A canonical graphical syntax for non-empty finite products and sums

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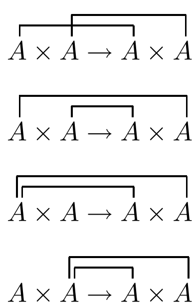
Cockett and Seely recently introduced a Lambek-style deductive system for finite products and sums, and proved decidability of equality of morphisms. The question remained * *as to whether one can present free categories with finite products and sums in a canonical way, i.e., as a category with morphisms and composition defined directly, rather than modulo equivalence relations. This paper shows that the non-empty case (i.e., omitting initial and final objects) can be treated in a surprisingly simple way: morphisms of the free category can be viewed as certain binary relations, with composition the usual composition of binary relations. In particular, there is a forgetful functor into the category* Rel *of sets and binary relations. The paper ends by relating these binary relations to proof nets.*

1 Introduction

In essence, this paper rigorises the following widely used informal graphical notation for the four canonical morphisms from a product $A \times A$ to itself:



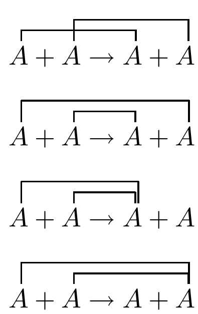
$\mathsf{id} = \lambda\langle x, y\rangle.\langle x, y\rangle$

$\mathsf{tw} = \lambda\langle x, y\rangle.\langle y, x\rangle$

$\mathsf{k}_0 = \lambda\langle x, y\rangle.\langle x, x\rangle$

$\mathsf{k}_1 = \lambda\langle x, y\rangle.\langle y, y\rangle$

Or dually:



$\mathsf{id} = \lambda\langle x, i\rangle.\langle x, i\rangle$

$\mathsf{tw} = \lambda\langle x, i\rangle.\langle x, \neg i\rangle$

$\mathsf{k}_0 = \lambda\langle x, i\rangle.\langle x, 0\rangle$

$\mathsf{k}_1 = \lambda\langle x, i\rangle.\langle x, 1\rangle$

We define a category $\mathcal{L}$ whose objects are formulas built from atoms $A_1, A_2, \ldots$ by non-emtpy finite product $\prod_{i \in I} X_i$ and non-empty sum $\sum_{i \in I} X_i$, and in which morphisms $X \rightarrow Y$ are certain binary relations between the leaves of X (when X is viewed as a parse tree) and the leaves of Y. For example, above, we depicted the four morphisms from $A \times A$ to itself, and the four morphisms from $A + A$ to itself. Composition in $\mathcal{L}$ is simply the usual composition of binary relations. In particular, there is a forgetful functor from $\mathcal{L}$ into the category Rel of sets and binary relations. $\mathcal{L}$ contains, as full subcategories, the category of non-empty finite sets, and its opposite.

Our main theorem (Theorem 1) is that $\mathcal{L}$ is isomorphic to the free category generated from the atoms of $\mathcal{L}$ by non-empty finite product and coproduct (sum). Since binary relations are graphs, we can thus view $\mathcal{L}$ as a canonical graphical syntax for non-empty finite products and sums.

The binary relations constituting the morphisms of $\mathcal{L}$ are akin to the graphs used by Kelly and MacLane to study coherence in symmetric monoidal closed categories [KM71], and to the linkings of proof nets for linear logic [Gir87]. In section 5 we discuss the relationship between the morphisms of $\mathcal{L}$ and proof nets for multiplicative-additive linear logic [Gir96, HG02].

This work was motivated by a recent paper of Cockett and Seely [CS01], which introduced a Lambek-style deductive system [Lam69] for finite products and sums, and proved decidability of equality of morphisms. The underlying sequent calculus is similar to additive linear logic [Gir87], without negation, and with exactly one formula either side of every turnstile. Our proof that $\mathcal{L}$ is the free category goes via the fragment of Cockett and Seely's deductive system without initial and final objects (empty sum and product). Additional motivation for this work came from the relationship between proof nets and coherence for monoidal closed categories [Blu93], and the two-sided proof nets (with units) used to tackle coherence for linearly (or weakly) distributive categories [BCST96].

Cockett and Seely suggest connections between their deductive system and Blass games [Bla92] (see also [San99]). Types can be viewed as finite games which lack the usual requirement that play should alternate strictly between player (product structure) and opponent (coproduct structure). The concurrent games of Abramsky and Melliès [AM99] (a strict extension of Blass games, constituting a fully complete model for multiplicative-additive linear logic) go a step further, dispensing altogether with a global schedule

*This paper essentially uses the resolution condition of MALL proof nets to characterise the free product-sum category. It was never published as it turned out Hongde Hu had already characterised the free category: *Contractible coherence spaces and maximal maps, ENTCS 20,* 1999. The proof in the present paper can be shortened considerably because of this prior art. Thanks to Cockett and Seely for pointing out this relationship. [Note: In footnote 1 overleaf, "ill-defined" refers to confluence, *i.e.*, lack of a unique normal form under the rewrite, not to the rewrite rule itself.]

between players. The similarity between the two is not surprising; afterall the underlying proof theory of Cockett and Seely's deductive system is essentially additive linear logic. Cockett and Seely remark that sum/product-games, in turn, relate to protocols for communication channels. Therefore, the characterisation of the free category presented in this paper may contribute to a foundation for understanding channel-based concurrent communication.

Open problems. Following a suggestion of Peter Selinger, Cockett and Seely [CS01, app. B] sketched a graphical decision procedure (called *cellular squares*) for the equality of cut-free proofs in their deductive system, in the special case of binary products and sums. It would be interesting to consider whether there is a 'correctness criterion' for picking out sound cellular squares (*i.e.*, those that interpret deductions), and to attempt to define a direct composition of cellular squares that preserves the criterion. It is conceivable that the two conditions presented in this paper for the well-formedness of the binary relations (morphisms) in $\mathcal{L}$ may map accross to yield a soundness criterion for cellular squares.

It would be interesting to make formal, concrete connections between the following: (a) Cockett and Seely's remarks that their deductive system [CS01] has a game-theoretic connection; (b) Santocanale's game-theoretic work on meet and join posets with fixed points [San99]; (c) the concurrent games of Abramsky and Melliès [AM99]; (d) conditions (1) and (2) characterising the binary relations that constitute the morphisms of the category of linkings $\mathcal{L}$ presented in this paper (section 3).

Work in progress aims to extend the approach presented here to units (*i.e.*, initial and final objects), and to an arbitrary base category (rather than a set of atoms, *i.e.*, discrete category). The former, if at all feasible, appears to be quite involved. This is evidenced by the fact that, when empty products and sums are present, there is no obvious confluent and terminating rewrite system for the cut-free proofs (or proof terms) of Cockett and Seely's deductive system.[1] If such a rewrite system can be found, it might provide useful clues towards extending the approach presented in this paper to the initial and final objects, yielding a canonical graphical syntax for finite products and sums.

