

# Proofs Without Syntax

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## Abstract

“Mathematicians care no more for logic than logicians for mathematics.”

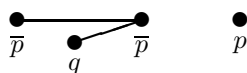
Augustus de Morgan, 1868

Proofs are traditionally syntactic, inductively generated objects. This paper presents an abstract mathematical formulation of propositional calculus (propositional logic) in which proofs are combinatorial (graph-theoretic), rather than syntactic. It defines a *combinatorial proof* of a proposition  $\phi$  as a graph homomorphism  $h : G \rightarrow G(\phi)$ , where  $G(\phi)$  is a graph associated with  $\phi$ , and  $G$  is a coloured graph. The main theorem is soundness and completeness:  $\phi$  is true iff there exists a combinatorial proof  $h : G \rightarrow G(\phi)$ .

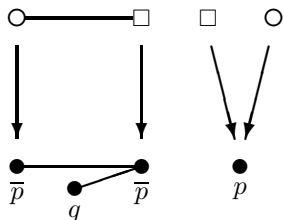
## 1 Introduction

In 1868, de Morgan lamented the rift between mathematics and logic [deM68]: “mathematicians care no more for logic than logicians for mathematics.” The dry syntactic manipulations of formal logic can be off-putting to mathematicians accustomed to beautiful symmetries, geometries, and rich layers of structure. Figure 1 shows a syntactic proof in a standard Hilbert system taught to mathematics undergraduates [Hil28, Joh87]. Although the system itself is elegant (e.g. just three axiom schemata suffice), the syntactic proofs generated in it need not be. Other systems such as [Fr1879, Gen35, Tai68] are also syntactic.

This paper presents an abstract mathematical formulation of propositional calculus (propositional logic) in which proofs are combinatorial (graph-theoretic), rather than syntactic. It defines a *combinatorial proof* of a proposition  $\phi$  as a graph homomorphism  $h : G \rightarrow G(\phi)$ , where  $G(\phi)$  is a graph associated with  $\phi$ , and  $G$  is a coloured graph. To illustrate, if  $\phi = ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$  then  $G(\phi)$  is



Here is a combinatorial proof  $h : G \rightarrow G(\phi)$  of  $\phi$ :



The colouring of  $G$  is indicated by vertex type ( $\circ$  or  $\square$ ) and  $h$  is given by the arrows. The same proposition was proved syntactically in Figure 1.

The main theorem is soundness and completeness:

*A proposition is true iff it has a combinatorial proof.*

As with conventional syntactic soundness and completeness, this theorem matches a universal quantification with an existential one: a proposition  $\phi$  is true if it evaluates to 1 for all 0/1 assignments of its variables, and  $\phi$  is provable if there exists a proof of  $\phi$ . However, where conventional completeness provides an inductively generated *syntactic* witness (e.g. Figure 1), this theorem provides an abstract *mathematical* witness for every true proposition (e.g. the homomorphism  $h$  drawn above).

Just three conditions suffice for soundness and completeness: a graph homomorphism  $h : G \rightarrow G(\phi)$  is a combinatorial proof of  $\phi$  if (1)  $G$  is a coloured graph of a certain type, (2)  $h$  is a *skew fibration*, a lax form of graph fibration, and (3) the image of each colour class is labelled appropriately.

**Prerequisites.** The paper should be accessible to a broad mathematical audience. An acquaintance with basic graph theory [Bol02] and propositional calculus [Joh87] would be helpful, though not strictly necessary.

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## 2 Notation and terminology

**Graphs.** An *edge* on a set  $V$  is a two-element subset of  $V$ . A *graph*  $(V, E)$  is a finite set  $V$ , whose elements are called *vertices*, and a set  $E$  of edges on  $V$ . Write  $V(G)$  and  $E(G)$  for the vertex set and edge set of a graph  $G$ , respectively, and  $vw$  for  $\{v, w\}$ . The *complement* of  $(V, E)$  is the graph  $(V, E^c)$  with  $vw \in E^c$  iff  $vw \notin E$ . A graph  $(V, E)$  is *coloured* if  $V$  is equipped with an equivalence relation  $\sim$  such that  $v \sim w$  only if  $vw \notin E$ ; each equivalence class is called a *colour class*. Given a set  $L$ , a graph is *L-labelled* if every vertex has an element of  $L$  associated with it, called its *label*. The *union*  $G \vee G'$  of graphs  $G = (V, E)$  and  $G' = (V', E')$  with no common vertex is  $(V \cup V', E \cup E')$  and the *join*  $G \wedge G'$  is  $(V \cup V', E \cup E' \cup \{vv' : v \in V, v' \in V'\})$ ;

Below is a proof of Peirce's law  $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$  in a standard Hilbert formulation of propositional logic, taught to mathematics undergraduates [Joh87], with axiom schemata

- (a)  $x \Rightarrow (y \Rightarrow x)$
- (b)  $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z))$
- (c)  $((x \Rightarrow \perp) \Rightarrow \perp) \Rightarrow x$

and where  $(m_j^i)$  marks *modus ponens* with hypotheses numbered  $i$  and  $j$ . Hilbert systems tend to emphasise the elegance of the schemata (e.g. just (a)–(c) suffice) over the elegance of the proofs generated by the schemata.

