došen and Petrić's proof-net categories – see §7.2.

As mentioned in the introduction, we hope to show that the proof-net category of Hughes and van Glabbeek (2005) is the free semi star-autonomous category with finite products (free in a 2-categorical sense[9]).

A Axioms for Monoidal Categories

This appendix describes some axioms for a symmetric monoidal category, and shows that this axiomatisation is equivalent to the usual one.[10] (Of course we are really interested in promonoidal categories: the arguments here can readily be transferred to the more general setting.)

A monoidal category is a category $\mathbb{C}$ equipped with a functor

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

and an object $I \in \mathbb{C}$, together with natural isomorphisms α, λ and ρ having components

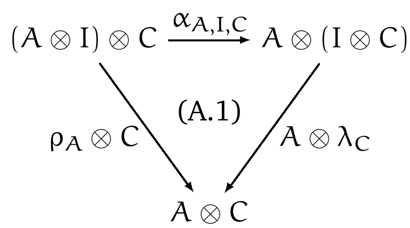
$$\begin{aligned} \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ \lambda_A &: I \otimes A \to A \\ \rho_A &: A \otimes I \to A \end{aligned}$$

such that the following diagrams commute for all $A,B,C,D \in \mathbb{C}$:

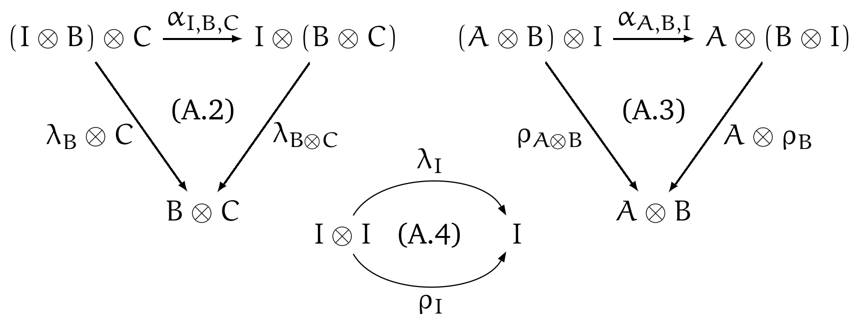
$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A\otimes B,C,D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C\otimes D}} & A \otimes (B \otimes (C \otimes D)) \\ \downarrow{\scriptstyle \alpha_{A,B,C} \otimes D} & & (\alpha) & & \uparrow{\scriptstyle A \otimes \alpha_{B,C,D}} \\ (A \otimes (B \otimes C)) \otimes D & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & A \otimes ((B \otimes C) \otimes D) \end{array}$$

[9]Since MALL formulas are implicitly quotiented by de Morgan duality.

[10]There is every chance that this axiomatisation is somewhere in the literature. I don't know a reference for it though.



These axioms have many interesting consequences. Most importantly, it follows that the following three diagrams commute for all $A,B,C \in \mathbb{C}$:

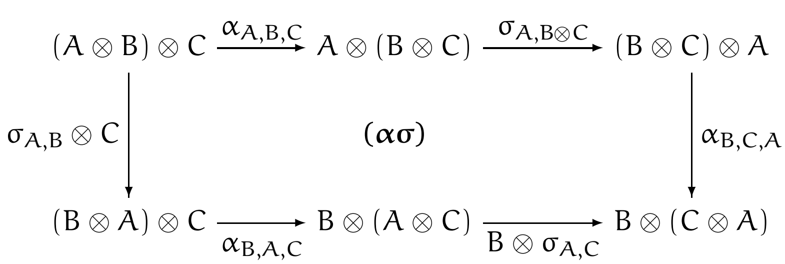


Joyal and Street (1993) give a simple and elegant proof. There are other possible axiomatisations: for example, conditions $(\boldsymbol{\alpha})$, (A.2) and (A.4) are also collectively sufficient.[11]

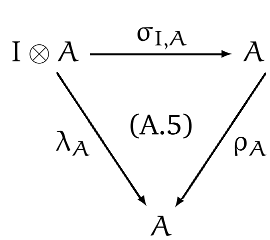
A *symmetric* monoidal category is a monoidal category $\mathbb{C}$ equipped with a natural isomorphism σ having components

$$\sigma_{A,B} : A \otimes B \to B \otimes A,$$

such that $\sigma_{A,B} = \sigma_{B,A}^{-1}$ and the following commutes for all $A,B,C \in \mathbb{C}$:



It then follows that



commutes for every $A \in \mathbb{C}$; again, see Joyal and Street (1993) for a proof.