[1] Following a suggestion of Santocanale, the term equivalences of [CS01] can be oriented to give normal forms in the non-nullary case [CS01, app. A]. Unfortunately, this orientation is ill-defined when the empty product (final object) is present: the rewrite $(p_k(f_i))_{i\in I} \implies p_k((f_i)_{i\in I})$ (the orientation given to conversion (10) of [CS01]) fails when I is empty, since k is undetermined. For example, in terms of proofs, this rewrite cannot be applied to $\overline{X_0\times X_1\vdash 1}\,^{tuple}$, since the right side pattern-matches two proofs, namely $\dfrac{\overline{X_k\vdash 1}\,^{tuple}}{X_0\times X_1\vdash 1}\,{}^{projection}$ for $k=0$ and $k=1$.

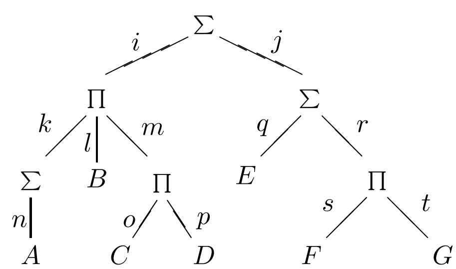
2 Notation for binary relations

We write Rel for the category of sets and binary relations, and write $R : U \nrightarrow V$ if R is a morphism from U to V, *i.e.*, a subset $R \subseteq U \times V$. We denote the composite of $R : U \nrightarrow V$ and $S : V \nrightarrow W$ by $R;S : U \nrightarrow W$. We write $\prod_{i\in I} X_i$ for biproduct in Rel (simultaneously product and sum), which acts as disjoint union on objects (sets).

3 The category $\mathcal{L}$ of linkings

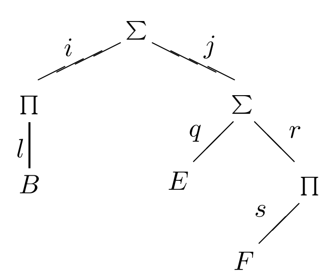
Fix a set $\mathcal{A} = \{A_1, A_2, \ldots\}$ of atoms. An object of $\mathcal{L}$ is a formula generated from the atoms of $\mathcal{L}$ by non-emtpy finite product $\prod_{i\in I} X_i$ and non-empty sum $\sum_{i\in I} X_i$. We shall refer to these objects simply as ***formulas***. We use $X_0 \times X_1$ as shorthand for $\prod_{i\in\{0,1\}} X_i$, and $X_0 + X_1$ as shorthand for $\sum_{i\in\{0,1\}} X_i$.

We identify a formula X with the corresponding labelled tree, *viz.*, with vertices labelled with $\prod$, $\sum$, and atoms, and with edges labelled with indices (*i.e.*, elements of the indexing sets occurring in X). For example:



By a ***leaf*** of a formula X we mean an occurrence of an atom in X, literally a leaf of X when X is viewed as a tree. The ***label*** of a leaf x is its atom, and is denoted $\widehat{x} \in \mathcal{A}$. If the vertex x is a child of y (so y is necessarily a $\prod$- or $\sum$-occurrence), then the subtree rooted at x (*i.e.*, x together with its descendents) is an ***argument*** of y.

A $\prod$***-strategy*** of a formula X is any result of deleting all but one argument of every $\prod$ in X. Thus every surviving $\prod$ of X becomes unary, and every surviving $\sum$ retains each of its arguments. For example, here is one of the 8 possible $\prod$-strategies of the formula depicted above:



(We emphasise that we are only interested in end results, not choices made along the way. For example, deletion of

C or D becomes irrelevant, given that we did not retain their parent Π.) Define a $\sum$***-strategy*** of X analogously, as any result of deleting all but one argument of every $\sum$. Thus every surviving $\sum$ of X becomes unary, and every surviving $\prod$ retains each of its arguments.

Henceforth we identify a strategy of X with the corresponding set of leaves. The following is immediate:

LEMMA 1 *Any $\prod$-strategy and $\sum$-strategy of X intersect in a single leaf of X. Conversely, every leaf of X is the intersection of some $\prod$-strategy and $\sum$-strategy of X.*

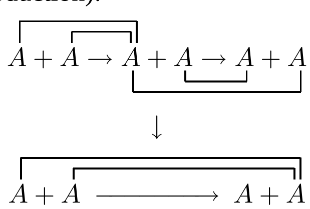
Write $|X|$ for the set of leaves of a formula X. A morphism $X \to Y$ in $\mathcal{L}$ is a ***linking*** from X to Y, namely, a binary relation $R \subseteq |X| \times |Y|$ between the leaves of X and the leaves of Y such that:

(1) R respects leaf labelling: $\langle x, y\rangle \in R$ only if $\widehat{x} = \widehat{y} \in \mathcal{A}$.

(2) for every $\sum$-strategy X' of X and $\prod$-strategy Y' of Y, R contains exactly one edge between X' and Y' (*i.e.*, $R \cap (X' \times Y')$ is a singleton).

The edges (elements) of a linking R are called ***links***. Note that, as distinct from Kelly-MacLane graphs for symmetric monoidal closed categories [KM71], and proof nets for multiplicative linear logic [Gir87], we do not demand that the links be disjoint.

Composition of linkings in $\mathcal{L}$ is simply the usual composition of binary relations. Identities are also inherited from Rel. For example, the following diagram illustrates the composition $\mathsf{k}_0; \mathsf{tw} = \mathsf{k}_1$, for k_0, tw and k_1 as shown in section 1 (Introduction):



To prove that composition is well-defined, *i.e.*, that composition preserves conditions (1) and (2) in the definition of linking, we shall use the following lemma.

LEMMA 2 *Let $X = \prod_{i\in I} X_i$, and let L be a set of leaves of X such that for all $\sum$-strategies X' of X, $L \cap X'$ is a singleton. Then L is contained entirely within one of the X_i. The dual result also holds, i.e., with $\prod$ and $\sum$ interchanged.*

Proof. Suppose $x_k \in X_k \cap L$ and $x_l \in X_l \cap L$ for $k, l \in I$ and $k \neq l$. For $i = k, l$ choose a $\sum$-strategy X_i' for X_i with $x_i \in X_i'$, and for remaining $i \in I$ choose an arbitrary $\sum$-strategy X_i' of X_i. Then $\bigcup_{i\in I} X_i$ is a $\sum$-strategy of X which intersects L in two leaves (x_k and x_l), a contradiction. □

PROPOSITION 1 *Composition and identities are well-defined in $\mathcal{L}$. In other words, the composite of any two linkings is a linking, and the identity binary relation on the leaves of a formula is a linking.*

Proof. Identities clearly satisfy condition (1), and (2) follows from Lemma 1. Given linkings $R : X \to Y$ and $S : Y \to Z$, the fact that the composite binary relation $R; S$ satisfies condition (1) is immediate. Condition (2) holds by induction on the number of vertices in Y: the base case is trivial, and the inductive step follows directly from Lemma 2. □

REMARK 1 Since the morphisms of $\mathcal{L}$ are binary relations between leaves, with composition and identities inherited from Rel, the map $|-|$ (taking a formula to its set of leaves) extends trivially to a forgetful functor $\mathcal{L} \to \mathsf{Rel}$.