- 1 (c)  $((q \Rightarrow \perp) \Rightarrow \perp) \Rightarrow q$
- 2 (a)  $((q \Rightarrow \perp) \Rightarrow \perp) \Rightarrow q \Rightarrow (\perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp) \Rightarrow q)$
- 3  $(m_2^1)$   $\perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp) \Rightarrow q$
- 4 (b)  $(\perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp) \Rightarrow q) \Rightarrow ((\perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp)) \Rightarrow (\perp \Rightarrow q))$
- 5  $(m_4^3)$   $(\perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp)) \Rightarrow (\perp \Rightarrow q)$
- 6 (a)  $\perp \Rightarrow ((q \Rightarrow \perp) \Rightarrow \perp)$
- 7  $(m_5^6)$   $\perp \Rightarrow q$
- 8 (a)  $(\perp \Rightarrow q) \Rightarrow (p \Rightarrow (\perp \Rightarrow q))$
- 9  $(m_8^7)$   $p \Rightarrow (\perp \Rightarrow q)$
- 10 (b)  $(p \Rightarrow (\perp \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))$
- 11  $(m_{10}^9)$   $(p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)$
- 12 (a)  $((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p))$
- 13 (b)  $((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 14 (a)  $((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))))$
- 15  $(m_{14}^{13})$   $((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 16 (b)  $((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)))$
- 17  $(m_{16}^{15})$   $((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow q) \Rightarrow p)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)))$

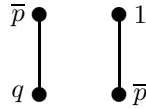
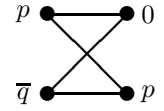
- 18  $(m_{17}^{12})$   $((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 19 (b)  $((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 20  $(m_{19}^{18})$   $((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 21 (a)  $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)))$
- 22 (a)  $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))))$
- 23  $(m_{22}^{20})$   $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 24 (b)  $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))))$
- 25  $(m_{24}^{23})$   $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q))) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)))$
- 26  $(m_{25}^{24})$   $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow q)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 27  $(m_{26}^{25})$   $(p \Rightarrow q) \Rightarrow p \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)$
- 28 (a)  $(p \Rightarrow \perp) \Rightarrow (((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp)) \Rightarrow (p \Rightarrow \perp))$
- 29 (b)  $((p \Rightarrow \perp) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp))) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp))$
- 30  $(m_{29}^{28})$   $(p \Rightarrow \perp) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp)) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp))$
- 31 (a)  $(p \Rightarrow \perp) \Rightarrow ((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp))$
- 32  $(m_{31}^{30})$   $(p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp)$
- 33 (b)  $((p \Rightarrow \perp) \Rightarrow (p \Rightarrow \perp)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow \perp))$
- 34  $(m_{33}^{32})$   $(p \Rightarrow \perp) \Rightarrow p \Rightarrow ((p \Rightarrow \perp) \Rightarrow \perp)$
- 35 (c)  $(p \Rightarrow \perp) \Rightarrow \perp$
- 36 (a)  $((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p))$
- 37  $(m_{36}^{35})$   $(p \Rightarrow \perp) \Rightarrow p \Rightarrow (((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p)$
- 38 (b)  $((p \Rightarrow \perp) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow \perp)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow \perp)) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p))$
- 39  $(m_{38}^{37})$   $((p \Rightarrow \perp) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow p)$
- 40  $(m_{39}^{38})$   $(p \Rightarrow \perp) \Rightarrow p \Rightarrow p$
- 41 (a)  $((p \Rightarrow \perp) \Rightarrow p) \Rightarrow p \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow p))$
- 42  $(m_{41}^{40})$   $(p \Rightarrow q) \Rightarrow p \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow p)$
- 43 (b)  $((p \Rightarrow q) \Rightarrow p) \Rightarrow (((p \Rightarrow \perp) \Rightarrow p) \Rightarrow p) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p)) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$
- 44  $(m_{43}^{42})$   $((p \Rightarrow q) \Rightarrow p) \Rightarrow ((p \Rightarrow \perp) \Rightarrow p) \Rightarrow (((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$
- 45  $(m_{44}^{43})$   $(p \Rightarrow q) \Rightarrow p \Rightarrow p$