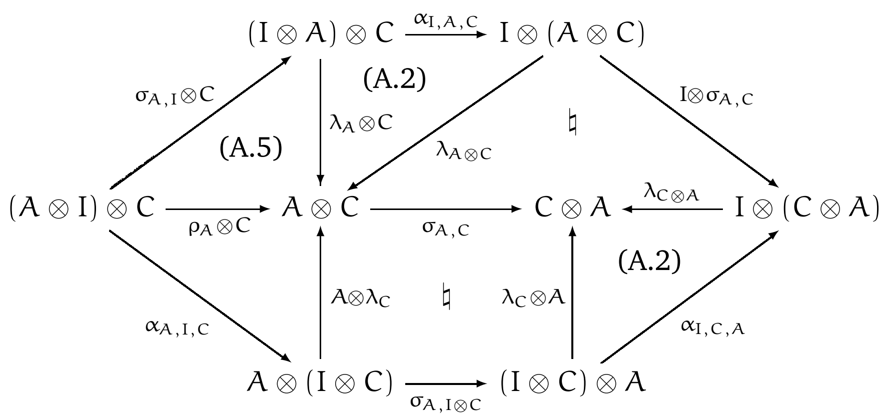
Conditions $(\boldsymbol{\alpha})$ and $(\boldsymbol{\alpha\sigma})$ are indispensable; the usual definition of symmetric monoidal category requires (A.1) in addition. However, it is sometimes convenient to eliminate the ρ isomorphism from the data: that

[11] This can be proved by the technique of Joyal and Street (1993).

is permissible, since (A.5) shows that ρ may be defined in terms of λ and σ. It turns out that, in this situation, we may require (A.2) in place of (A.1). Specifically:

Proposition A.1. *If* (*A.2*), $(\boldsymbol{\alpha\sigma})$ *and* (*A.5*) *hold then so does* (*A.1*).

Proof. Consider the diagram



The outside is an instance of $(\boldsymbol{\alpha\sigma})$, and the labelled regions commute for the reasons marked. Since all the morphisms are invertible, it follows that the unlabelled region at lower left commutes. This region is just (A.1). □

In summary, we may define a symmetric monoidal category to be a category $\mathbb{C}$ with a functor $\otimes$ and a unit object I, together with natural isomorphisms α, λ and σ such that $\sigma_{A,B} = \sigma^{-1}_{B,A}$ and diagrams $(\boldsymbol{\alpha})$, $(\boldsymbol{\alpha\sigma})$ and (A.2) commute.

B The Lamarche-Straßburger Definition

In their draft, Lamarche and Straßburger (2005) give a very interesting discussion of autonomous categories without units, and the following definition of (*unitless*) *autonomous category*, based on the notion of a *virtual object*. In Section 7.1 we presented the definition in our own notation (i.e., promonoidal style); for the sake of completeness, here is (a condensed presentation of) the original definition of Lamarche and Straßburger (2005), in terms of virtual objects.

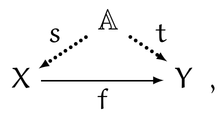
A category $\mathcal{C}$ *has tensors* if it is equipped with a bifunctor $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ with the usual associativity and symmetry isomorphisms

$$\begin{aligned} \mathsf{assoc}_{A,B,C} &: \ A \otimes (B \otimes C) \to (A \otimes B) \otimes C \\ \mathsf{twist}_{A,B} &: \ A \otimes B \to B \otimes A \end{aligned}$$

obeying the usual associated coherence laws. When it exists, write $-\!\!\multimap- : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ for internal hom, defined by adjointness to tensor as in the usual case with units. Write $h_X = \mathcal{C}(-, X)$ and $h^X = \mathcal{C}(X, -)$ for the hom functors, and $H^X = X \multimap - : \mathcal{C} \to \mathcal{C}$. Writing a functor $\mathcal{C} \to \mathsf{Set}$ as $h^{\mathbb{A}}$ for a symbol $\mathbb{A}$ allows one to write an element s of the set $h^{\mathbb{A}}(s)$, corresponding (by Yoneda's Lemma) to a natural transformation $h^X \to h^{\mathbb{A}}$, as