REMARK 2 The full subcategory of $\mathcal{L}$ whose objects are of the form $\sum_{i\in I} A$ for a fixed atom A is isomorphic to the category of non-empty finite sets. Substituting $\prod$ for $\sum$, we obtain the opposite category as a full subcategory of $\mathcal{L}$.

Non-empty finite products and sums. Recall that we identify a strategy X' of X with a subset $X' \subseteq |X|$ of the set of leaves of X. The following is immediate:

LEMMA 3 *Every $\prod$-strategy of $\sum_{i\in I} X_i$ is a union $\bigcup_{i\in I} X_i'$ of a $\prod$-strategies X_i' of X. Dually, every $\sum$-strategy of $\prod_{i\in I} X_i$ is a union $\bigcup_{i\in I} X_i'$ of a $\sum$-strategies X_i' of X.*

PROPOSITION 2 *$\mathcal{L}$ has all non-empty finite products and non-empty finite coproducts.*

Proof. Define product and sum in the obvious way on formulas, *i.e.*, $\langle X_i\rangle_{i\in I} \mapsto \prod_{i\in I} X_i$ and $\sum_{i\in I} X_i$ respectively. Projections and injections

$$\begin{array}{llcll} \mathsf{in}_k & : & X_k & \to & \sum_{i\in I} X_i \quad \text{(injections)} \\ \mathsf{pr}_k & : & \prod_{i\in I} X_i & \to & X_k \quad \text{(projections)} \end{array}$$

are inherited from Rel, *i.e.*, the underlying binary relations on leaves are the corresponding canonical morphisms in Rel:

$$\begin{array}{llcll} \mathsf{in}_k & : & |X_k| & \nrightarrow & \coprod_{i\in I} |X_i| \quad \text{(injections)} \\ \mathsf{pr}_k & : & \coprod_{i\in I} |X_i| & \nrightarrow & |X_k| \quad \text{(projections)} \end{array}$$

(using the fact that $|\prod_{i\in I} X_i| = |\sum_{i\in I} X_i| = \coprod_{i\in I} |X_i|$). These canonical binary relations clearly satisfy conditions (1) and (2) of the definition of linking. The natural isomorphisms

$$\begin{array}{rcl} \mathsf{hom}(Z, \prod_{i\in I} X_i) & \cong & \prod_{i\in I} \mathsf{hom}(Z, X_i) \\ \mathsf{hom}(\sum_{i\in I} X_i, Z) & \cong & \prod_{i\in I} \mathsf{hom}(X_i, Z) \end{array}$$

follow from condition (2) of linking, using Lemma 3. □

REMARK 3 The forgetful functor $|-| : \mathcal{L} \to \mathsf{Rel}$ preserves binary products and sums, since $|\prod_{i\in I} X_i| = |\sum_{i\in I} X_i| = \coprod_{i\in I} |X_i|$.

Softness. Recall that a category with products $\prod$ and sums $\sum$ is ***soft*** [Joy95] if every morphism $\prod_{i\in I} X_i \to \sum_{j\in J} Y_j$ factors either through a projection from one of the X_i, or through an injection into one of the Y_j. The softness of $\mathcal{L}$ will follow directly from the following simple combinatorial lemma:

LEMMA 4 *Let r be a binary relation $r \subseteq U \times V$ between sets U and V each with at least two elements, and suppose every $u \in U$ and $v \in V$ is in some edge of r. Then r has a pair of disjoint edges, i.e., there exist distinct $u, u' \in U$ and distinct $v, v' \in V$ such that $\langle u, v\rangle \in r$ and $\langle u', v'\rangle \in r$.*

PROPOSITION 3 *$\mathcal{L}$ is soft.*

Proof. Suppose $R : \prod_{i\in I} X_i \to \sum_{j\in J} Y_j$ was not soft. Then each X_i and Y_j has an atom in a link of R. By Lemma 4, R contains disjoint edges $\langle x, y\rangle$ and $\langle x', y'\rangle$ with x and x' in distinct X_i, say X_k and $X_{k'}$, and y, y' in distinct Y_j, say Y_l and $Y_{l'}$. Pick $\sum$-strategies of X_k and $X_{k'}$ containing x and x', and $\prod$-strategies of Y_l and $Y_{l'}$ containing y and y'. Pick arbitrary $\sum$-strategies and $\prod$-strategies for the remaining X_i and Y_j, respectively. By Lemma 3, the unions yield a $\sum$-strategy of $\prod_{i\in I} X_i$ and a $\prod$-strategy of $\sum_{j\in J} Y_j$. By construction, there are two edges of R between these strategies (namely $\langle x_0, y_0\rangle$ and $\langle x_1, y_1\rangle$), contradicting condition (2) of linking. □

4 Isomorphism with free category

In this section we prove our main theorem (Theorem 1), that $\mathcal{L}$ is isomorphic to the free category generated by non-empty finite product and sum from its atoms.

Throughout this section, $\mathcal{L}(\mathcal{A})$ denotes the category of linkings over a set $\mathcal{A}$ of atoms, and $\mathcal{F}(\mathcal{A})$ denotes the category generated freely from $\mathcal{A}$ by non-empty finite product and non-empty finite coproduct. In section 4.1, we define $\mathcal{D}(\mathcal{A})$ to be Cockett and Seely's deductive system for finite products and sums [CS01], restricted to the non-empty case. This category is isomorphic to the free category $\mathcal{F}(\mathcal{A})$. Section 4.2 defines a functor $F : \mathcal{D}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$ witnessing an isomorphism between $\mathcal{D}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$. Therefore, by transitivity ($\mathcal{L}(\mathcal{A}) \cong \mathcal{D}(\mathcal{A}) \cong \mathcal{F}(\mathcal{A})$), we obtain our main theorem (Theorem 1): $\mathcal{L}(\mathcal{A})$ is isomorphic to the free category $\mathcal{F}(\mathcal{A})$.

4.1 Cockett and Seely's deductive system

We write $\mathcal{D}(\mathcal{A})$ for Cockett and Seely's deductive system for finite sums and products [CS01], restricted to the non-empty case (*i.e.*, without initial and final object), and generated from the set $\mathcal{A}$ of atoms. The underlying sequent calculus is similar to additive linear logic [Gir87], without negation, and with exactly one formula either side of every turnstile. The inference rules are shown in Figure 1. Throughout the paper, we adopt the convention that $A, B, \ldots$ range over atoms and $X, Y, \ldots$ range over formulas.