colourings/labellings of  $G$  and  $G'$  are inherited. A **homomorphism**  $h : G \rightarrow G'$  from  $G$  to  $G'$  (either of which may be coloured or labelled) is a function  $h : V(G) \rightarrow V(G')$  such that  $vw \in E(G)$  implies  $h(v)h(w) \in E(G')$ . A graph  $(V, E)$  is a **cograph** (see e.g. [BLS99]) if it has at least one vertex and for any distinct  $v, w, x, y \in V$ , the restriction of  $E$  to edges on  $\{v, w, x, y\}$  is not  $\{vw, wx, xy\}$ . A set  $W \subseteq V(G)$  **induces a matching** if it is non-empty and for all  $w \in W$  there is a unique  $w' \in W$  with  $ww' \in E(G)$ .

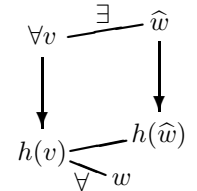
**Propositions.** Fix a set  $\mathcal{V}$  of **variables**. A **proposition** is any expression generated freely from variables by the binary operations **and**  $\wedge$ , **or**  $\vee$ , and **implies**  $\Rightarrow$ , the unary operation **not**  $\neg$ , and the constants (nullary operations) **true** 1 and **false** 0. A **valuation** is a function  $f : \mathcal{V} \rightarrow \{0, 1\}$ . Write  $\hat{f}$  for the extension of a valuation  $f$  to propositions defined by  $\hat{f}(0) = 0$ ,  $\hat{f}(1) = 1$ ,  $\hat{f}(\neg\phi) = 1 - \hat{f}(\phi)$ ,  $\hat{f}(\phi \wedge \rho) = \min\{\hat{f}(\phi), \hat{f}(\rho)\}$ ,  $\hat{f}(\phi \vee \rho) = \max\{\hat{f}(\phi), \hat{f}(\rho)\}$ ,  $\hat{f}(\phi \Rightarrow \rho) = \hat{f}(\rho \vee \neg\phi)$ . A proposition  $\phi$  is **true** if  $\hat{f}(\phi) = 1$  for all valuations  $f$ , and **false** otherwise. Variables  $p \in \mathcal{V}$  and their negations  $\bar{p} = \neg p$  are **literals**;  $p$  and  $\bar{p}$  are **complementary**, as are 0 and 1. An **atom** is a literal or constant, and  $\mathcal{A}$  denotes the set of atoms.

### 3 Combinatorial proofs

Given an  $\mathcal{A}$ -labelled graph  $G$ , define  $\neg G$  as the result of complementing  $G$  and every label of  $G$ . For example, if  $G$  is the graph shown right, then  $\neg G$  is the graph below left. Define  $G \Rightarrow G' = (\neg G) \vee G'$ . Identify each atom  $a$  with a single vertex labelled  $a$ ; thus, having defined operations  $\neg$ ,  $\vee$ ,  $\wedge$  and  $\Rightarrow$  on  $\mathcal{A}$ -labelled graphs, every proposition  $\phi$  determines an  $\mathcal{A}$ -labelled graph, denoted  $G(\phi)$ . For example,  $G((p \vee \neg q) \wedge (0 \vee p))$  is above right,  $G((q \wedge \neg p) \vee (1 \wedge \neg p))$  is above left, and  $G(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$  is the first graph on page 1.



A colouring is **nice** if every colour class has at most two vertices and no union of two-vertex colour classes induces a matching.<sup>1</sup> A graph homomorphism<sup>2</sup>  $h : G \rightarrow G'$  is a **skew fibration** (see figure right) if for all  $v \in V(G)$  and  $h(v)w \in E(G')$  there exists  $v\hat{w} \in E(G)$  with  $h(\hat{w})w \notin E(G')$ . Given a graph homomorphism  $h : G \rightarrow G'$  with  $G'$



<sup>1</sup>I.e., every colour class has 1 or 2 vertices, and if  $c_1, \dots, c_n$  are two-vertex colour classes then  $c_1 \cup \dots \cup c_n$  does not induce a matching.

<sup>2</sup>Recall that our definition of homomorphism ignores any possible colouring/labelling of  $G$  or  $G'$ .

an  $\mathcal{A}$ -labelled graph, a vertex  $v \in V(G)$  is **self-evident** if  $h(v)$  is labelled 1, and a two-vertex set  $\{v, w\} \subseteq V(G)$  is self-evident if  $h(v)$  and  $h(w)$  are labelled by complementary literals.