$$\mathbb{A} \overset{s}{\dashrightarrow} X$$

When the symbol $\mathbb{A}$ is an object of $\mathcal{C}$, the functor $h^{\mathbb{A}}$ is representable, in the usual sense; when $\mathbb{A}$ is not an object of $\mathcal{C}$, it is a *virtual object*. In general a dotted arrow will mean at least one of the source or target is virtual, and should be interpreted as a reverse-direction natural transformation between the corresponding functors. For example, given $f : X \to Y$ and $t = (h^{\mathbb{A}} f)(s)$, one can draw the 'commutative diagram'

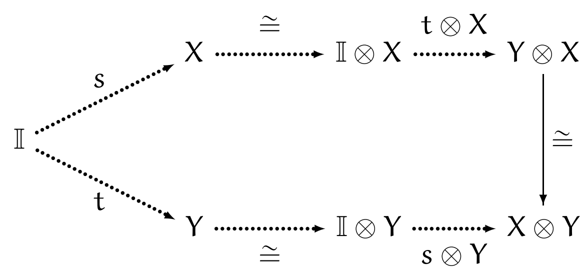


justifying the notation $t = f \circ s$, or simply $t = fs$. Define $\mathbb{A} \otimes X$ in the obvious way, i.e., $h^{\mathbb{A} \otimes X} = h^{\mathbb{A}} H^X$. This construction is natural in both variables: given $s : \mathbb{A} \dashrightarrow \mathbb{B}$ (between virtual objects) and a morphism $f : X \to Y$, there is an obvious $s \otimes f : \mathbb{A} \otimes X \dashrightarrow \mathbb{B} \otimes Y$.

Definition B.1. A category $\mathcal{C}$ with tensors is an *autonomous category* if it has an internal hom $\multimap$ and a functor $h^{\mathbb{I}}$ with a natural isomorphism

$$h^{\mathbb{I}}(X \multimap Y) \;\cong\; \mathcal{C}(X, Y)$$

such that the following diagram (of mostly virtual arrows) commutes:



This diagram corresponds to diagram (A.4) of the previous appendix, translated into promonoidal style.

References

Michael Barr. ⋆-*Autonomous Categories*, volume 752 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979.

Mario Càccamo, Martin Hyland, and Glynn Winskel. Lecture notes in category theory. URL `http://www.brics.dk/~mcaccamo/catnotes.ps.gz`. BRICS LS-02-catnotes (draft), 2002.

Brian Day. On closed categories of functors. In Saunders Mac Lane, editor, *Reports of the Midwest Category Seminar*, volume 137 of *Lecture Notes in Mathematics*, pages 1–38, Berlin–New York, 1970. Springer-Verlag.

Brian Day. Note on monoidal monads. *Journal of the Australian Mathematical Society*, 23:292–311, 1977.

Brian Day and Ross Street. Kan extensions along promonoidal functors. *Theory and Applications of Categories*, 1(4):72–77, 1995.

Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129:99–157, 1997.

Brian Day and Ross Street. Quantum categories, star autonomy, and quantum groupoids. In *Galois Theory, Hopf Algebras, and Semiabelian Categories*, volume 43 of *Fields Institute Communications*, pages 187–226. American Mathematical Society, 2004. `arXiv:math.CT/0301209`.

Kosta Došen and Zoran Petrić. Coherence of proof-net categories. `arXiv:math.CT/0503301`, March 2005.

Samuel Eilenberg and Max Kelly. Closed categories. In *Conference on Categorical Algebra, La Jolla*, pages 421–562. Springer-Verlag, 1965.

Samuel Eilenberg and Max Kelly. A generalization of the functorial calculus. *J. Algebra*, 3:366–375, 1966.

Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

Dominic Hughes and Rob van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic. *ACM Transactions on Computational Logic*, 2005. To appear. Invited submission November 2003, revised January 2005.

André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 1993.

Max Kelly and Saunders Mac Lane. Coherence in closed categories. *J. Pure and Applied Algebra*, 1:97–140, 1971.

François Lamarche and Lutz Straßburger. Constructing free boolean categories. Submission for *Logic in Computer Science*, January 2005.