An object of $\mathcal{D}(\mathcal{A})$ is a formula generated from the atoms of $\mathcal{A}$ by non-empty finite product $\prod_{i\in I} X_i$ and non-empty finite sum $\sum_{i\in I} X_i$ (so the objects of $\mathcal{D}(\mathcal{A})$, $\mathcal{F}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ are the same). A morphism $X \to Y$ is an equivalence class of cut-free proofs of $X \vdash Y$, modulo conversions (9)–(12) shown in Figure 2 (more precisely, the symmetric and transitive closure of (9)–(12)). (We follow the enumeration given in [CS01].) We write $[\pi]$ for the equivalence class of the cut-free proof π.

Composition is by cut elimination, using conversions (1)–(8) shown Figure 3. More precisely, the composite of $[\pi] : X \to Y$ and $[\pi'] : Y \to Z$ is the equivalence class of the proof resulting from taking π and π' as hypotheses of a cut rule, yielding (say) the proof π^*, and then eliminating the cuts from π^* using conversions (1)–(8) of Figure 3.

The identity $\mathsf{id}_X : X \to X$ is defined by induction on the number of connectives ($\prod$ or $\sum$) in X, as follows. Given an atom A, let 1_A be the derivation consisting of the identity rule with conclusion $A \vdash A$. The identity derivation $1_{\prod_{i\in I} X_i}$ is given by:

$$\dfrac{\left\{ \dfrac{\dfrac{1_{X_i}}{X_i \vdash X_i}}{\prod_{j\in I} X_j \vdash X_i} \textstyle\prod l \right\}_{i\in I}}{\prod_{i\in I} X_i \vdash \prod_{i\in I} X_i} \textstyle\prod r$$

The identity $1_{\Sigma_{i\in I} X_i}$ is obtained dually, *i.e.*, exchanging left and right around the turnstile, and exchanging $\prod$ and $\sum$. Define the identity morphism $\mathsf{id}_X : X \to X$ to be the equivalence class $[1_X]$ of 1_X.

PROPOSITION 4 *The free category $\mathcal{F}(\mathcal{A})$ is isomorphic to the deductive system $\mathcal{D}(\mathcal{A})$.*

Proof. Proposition 4.6 of [CS01], restricted to the non-empty case, and to a generating set (discrete category). □

4.2 The functor

We define a functor $F : \mathcal{D}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$ as follows. On objects, define F to be the identity. Given a cut-free derivation π, define $F(\pi)$ by induction on the depth of the derivation, as follows. For the base case, π is an identity rule $\overline{A \vdash A}^{\,\mathsf{id}}$; define $F(\pi)$ to be the identity linking $\mathsf{id}_A : A \to A$. The four cases of the inductive step (corresponding to the last rule of π) are shown in Figure 4. To aid clarity, this presentation leaves various canonical maps of Rel implicit; they are shown explicitly in Figure 5.

$$\frac{}{A \vdash A}\,\mathsf{id} \qquad \frac{\{X_i \vdash Y\}_{i\in I}}{\sum_{i\in I} X_i \vdash Y}\,{\textstyle\sum} l \qquad \frac{X_k \vdash Y}{\prod_{i\in I} X_i \vdash Y}\,{\textstyle\prod} l \quad (k\in I)$$

$$\frac{X \vdash Y \qquad Y \vdash Z}{X \vdash Z}\,\mathsf{cut} \qquad \frac{\{Y \vdash X_i\}_{i\in I}}{Y \vdash \prod_{i\in I} X_i}\,{\textstyle\prod} r \qquad \frac{Y \vdash X_k}{Y \vdash \sum_{i\in I} X_i}\,{\textstyle\sum} r \quad (k\in I)$$

Figure 1. Inference rules.

$$\dfrac{\dfrac{\left\{\dfrac{\pi_i}{X_i\vdash Y_k}\right\}_{i\in I}}{\sum_{i\in I} X_i \vdash Y_k}\,{\textstyle\sum} l}{\sum_{i\in I}X_i\vdash\sum_{j\in J}Y_j}\,{\textstyle\sum} r \quad \stackrel{(9)}{\longrightarrow} \quad \dfrac{\left\{\dfrac{\dfrac{\pi_i}{X_i\vdash Y_k}}{X_i\vdash\sum_{j\in J}Y_j}\,{\textstyle\sum} r\right\}_{i\in I}}{\sum_{i\in I}X_i\vdash\sum_{j\in J}Y_j}\,{\textstyle\sum} l$$

$$\dfrac{\left\{\dfrac{\dfrac{\pi_j}{X_k\vdash Y_j}}{\prod_{i\in I}X_i\vdash Y_j}\,{\textstyle\prod} l\right\}_{j\in J}}{\prod_{i\in I}X_i\vdash\prod_{j\in J}Y_j}\,{\textstyle\prod} r \quad \stackrel{(10)}{\longrightarrow} \quad \dfrac{\dfrac{\left\{\dfrac{\pi_j}{X_k\vdash Y_j}\right\}_{j\in J}}{X_k\vdash\prod_{j\in J}Y_j}\,{\textstyle\prod} r}{\prod_{i\in I}X_i\vdash\prod_{j\in J}Y_j}\,{\textstyle\prod} l$$

$$\dfrac{\dfrac{\dfrac{\pi}{X_k\vdash Y_m}}{\prod_{i\in I}X_i\vdash Y_m}\,{\textstyle\prod} l}{\prod_{i\in I}X_i\vdash\sum_{j\in J}Y_j}\,{\textstyle\sum} r \quad \stackrel{(11)}{\longrightarrow} \quad \dfrac{\dfrac{\dfrac{\pi}{X_k\vdash Y_m}}{X_k\vdash\sum_{j\in J}Y_j}\,{\textstyle\sum} r}{\prod_{i\in I}X_i\vdash\sum_{j\in J}Y_j}\,{\textstyle\prod} l$$

$$\dfrac{\left\{\dfrac{\left\{\dfrac{\pi_{ij}}{X_i\vdash Y_j}\right\}_{i\in I}}{\sum_{i\in I}X_i\vdash Y_j}\,{\textstyle\sum} l\right\}_{j\in J}}{\sum_{i\in I}X_i\vdash\prod_{j\in J}Y_j}\,{\textstyle\prod} r \quad \stackrel{(12)}{\longrightarrow} \quad \dfrac{\left\{\dfrac{\left\{\dfrac{\pi_{ij}}{X_i\vdash Y_j}\right\}_{j\in J}}{X_i\vdash\prod_{j\in J}Y_j}\,{\textstyle\prod} r\right\}_{i\in I}}{\sum_{i\in I}X_i\vdash\prod_{j\in J}Y_j}\,{\textstyle\sum} l$$