**DEFINITION 1** A **combinatorial proof** of a proposition  $\phi$  is a skew fibration  $h : G \rightarrow G(\phi)$  from a nicely coloured cograph  $G$  to the graph  $G(\phi)$  of  $\phi$ , such that every colour class of  $G$  is self-evident.

A combinatorial proof of  $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$  was shown on page 1. The reader may find it instructive to consider why  $p \wedge \neg p$  has no combinatorial proof.

**THEOREM 1 (SOUNDNESS AND COMPLETENESS)**  
A proposition is true iff it has a combinatorial proof.

Section 4 reformulates this theorem in terms of combinatorial (non-syntactic, non-inductive) notions of *proposition* and *truth*. Section 5 proves the reformulated theorem.

**Notes.** The translation  $\phi \mapsto G(\phi)$  is a well-understood translation of a boolean formula into a graph, interpreting  $\vee$  and  $\neg$  as union and complement (see e.g. [CLS81]), and identifies propositions modulo associativity of  $\wedge$  and  $\vee$ , double negation  $\neg\neg\phi = \phi$ , de Morgan duality  $\neg(\phi \wedge \rho) = (\neg\phi) \vee (\neg\rho)$  and  $\neg(\phi \vee \rho) = (\neg\phi) \wedge (\neg\rho)$ , and  $\phi \Rightarrow \rho = (\neg\phi) \vee \rho$ . Perhaps the earliest graphical representation of propositions is due to Peirce [Pei33, vol. 4:2], dating from the late 1800s.

A skew fibration is a lax notion of graph fibration. A graph homomorphism  $h : G \rightarrow G'$  is a **graph fibration** (see e.g. [BV02]) if for all  $v \in V(G)$  and  $h(v)w \in E(G')$  there is a unique  $v\hat{w} \in E(G)$  with  $h(\hat{w}) = w$ .<sup>3</sup> The definition of skew fibration drops uniqueness and relaxes  $h(\hat{w}) = w$  to ‘skewness’  $h(\hat{w})w \notin E(G')$ .

In the example of a combinatorial proof drawn on page 1, observe that the image of the colour class  $\circ \circ$  under  $h$  is  $\overset{\bullet}{\bar{p}} \overset{\bullet}{p}$ . Think of the colour class as actively pairing an occurrence of a variable  $p$  with an occurrence of its dual  $\bar{p}$ . The idea of pairing dual variable occurrences has arisen independently in the study of various forms of syntax, such as closed categories [KM71], contraction-free predicate calculus [KW84], and linear logic [Gir87].

A partially combinatorial notion of proof for classical logic, called a *proof net*, was presented in [Gir91]. Proof nets are rather syntactic: a proof net of a proposition  $\phi$  has an underlying syntax tree which may contain not only more  $\wedge$ ’s and  $\vee$ ’s than  $\phi$  itself, but also auxiliary syntactic connectives which are not even boolean operations (*contraction* and *weakening*).

<sup>3</sup>This is simply a convenient restatement of the familiar notions of fibration in topology [Whi78] and category theory [Gro59, Gra66]: a graph homomorphism is a graph fibration iff it satisfies the homotopy lifting property (when viewed as a continuous map by identifying each graph edge with a copy of the unit interval) iff it has all requisite cartesian liftings (when viewed as a functor by identifying each graph with its path category).

Nicely coloured cographs relate to a class of multi-graphs studied in [Ret03], *chorded R&B-cographs*. Unlabelled chorded R&B-cographs are in bijection with nicely coloured cographs in which every colour class has two vertices.

## 4 Combinatorial propositions and truth

We begin by recalling more standard material on graphs. A graph  $G$  is **complete** if  $E(G)$  contains every edge possible on  $V(G)$ , and  $G$  is a **subgraph** of  $G'$ , denoted  $G \subseteq G'$ , if  $V(G) \subseteq V(G')$  and  $E(G) \subseteq E(G')$ . A **clique** is a maximal complete subgraph. A graph is a cograph (defined non-inductively on page 2) iff it is derivable from individual vertices by union and join, or equivalently, by union, join and complement [BLS99, §11.3].

A **combinatorial proposition** is an  $\mathcal{A}$ -labelled cograph. The translation  $\phi \mapsto G(\phi)$  of a syntactic proposition into a graph was defined in terms of graph union, join and complement; thus  $\{G(\phi) : \phi \text{ is a syntactic proposition}\}$  is precisely the set of combinatorial propositions.