Figure 2. Conversions of cut-free derivations.

$$\dfrac{\dfrac{\pi}{X \vdash A} \qquad \dfrac{}{A \vdash A}\,\mathsf{id}}{X \vdash A}\,\mathsf{cut} \quad \overset{(1)}{\longrightarrow} \quad \dfrac{\pi}{X \vdash A}$$

$$\dfrac{\dfrac{}{A \vdash A}\,\mathsf{id} \qquad \dfrac{\pi}{A \vdash X}}{A \vdash X}\,\mathsf{cut} \quad \underset{\text{(dual to (1))}}{\overset{(2)}{\longrightarrow}} \quad \dfrac{\pi}{A \vdash X}$$

$$\dfrac{\dfrac{\pi}{X \vdash Y} \qquad \dfrac{\dfrac{\pi'}{Y \vdash Z_k}}{Y \vdash \sum_{i \in I} Z_i}\,\textstyle\sum r}{X \vdash \sum_{i \in I} Z_i}\,\mathsf{cut} \quad \overset{(3)}{\longrightarrow} \quad \dfrac{\dfrac{\dfrac{\pi}{X \vdash Y} \qquad \dfrac{\pi'}{Y \vdash Z_k}}{X \vdash Z_k}\,\mathsf{cut}}{X \vdash \sum_{i \in I} Z_i}\,\textstyle\sum r$$

$$\dfrac{\dfrac{\dfrac{\pi}{Z_k \vdash Y}}{\prod_{i \in I} Z_i \vdash Y}\,\textstyle\prod l \qquad \dfrac{\pi'}{Y \vdash X}}{\prod_{i \in I} Z_i \vdash X}\,\mathsf{cut} \quad \underset{\text{(dual to (3))}}{\overset{(4)}{\longrightarrow}} \quad \dfrac{\dfrac{\dfrac{\pi}{Z_k \vdash Y} \qquad \dfrac{\pi'}{Y \vdash X}}{Z_k \vdash X}\,\mathsf{cut}}{\prod_{i \in I} Z_i \vdash X}\,\textstyle\prod l$$

$$\dfrac{\dfrac{\pi}{X \vdash Y} \qquad \dfrac{\left\{\dfrac{\pi_i}{Y \vdash Z_i}\right\}_{i \in I}}{Y \vdash \prod_{i \in I} Z_i}\,\textstyle\prod r}{X \vdash \prod_{i \in I} Z_i}\,\mathsf{cut} \quad \overset{(5)}{\longrightarrow} \quad \dfrac{\left\{\dfrac{\dfrac{\pi}{X \vdash Y} \qquad \dfrac{\pi_i}{Y \vdash Z_i}}{X \vdash Z_i}\,\mathsf{cut}\right\}_{i \in I}}{X \vdash \prod_{i \in I} Z_i}\,\textstyle\prod r$$

$$\dfrac{\dfrac{\left\{\dfrac{\pi_i}{Z_i \vdash Y}\right\}_{i \in I}}{\sum_{i \in I} Z_i \vdash Y}\,\textstyle\sum l \qquad \dfrac{\pi}{Y \vdash X}}{\sum_{i \in I} Z_i \vdash X}\,\mathsf{cut} \quad \underset{\text{(dual to (5))}}{\overset{(6)}{\longrightarrow}} \quad \dfrac{\left\{\dfrac{\dfrac{\pi_i}{Z_i \vdash Y} \qquad \dfrac{\pi}{Y \vdash X}}{Z_i \vdash X}\,\mathsf{cut}\right\}_{i \in I}}{\sum_{i \in I} Z_i \vdash X}\,\textstyle\sum l$$

$$\dfrac{\dfrac{\left\{\dfrac{\pi_i}{X \vdash Y_i}\right\}_{i \in I}}{X \vdash \prod_{i \in I} Y_i}\,\textstyle\prod r \qquad \dfrac{\dfrac{\pi}{Y_k \vdash Z}}{\prod_{i \in I} Y_i \vdash Z}\,\textstyle\prod l}{X \vdash Z}\,\mathsf{cut} \quad \overset{(7)}{\longrightarrow} \quad \dfrac{\dfrac{\pi_k}{X \vdash Y_k} \qquad \dfrac{\pi}{Y_k \vdash Z}}{X \vdash Z}\,\mathsf{cut}$$

$$\dfrac{\dfrac{\dfrac{\pi}{Z \vdash Y_k}}{Z \vdash \sum_{i \in I} Y_i}\,\textstyle\sum r \qquad \dfrac{\left\{\dfrac{\pi_i}{Y_i \vdash X}\right\}_{i \in I}}{\sum_{i \in I} Y_i \vdash X}\,\textstyle\sum l}{Z \vdash X}\,\mathsf{cut} \quad \underset{\text{(dual to (7))}}{\overset{(8)}{\longrightarrow}} \quad \dfrac{\dfrac{\pi}{Z \vdash Y_k} \qquad \dfrac{\pi_k}{Y_k \vdash X}}{Z \vdash X}\,\mathsf{cut}$$

Figure 3. Cut elimination rewrites.

$$\frac{\{R_i : X_i \to Y\}_{i\in I}}{\bigcup_{i\in I} R_i : \sum_{i\in I} X_i \to Y}\ \textstyle\sum l$$

$$\frac{\{R_i : Y \to X_i\}_{i\in I}}{\bigcup_{i\in I} R_i : Y \to \prod_{i\in I} X_i}\ \textstyle\prod r$$

$$\frac{R : X_k \to Y}{R : \prod_{i\in I} X_i \to Y}\ \textstyle\prod l$$

$$\frac{R : Y \to X_k}{R : Y \to \sum_{i\in I} X_k}\ \textstyle\sum r$$

Figure 4. The inductive step for the definition of $F : \mathcal{D}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$, with canonical maps in Rel left implicit.

$$\mathsf{pr}_k : |\textstyle\prod_{i\in I} X_i| = |\sum_{i\in I} X_i| = \coprod_{i\in I} |X_i| \nrightarrow |X_k|$$

$$\mathsf{in}_k : |X_k| \nrightarrow \textstyle\coprod_{i\in I} |X_i| = |\prod_{i\in I} X_i| = |\sum_{i\in I} X_i|$$

$$\frac{\{R_i : X_i \to Y\}_{i\in I}}{\bigcup_{i\in I}(\mathsf{pr}_i; R_i)\ :\ \sum_{i\in I} X_i \to Y}\ \textstyle\sum l$$

$$\frac{\{R_i : Y \to X_i\}_{i\in I}}{\bigcup_{i\in I}(R_i; \mathsf{in}_i)\ :\ Y \to \prod_{i\in I} X_i}\ \textstyle\prod r$$

$$\frac{R : X_k \to Y}{\mathsf{pr}_k; R\ :\ \prod_{i\in I} X_i \to Y}\ \textstyle\prod l$$

$$\frac{R : Y \to X_k}{R; \mathsf{in}_k\ :\ Y \to \sum_{i\in I} X_i}\ \textstyle\sum r$$

Figure 5. The inductive step for the definition of $F : \mathcal{D}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$, with canonical maps in Rel made explicit.

4.2.1 Well-definedness

For F to be well-defined on morphisms, we must verify that the binary relation below each rule of Figure 4 is a linking whenever the hypotheses relation(s) are linkings, and that the definition of F respects equivalence of cut-free derivations. The former follows from the fact that a $\sum$-strategy of $\sum_{i\in I} X_i$ corresponds to a choice of $k \in I$ together with a $\sum$-strategy of X_k, and a $\prod$-strategy of $\sum_{i\in I} X_i$ is the union $\bigcup_{i\in I} X'_i$ of $\prod$-strategies X'_i of X_i (and dually, with $\prod$ and $\sum$ exchanged).

The verification of the latter is equally simple: either side of each of the conversions (9)–(12) of Figure 2 results in the same linking. For example, both sides of conversion (9) result in $\bigcup_{i\in I} F(\pi_i) : \sum_{i\in I} X_i \to \sum_{j\in J} Y_j$.

4.2.2 Functoriality

We must verify that F respects identities and composition. The former follows by induction, since the identities of $\mathcal{D}(\mathcal{A})$ are defined inductively (section 4.1). To verify the latter, we must show that for all cut-free derivations π_1 of $X \vdash Y$ and π_2 of $Y \vdash Z$, we have

$$F(\pi_1); F(\pi_2)\ =\ F\big(\mathsf{elim}(\mathsf{cut}(\pi_1, \pi_2))\big)$$

where $\mathsf{cut}(\pi_1, \pi_2)$ is the derivation obtained by placing π_1 and π_2 above a cut-rule, and $\mathsf{elim}(\pi)$ is the result of elminating the cuts of π using conversions (1)–(8) of Figure 3.

We proceed by induction on $\mathsf{size}(\pi_1) \times \mathsf{size}(\pi_2)$, where $\mathsf{size}(\pi)$ is the number of non-id rules in π. (Thus $\mathsf{size}(\pi) = 0$ iff π consists of a single id-rule.) In the base case, one of π_1 or π_2 is an identity rule, so $\mathsf{elim}(\mathsf{cut}(\pi_1, \pi_2))$ is equal to the other. The result follows since F preserves identities.

For the inductive step, we must verify the translations of conversions (3)–(8) of Figure 3 using the definition of F given in Figure 5. (Conversions (1) and (2) correspond to the base case, already dealt with above.) These follow from the equalities between binary relations shown in Table 1: take $R = F(\pi)$, $R' = F(\pi')$ and $R_i = F(\pi_i)$, where π, π' and π_i are the cut-free proofs parameterising (3)–(8). (To aid pattern-matching, we have labelled the equation corresponding to conversion (n) by n.) Equalitites 3 and 4 are instances of the associativity of the composition of binary relations; equalities 5–8 hold by unfolding the definitions of injections and projections in Rel.

4.2.3 Fullness

The fullness of F is essentially a corollary of the softness of $\mathcal{L}(\mathcal{A})$ (Proposition 3).

Given a linking $R : X \to Y$ we must show that there exists a cut-free proof π such that $F(\pi) = R$. We proceed by induction on the sum of the number of connectives ($\prod$ or $\sum$) in X and the number of connectives in Y. The base case (0 connectives) is $R = \mathsf{id}_A : A \to A$ for some atom A, which is the image of the identity rule with atom A. For the induction step:

1. If $X = \sum_{i\in I} X_i$, then, by condition (2) in the definition of linking, and the nature of a $\sum$-strategy of a sum $\sum_{i\in I} X_i$, R is the union over $i \in I$ of linkings $R_i : X_i \to Y$. By induction hypothesis, $R_i = F(\pi_i)$ for cut-free proofs π_i. Now $R = F(\pi)$ for π the proof

$$R;(R';\mathsf{in}_k) \overset{3}{=} (R;R');\mathsf{in}_k$$

$$(\mathsf{pr}_k;R);R' \overset{4}{=} \mathsf{pr}_k;(R;R')$$

$$R;\left(\bigcup_{i\in I} R_i;\mathsf{in}_i\right) \overset{5}{=} \bigcup_{i\in I}\left((R;R_i);\mathsf{in}_i\right)$$

$$\left(\bigcup_{i\in I} \mathsf{pr}_i;R_i\right);R \overset{6}{=} \bigcup_{i\in I}\left(\mathsf{pr}_i;(R_i;R)\right)$$

$$\left(\bigcup_{i\in I} R_i;\mathsf{in}_i\right);(\mathsf{pr}_k;R) \overset{7}{=} R_k;R$$

$$(R;\mathsf{in}_k);\left(\bigcup_{i\in I} \mathsf{pr}_i;R_i\right) \overset{8}{=} R;R_k$$

Table 1. Equations on binary relations corresponding to conversions (3)–(8).

consisting of a $\sum l$-rule with hypotheses $\{\pi_i\}_{i\in I}$.

2. If $Y = \prod_{j\in J} Y_j$, then $R = F(\pi)$ for a proof π ending in a $\prod r$-rule, by an argument similar to 1.

3. If X is an atom and $Y = \sum_{j\in J} Y_j$, then by condition (2) of linking and the nature of $\prod$-strategies of a sum $\sum_{j\in J} Y_j$, R intersects exactly one of the Y_i, say Y_k. Thus $R = R';\mathsf{in}_k$ for some $R' : X \to Y_k$. By induction hypothesis $R' = F(\pi')$, hence $R = F(\pi)$ for π the extension of π' with a $\sum r$-rule.

4. If Y is an atom and $X = \prod_{i\in I} X_i$, then $R = F(\pi)$ for a proof π ending in a $\prod l$-rule, by an argument similar to 3.

5. Otherwise $X = \prod_{i\in I} X_i$ and $Y = \sum_{j\in J} Y_j$. By softness (Proposition 3), R factorises either as $R = R';\mathsf{in}_k$ or $R = \mathsf{pr}_m;R'$. In the former case, $R = F(\pi)$ for a proof π ending in a $\sum r$-rule (by reasoning as in 3), and in the latter case, $R = F(\pi)$ for a proof π ending in a $\prod l$-rule (by reasoning as in 4).

4.2.4 Faithfulness

To show that F is faithful, we prove that if π and π' are distinct normal forms with respect to the cut-free rewrites (9)–(12) in Figure 2, then the linkings $F(\pi)$ and $F(\pi')$ are distinct.