A **1-clique** is a  $\{1\}$ -labelled clique. Given a combinatorial proposition  $P$  and a valuation  $f : \mathcal{V} \rightarrow \{0, 1\}$ , define  $P^f$  by replacing every label  $a$  of  $P$  by  $\hat{f}(a) \in \{0, 1\}$ ;  $P$  is **true** if  $P^f$  contains a 1-clique for all valuations  $f$ , and **false** otherwise. For example, let  $P = \overset{\bullet}{\bar{p}} \overset{\bullet}{p} \overset{\bullet}{1}$  (which is  $G(p \Rightarrow (p \wedge 1))$ ). If  $f(p) = 1$  then  $P^f$  is  $\overset{\bullet}{0} \overset{\bullet}{1} \overset{\bullet}{1}$  containing the 1-clique  $\overset{\bullet}{1} \overset{\bullet}{1}$ ; if  $f(p) = 0$  then  $P^f$  is  $\overset{\bullet}{1} \overset{\bullet}{0} \overset{\bullet}{1}$  containing the 1-clique  $\overset{\bullet}{1}$  (the left-most vertex); so  $P$  is true.

**LEMMA 1** A proposition  $\phi$  is true iff its combinatorial proposition  $G(\phi)$  is true.

*Proof.* A routine induction, relegated to the appendix on page 6.  $\square$

**DEFINITION 2** A **combinatorial proof** of a combinatorial proposition  $P$  is a skew fibration  $h : G \rightarrow P$  from a nicely coloured cograph  $G$ , such that every colour class of  $G$  is self-evident.

Thus a combinatorial proof of  $P$  is a combinatorial proof of  $\phi$  for any choice of syntactic proposition  $\phi$  with  $P = G(\phi)$ . By Lemma 1, the following is equivalent to Theorem 1 (Soundness and Completeness).

**THEOREM 2 (COMBINATORIAL SOUNDNESS AND COMPLETENESS)** A combinatorial proposition is true iff it has a combinatorial proof.

## 5 Proof of Theorem 2

The diagram right shows the dependency between the Lemmas (1–9) and Theorems (T1–T4) in this paper.

Given a graph homomorphism  $h : G \rightarrow G'$ , an edge  $v\hat{w} \in E(G)$  is a **skew lifting of**  $h(v)w \in E(G')$

**at**  $v$  if  $h(\hat{w})w \notin E(G')$ . Thus  $h$  is a skew fibration iff every edge  $h(v)w \in E(G')$  has a skew lifting at  $v$ .

The subgraph  $G[W]$  **induced by**  $W \subseteq V(G)$  is  $(W, \{vw \in E(G) : v, w \in W\})$ . Let  $h : G \rightarrow H$  be a graph homomorphism and let  $G', H'$  be induced subgraphs of  $G, H$ , respectively. Write  $h(G')$  for the induced subgraph  $H[h(V(G'))]$  and  $h^{-1}(H')$  for the induced subgraph  $G[h^{-1}(V(H'))]$ . The **restriction**  $h|_{H'}$  is the graph homomorphism  $h^{-1}(H') \rightarrow H'$  defined by  $h|_{H'}(v) = h(v)$ .

LEMMA 2 Let  $\diamond \in \{\wedge, \vee\}$ . If  $h : G \rightarrow H_1 \diamond H_2$  is a skew fibration then both restrictions  $h|_{H_i}$  are skew fibrations.

*Proof.* We prove that if  $v\hat{w}$  is a skew lifting of  $h|_{H_i}(v)w = h(v)w \in E(H_i)$  at  $v$  with respect to  $h$ , then  $h(\hat{w}) \in H_i$ ; hence  $v\hat{w}$  is a well defined skew lifting with respect to  $h|_{H_i}$ . Suppose  $h(\hat{w}) \in H_j$  and  $j \neq i$ . If  $\diamond = \vee$ , since  $h$  is a homomorphism,  $h(v)h(\hat{w})$  is an edge between  $H_1$  and  $H_2$  in  $H_1 \vee H_2$ , a contradiction; if  $\diamond = \wedge$ , since  $H_1 \wedge H_2$  has all edges between  $H_1$  and  $H_2$ ,  $h(\hat{w})w$  is an edge, contradicting  $v\hat{w}$  being a skew lifting with respect to  $h$ .  $\square$

LEMMA 3 Let  $h : (G_1 \wedge G_2) \vee (H_1 \vee H_2) \rightarrow (K_1 \wedge K_2) \vee L$  be a skew fibration with  $h(G_i) \subseteq K_i$  and  $h(H_j) \subseteq L$ . Then  $h_i : G_i \vee H_i \rightarrow K_i \vee L$  defined by  $h_i(v) = h(v)$  is a skew fibration.