Let π and π' be distinct normal cut-free proofs of the sequent $X \vdash Y$. We argue that $F(\pi) \neq F(\pi')$ by induction on the sum of the number of connectives ($\prod$ or $\sum$) in X and the number of connectives in Y.

Base case. X and Y are atoms. Necessarily $X = Y$ and π and π' are uniquely determined as the same id-rule, contradicting the fact that π and π' are distinct.

Induction step. Assume π and π' finish with distinct rules, otherwise we can appeal immediately to the induction hypothesis with the branches of π and π'.

1. Case: one of X or Y is an atom. Then π and π' necessarily finish with the same rule, contradicting our assumption.

2. Case: $X = \prod_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$. Thus π and π' each finish with one of the following rules:

$$\frac{\left\{\prod_{i\in I} X_i \vdash Y_j\right\}_{j\in J}}{\prod_{i\in I} X_i \vdash \prod_{j\in J} Y_j}\ \textstyle\prod r$$

$$\frac{X_k \vdash \prod_{j\in J} Y_j}{\prod_{i\in I} X_i \vdash \prod_{j\in J} Y_j}\ \textstyle\prod l$$

By our earlier assumption, π and π' end with distinct rules. If the final rules are both $\prod l$, with $k = m$ and $k = m'$ respectively (and necessarily $m \neq m'$), then $F(\pi)$ and $F(\pi')$ are distinct: all links of the former intersect X_m, and all links of the latter intersect $X_{m'}$, so to be equal, $F(\pi)$ and $F(\pi')$ must both be empty. However, every linking is non-empty (by condition (2) of the definition of linking), so this is a contradiction.

Thus one of π and π' ends with $\prod r$, and the other with $\prod l$. Without loss of generality, π ends with $\prod r$ and π' ends with $\prod l$. For a contradiction, assume the linkings $F(\pi)$ and $F(\pi')$ are equal. Let ρ be (one of) the highest occurrences in π (measured in terms of the number of rules below ρ) of a $\prod l$-rule introducing $\prod_{i\in I} X_i$. Since π' ends with $\prod l$, each of its links intersect X_k, thus ρ in π has hypothesis $X_k \vdash Z$ and conclusion $\prod_{i\in I} X_i \vdash Z$, for some subformula Z of Y. Let ρ' be the rule immediately following ρ, necessarily introducing a connective on the right, with Z as one of its arguments. This connective must be a $\prod$, and ρ' an occurrence of $\prod r$, otherwise ρ and ρ' together would constitute a redex for conversion (11) of Figure 2, contradicting the normality of π. Thus ρ' has hypothesis $\left\{\prod_{i\in I} X_i \vdash Z_m\right\}_{m\in M}$ and conclusion $\prod_{i\in I} X_i \vdash \prod_{m\in M} Z_m$, and $Z = Z_q$ for some $q \in M$. Since ρ is highest, each proof π_m of $\prod_{i\in I} X_i \vdash Z_m$ ends with an instance of $\prod l$, with hypothesis $X_{k(m)} \vdash Z_m$. (There must be a rule somewhere in π_m introducing $\prod_{i\in I} X_i$; it cannot be any higher than ρ, since ρ is highest.) Furthermore, $k(m) = k$ for all m, since it is X_k that is in the hypothesis of the last rule of π' (hence this is the only one of the X_i which intersects with a link). Thus in π

we have:

$$\dfrac{\left\{\begin{array}{c}\pi_m \,\vdots \\ \dfrac{X_k \vdash Z_m}{\prod_{i\in I} X_i \vdash Z_m}\,\prod l\end{array}\right\}_{m\in M}}{\prod_{i\in I} X_i \vdash \prod_{m\in M} Z_m}\,\prod r \quad (=\rho')$$

for $\prod_{m\in M} Z_m$ a subformula of Y. This is a redex of conversion (10), contradicting the normality of π.

3. Case: $X = \sum_{i\in I} X_i$ and $Y = \sum_{j\in J} Y_j$. Dual to the previous case: exchange left/right, and $\prod$ / $\sum$.

4. Case: $X = \prod_{i\in I} X_i$ and $Y = \sum_{j\in J} Y_j$. Thus π and π' each finish with one of the following rules:

$$\dfrac{\prod_{i\in I} X_i \vdash Y_m}{\prod_{i\in I} X_i \vdash \sum_{j\in J} Y_j}\,\sum r$$

$$\dfrac{X_k \vdash \sum_{j\in J} Y_j}{\prod_{i\in I} X_i \vdash \sum_{j\in J} Y_j}\,\prod l$$

By assumption, π and π' end with distinct rules. If the final rules are both $\prod l$, we obtain a contradiction (as in case 2). Similarly, both $\sum r$ leads to a contradiction. Therefore, without loss of generality, π finishes with $\sum r$ and π' finishes with $\prod l$. Now the reasoning of case 2 applies directly (since π' finishes with $\prod l$), yielding a contradiction if $F(\pi) = F(\pi')$.

5. Case: $X = \sum_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$. Thus π and π' each finish with one of the following rules:

$$\dfrac{\left\{\sum_{i\in I} X_i \vdash Y_j\right\}_{j\in J}}{\sum_{i\in I} X_i \vdash \prod_{j\in J} Y_j}\,\prod r$$

$$\dfrac{\left\{X_i \vdash \prod_{j\in J} Y_j\right\}_{i\in I}}{\sum_{i\in I} X_i \vdash \prod_{j\in J} Y_j}\,\sum l$$

By assumption, π and π' end with distinct rules, so without loss of generality, π ends with $\prod r$, and π' ends with $\sum l$.

For a contradiction, assume the linkings $F(\pi)$ and $F(\pi')$ are equal. Let ρ be (one of) the highest rules in π (measured in terms of the number of proof rules below it) introducing $\sum_{i\in I} X_i$ (so ρ is an instance of $\sum l$), say with conclusion $\sum_{i\in I} X_i \vdash Z$. Let ρ' be the rule immediately following ρ. If ρ' is an instance of $\sum r$, then ρ and ρ' together form a redex for conversion (9) of Figure 2, contradicting the normality of π. Hence ρ' must be an instance of $\prod r$, introducing $\prod_{m\in M} Z_m$, and with Z one of the Z_m.

Since ρ is highest, each proof π_m of $\sum_{i\in I} X_i \vdash Z_m$ ends with an instance of $\sum l$, with hypothesis $\{X_i \vdash Z_m\}_{i\in I}$ and conclusion $\sum_{i\in I} X_i \vdash Z_m$. (There must be a rule somewhere in π_m introducing $\sum_{i\in I} X_i$; it cannot be any higher than ρ, since ρ is highest.) Thus in π we have:

$$\dfrac{\left\{\dfrac{\left\{\dfrac{\pi_{im}}{X_i \vdash Z_m}\right\}_{i\in I}}{\sum_{i\in I} X_i \vdash Z_m}\,\sum l\right\}_{m\in M}}{\sum_{i\in I} X_i \vdash \prod_{m\in M} Z_m}\,\prod r \quad (=\rho')$$

for a subformula $\prod_{m\in M} Z_m$ of Y, and where the π_{im}, as i ranges over I, together with the following $\sum l$-rule, constitute π_m. This is a redex of conversion (12), contradicting the normality of π.

4.2.5 Preservation of non-empty finite products, sums

The fact that $F : \mathcal{D}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$ preserves non-empty finite products and sums is easily verified, since the objects of $\mathcal{D}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ coincide, as do the definitions of $\prod_{i\in I} X_i$ and $\sum_{i\in I} X_i$ on objects.

4.3 Main theorem

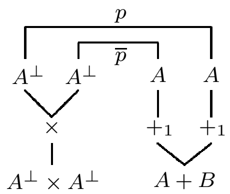
THEOREM 1 *The category $\mathcal{L}$ of linkings is isomorphic to the category generated freely from the atoms of* $linkings$ *by non-empty finite product and non-empty finite sum (coproduct).*

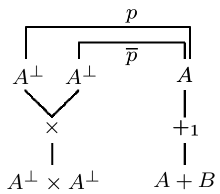
Proof. $\mathcal{F}(\mathcal{A}) \cong \mathcal{D}(\mathcal{A})$ by Proposition 4. In section 4.2 we defined a structure-preserving full and faithful functor $F : \mathcal{D}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$, which witnesses an isomorphism $\mathcal{D}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A})$, since F acts trivially on objects. Thus $\mathcal{F}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A})$. □

5 Relationship with proof nets

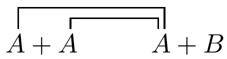
In [Gir96], Girard defines a notion of proof net for multiplicative-additive linear logic. The definition is somewhat involved, so we do not reproduce it here. We substitute the standard categorical notation $\times/+$ for Girard's & $/\oplus$.

To relate the category of linkings to proof nets, we restrict products and sums to the binary case. Let X and Y be $\times/+$-formulas over the set $\mathcal{A}$ of atoms. Viewing atoms as literals, $X^\perp$ and Y are well-defined formulas of additive linear logic. Let Θ be a cut-free proof net with conclusions $X^\perp$ and Y. Every axiom link of Θ determines an edge between a literal of $X^\perp$ and a literal of Y, whence Θ determines a binary relation $R(\Theta) \subset |X| \times |Y|$ between the leaves of X and the leaves of Y. Note that distinct cut-free proof nets can yield the same binary relation, for example:





both yield the following binary relation between leaves:



Define Θ to be ***tight*** if every formula occurrence $U \times V$ of Θ has exactly one $\times$-link immediately above it, and every formula occurrence $U + V$ has exactly one $+_1$-link and one $+_2$-link immediately above it.

PROPOSITION 5 *Let X and Y be $\times/+$-formulas over the atoms of $\mathcal{L}$. Morphisms $X \to Y$ in $\mathcal{L}$ are in bijection with tight, cut-free Girard proof nets with conclusions $X^\perp$ and Y.*

Proof. First we show that $R(\Theta)$ is a linking. Condition (1) of linking is trivial. Every valuation φ of the eigenvariables of Θ determines a $+$-strategy $X'(\varphi)$ of X and a $\times$-strategy $Y'(\varphi)$ of Y, and every pair $\langle X', Y' \rangle$ consisting of a $+$-strategy of X and a $\times$-strategy of Y arises in this manner.

We must show that, given any $+$-strategy X' of X and $\times$-strategy Y' of Y, $R(\Theta)$ contains a unique edge between X' and Y'. By the previous paragraph, there exists a valuation φ such that $X' = X'(\varphi)$ and $Y' = Y'(\varphi)$. Since the slice $\varphi(\Theta)$ gives rise to an edge e_φ of $R(\Theta)$ between $X'(\varphi)$ and $Y'(\varphi)$, it remains to verify that there is at most one edge in $R(\Theta)$ between X' and Y'. For a contradiction, suppose there are distinct edges $e, e' \in R(\Theta)$ between X' and Y'. Without loss of generality, $e = e_\varphi$. Let φ' be a valuation such that $e' = e_{\varphi'}$. Let $e = \langle x, y \rangle$, let $e' = \langle x', y' \rangle$, let x^* be the unique vertex of X such x and x' are leaves of distinct arguments of x^*, and let y^* be the unique vertex of Y such that y and y' are leaves of distinct arguments of y^*. Since e and e' are both between the strategies X' and Y', neither of x^* and y^* is a $\times$.

Let a and a' be the axiom links in Θ that gave rise to e and e', respectively. Let $\tilde{x}$ and $\tilde{y}$ be the vertices occurrences of Θ corresponding to x^* and y^*. Recall from [AM99] that since Θ is a proof net, if $p_L.w$ is any weight occurring in Θ, then $w \subset w(L)$, for any $\times$-link in Θ. Therefore, since neither $\tilde{x}$ nor $\tilde{y}$ is a $\times$-link, the weight $w(a)$ of a does not depend on any of the $\times$'s on which φ and φ' differ. Hence a is present not only in $\varphi(\Theta)$, but also $\varphi'(\Theta)$, a contradiction (the weight of one of $\tilde{x}$ or $\tilde{y}$ fails to be the disjoint sum of the weights of its children). This completes the proof that $R(\Theta)$ is a linking.

It remains to show that $R(-)$ is injective and surjective. Injectivity follows from the tightness assumption. Surjectivity follows from the fact that every morphism R determines a tight proof net $\Theta(R)$. By tightness, the link and formula-occurrence structure is fully determined by X and Y, with the literal occurrences of $\Theta(R)$ being in bijection with the leaves involved in edges of R. The weights of $\Theta(R)$ are determined by specifying weights on axiom links a: identifying a valuation with a monomial, the weight of a is the union of the valuations φ such that the edge of R corresponding to a is between the strategies induced by φ. It is easy to check that each of these weights will be a monomial, and the remaining requirements for $\Theta(R)$ to be a proof net. It is routine to verify that $\Theta(-)$ is inverse to $R(-)$. □

The present author, with Rob van Glabbeek collaborating as second author, recently introduced an alternative notion of proof net for multiplicative-additive linear logic [HG02].

PROPOSITION 6 *Let X and Y be $\times/+$-formulas over the atoms of $\mathcal{L}$. Morphisms $X \to Y$ in $\mathcal{L}$ are in bijection with cut-free proof nets, as defined in [HG02], on the sequent $\vdash X^\perp, Y$.*

The proof is far less involved than that of Proposition 5, since $\prod$- and $\sum$-strategies are directly related to the notion of resolution defined in [HG02].

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