*Proof.* Since a graph union  $X_1 \vee X_2$  has no edges between  $X_1$  and  $X_2$ , (a) if  $k : X_1 \vee X_2 \rightarrow Y$  is a skew fibration, so also is  $k|^{X_i} : X_i \rightarrow Y$  defined by  $k|^{X_i}(x) = k(x)$ , and (b) if  $k_i : Z_i \rightarrow X_i$  is a skew fibration for  $i = 1, 2$ , so also is  $k_1 \vee k_2 : Z_1 \vee Z_2 \rightarrow X_1 \vee X_2$  defined by  $(k_1 \vee k_2)(z) = k_i(z)$  iff  $z \in V(Z_i)$ . Since  $h_i = ((h|_{(K_1 \wedge K_2)})|_{K_i}) \vee ((h|_{L_i})|^{H_i})$ ,  $h_i$  is a skew fibration by (a), (b) and Lemma 2.  $\square$

A **01-cograph** is a  $\{0,1\}$ -labelled cograph (hence a combinatorial proposition). A 01-cograph  $C$  is true iff it contains a 1-clique (since  $C^f = C$  for all valuations  $f$ ).

LEMMA 4 Let  $h : G \rightarrow C$  be a skew fibration from a cograph  $G$  into a 01-cograph  $C$ . If  $h(G)$  is true then  $C$  is true.

*Proof.* By induction on the number of vertices in  $C$ . If  $C$  is a vertex the result is trivial. Otherwise  $C = C_1 \diamond C_2$  for  $\diamond \in \{\wedge, \vee\}$  and 01-cographs  $C_i$ . Let  $G_i = h^{-1}(C_i)$  and  $h_i = h|_{C_i} : G_i \rightarrow C_i$ , a skew fibration (by Lemma 2). Let

$K \subseteq h(G) = h_1(G_1) \diamond h_2(G_2)$  be a 1-clique. If  $\diamond = \vee$  then  $K \subseteq h_j(G_j)$  for  $j = 1$  or  $2$ , so  $h_j(G_j)$  is true; by induction  $C_j$  is true, hence  $C = C_1 \vee C_2$  is true (since a clique of  $C_j$  is a clique of  $C_1 \vee C_2$ ). If  $\diamond = \wedge$  then  $K = K_1 \wedge K_2$  for 1-cliques  $K_i \subseteq h_i(G_i)$ , so each  $h_i(G_i)$  is true; by induction each  $C_i$  is true, hence  $C = C_1 \wedge C_2$  is true (since a join of cliques of the  $C_i$  is a clique of  $C_1 \wedge C_2$ ).  $\square$

LEMMA 5 Let  $h : G \rightarrow P$  be a skew fibration from a cograph  $G$  into a combinatorial proposition  $P$ . If  $h(G)$  is true then  $P$  is true.

*Proof.* Define  $h^f : G \rightarrow P^f$  by  $h^f(v) = h(v)$ , a skew fibration since  $V(P^f) = V(P)$  and  $E(P^f) = V(P)$ . Then:  $h(G)$  is true  $\stackrel{\text{def}}{\iff} h(G)^f = h^f(G)$  is true for all valuations  $f \stackrel{\text{Lemma 4}}{\implies} P^f$  is true for all valuations  $f \stackrel{\text{def}}{\iff} P$  is true.  $\square$

The **empty** graph is the graph with no vertices. A graph is **disconnected** if it is a union of non-empty graphs, and **connected** otherwise. A **component** is a maximal non-empty connected subgraph. A graph homomorphism  $h : G \rightarrow H$  is **shallow** if  $h^{-1}(K)$  has at most one component for every component  $K$  of  $H$ .

LEMMA 6 For any combinatorial proof  $h : G \rightarrow P$  there exists a shallow combinatorial proof  $h' : G \rightarrow P'$  such that  $P$  is true iff  $P'$  is true.

*Proof.* Let  $G_1, \dots, G_n$  be the components of  $G$ , and let  $P'$  be the union of  $n$  copies of  $P$  defined by  $V(P') = V(P) \times \{1, \dots, n\}$  and  $\langle v, i \rangle \langle w, j \rangle \in E(P')$  iff  $vw \in E(P)$  and  $i = j$ , and the label of  $\langle v, i \rangle$  in  $P'$  equal to the label of  $v$  in  $P$ . Define  $h' : G \rightarrow P'$  on  $v \in V(G_i)$  by  $h'(v) = \langle h(v), i \rangle$ . Since  $P'$  is a union of copies of  $P$ , it is true iff  $P$  is true (every 1-clique of  $(P')^f$  is a copy of a 1-clique of  $P^f$ ), and  $h'$  is a combinatorial proof (with skew liftings copied from those of  $h$ ).  $\square$

A set of vertices  $W \subseteq V(G)$  is a **portion** of  $G$  if  $uv \in E(G)$  implies  $[u \in W \text{ iff } v \in W]$  (i.e., there is no edge between  $W$  and  $V(G) \setminus W$ ). A **fusion** of  $G$  and  $H$  is any graph obtained from  $G \vee H$  by selecting portions  $U$  of  $G$  and  $W$  of  $H$  and adding edges between every vertex of  $U$  and every vertex of  $W$ . Thus union and join are extremal cases of fusion: union with  $U, W$  empty; join with  $U = V(G), W = V(H)$ . On coloured graphs, the converse does not hold: fusion cannot be reduced to union and join. For example, the (nicely) coloured cograph  $\circ \text{---} \square \square$  on page 1 is a fusion of  $\circ \circ$  and  $\square \square$ , but is neither a union nor a join of coloured graphs. For given coloured graphs  $G, G'$  whose colourings are the equivalence relations  $\sim, \sim'$ , by definition  $G \vee G'$  (and  $G \wedge G'$ ) has the colouring  $\sim \cup \sim'$ ; thus every colour class of  $G \vee G'$  (and  $G \wedge G'$ ) is entirely in  $G$  or entirely in  $G'$ .

LEMMA 7 *A fusion of nicely coloured cographs is a nicely coloured cograph.*

*Proof.* Let  $G$  be the fusion of nicely coloured cographs  $G_1$  and  $G_2$  obtained by joining portions  $W_i$  of  $G_i$ . Suppose  $U$  is a union of two-vertex colour classes in  $G$  inducing a matching. Let  $U_i = U \cap V(G_i)$  and  $U'_i = U \cap W_i$ . By definition of fusion, the only edges in  $G$  between  $U_1$  and  $U_2$  are between  $U'_1$  and  $U'_2$ , and there are edges between all vertices of  $U'_1$  and all vertices of  $U'_2$ ; thus  $(\star)$  there is at most one edge between  $U_1$  and  $U_2$ , or else two edges of  $G$  on  $U$  would intersect. Since  $U$  is a union of two-vertex colour classes, each either in  $U_1$  or  $U_2$ , each  $U_i$  contains an even number of vertices. Therefore, since  $U$  induces a matching,  $(\dagger)$  there must be an even number of edges between  $U_1$  and  $U_2$ . Together  $(\star)$  and  $(\dagger)$  imply there is no edge between  $U_1$  and  $U_2$ , hence, for whichever  $U_i$  is non-empty (perhaps both),  $U_i$  is a union of two-vertex colour classes inducing a matching in  $G_i$ , contradicting  $G_i$  being nicely coloured.  $\square$

LEMMA 8 *Every nicely coloured cograph with more than one colour class is a fusion of nicely coloured cographs.*

*Proof.* Let  $G$  be a nicely coloured cograph. Since  $G$  is a cograph, its underlying (uncoloured) graph has the form  $(G_1 \wedge G_2) \vee (G_3 \wedge G_4) \vee \dots \vee (G_{n-1} \wedge G_n) \vee H$  where  $H$  has no edges. Assume  $n \neq 0$ , otherwise the result is trivial. Let  $\widehat{G}$  be the graph whose vertices are the  $G_i$ , with  $G_i G_j \in E(\widehat{G})$  iff there is an edge or colour class  $\{v, w\}$  in  $G$  with  $v \in V(G_i)$  and  $w \in V(G_j)$  (cf. the proof of Theorem 4 in [Ret03]). A *perfect matching* is a set of pairwise disjoint edges whose union contains all vertices. Since  $G$  is nicely coloured,  $M = \{G_1 G_2, G_3 G_4, \dots, G_{n-1} G_n\}$  is the only perfect matching of  $\widehat{G}$ . For if  $M'$  is another perfect matching, then  $M' \setminus M$  determines a set of two-vertex colour classes in  $G$  whose union induces a matching in  $G$ : for each  $G_i G_j \in M' \setminus M$  pick a colour class  $\{v, w\}$  with  $v \in V(G_i)$  and  $w \in V(G_j)$ . Since  $\widehat{G}$  has a unique perfect matching, some  $G_i G_{i+1} \in M$  is a bridge (see [Kot59], or Corollary 2.3 in [Bol78], derived from Hall's Marriage Theorem and (a proof of) Tutte's Theorem), i.e.,  $(V(\widehat{G}), E(\widehat{G}) \setminus G_i G_{i+1}) = X \vee Y$  with  $G_i \in V(X)$  and  $G_{i+1} \in V(Y)$ . Let  $W$  be the union of all colour classes of  $G$  coincident with any  $G_j$  in  $V(X)$ , and let  $W' = V(G) \setminus W$ . Then  $G[W]$  and  $G[W']$  are nicely coloured (since  $W$  and  $W'$  are unions of colour classes), and  $G$  is the fusion of  $G[W]$  and  $G[W']$  obtained by joining portions  $V(G_i)$  of  $G[W]$  and  $V(G_{i+1})$  of  $G[W']$ .  $\square$

LEMMA 9 *Let  $P_1, P_2$  be combinatorial propositions and  $Q$  a combinatorial proposition or the empty graph. Then  $(P_1 \wedge P_2) \vee Q$  is true iff  $P_1 \vee Q$  and  $P_2 \vee Q$  are true.*

*Proof.* We must prove  $(P_1^f \wedge P_2^f) \vee Q^f$  contains a 1-clique iff  $P_1^f \vee Q^f$  and  $P_2^f \vee Q^f$  do, for any assignment  $f$ . (Interpret  $Q^f$  as empty if  $Q$  is empty.) If  $Q^f$  contains a 1-clique this is trivial; otherwise the result reduces to showing that  $P_1^f \wedge P_2^f$  contains a 1-clique iff  $P_1^f$  and  $P_2^f$  do; this holds because a clique in a join  $G \wedge H$  of graphs  $G$  and  $H$  is a join of cliques in  $G$  and  $H$ .  $\square$

THEOREM 3 (COMBINATORIAL SOUNDNESS) *If a combinatorial proposition has a combinatorial proof, it is true.*

*Proof.* Let  $h : G \rightarrow P$  be a combinatorial proof. We prove that  $P$  is true by induction on the number of colour classes in  $G$ . In the base case,  $G$  is a single colour class. If  $v \in V(G)$  then  $h(v)$  is a one-vertex clique of  $P$ , for if  $h(v)w \in E(P)$  then its skew lifting at  $v$  is an edge in  $G$ , a contradiction. Let  $f$  be a valuation. If  $G$  has one vertex  $v$ , by definition of combinatorial proof  $h(v)$  is labelled 1, hence  $P^f$  contains a 1-clique ( $h(v)$ ); if  $G$  has two vertices  $v, w$ , by definition of combinatorial proof  $h(v)$  and  $h(w)$  are labelled by complementary literals, hence  $P^f$  contains a 1-clique (one of  $h(v)$  or  $h(w)$ ).

*Induction step.* By Lemmas 5 and 6, we may assume  $h$  is shallow and surjective. By Lemma 8,  $G$  is a fusion of nicely coloured cographs  $G_1$  and  $G_2$ , obtained from  $G_1 \vee G_2$  by joining portions  $W_i$  of  $G_i$ . If either  $W_i$  is empty then  $G = G_1 \vee G_2$ , hence  $h' : G_1 \rightarrow P$  defined by  $h'(v) = h(v)$  is a combinatorial proof, and  $P$  is true by induction hypothesis. Otherwise both  $W_i$  are non-empty. Let  $P_i = P[h(W_i)]$ . Since  $h$  is a shallow surjection,  $P_1 \wedge P_2$  is a component of  $P$ , say  $P = (P_1 \wedge P_2) \vee Q$ . Define  $h_i : G_i \rightarrow P_i \vee Q$  by  $h_i(v) = h(v)$ , a combinatorial proof:  $G_i$  is a nicely coloured cograph, the self-evident colour class property is inherited from  $h$ , and  $h_i$  is a skew fibration by Lemma 3 (applied after forgetting colourings). By induction hypothesis,  $P_i \vee Q$  is true, hence  $P$  is true by Lemma 9.  $\square$

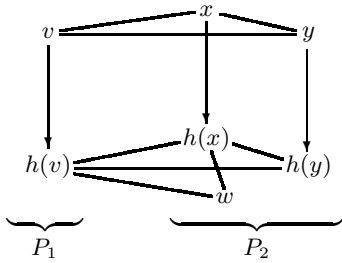
THEOREM 4 (COMBINATORIAL COMPLETENESS) *Every true combinatorial proposition has a combinatorial proof.*

*Proof.* Let  $P$  be a true combinatorial proposition. We construct a combinatorial proof of  $P$  by induction on the number of edges in  $P$ . In the base case,  $P$  is a union of vertices labelled by atoms. Since  $P$  is true, either (a) there exist  $v_1, v_2 \in V(P)$  labelled by complementary literals or (b) there exists  $v_1 \in V(P)$  labelled 1. Let  $G$  comprise a single colour class,  $\{w_1, w_2\}$  in case (a) and  $\{w_1\}$  in case (b). Define  $h : G \rightarrow P$  by  $h(w_i) = v_i$ .

*Induction step.* Since  $P$  is a cograph with an edge,  $P = (P_1 \wedge P_2) \vee Q$  for combinatorial propositions  $P_1, P_2$  and  $Q$  a combinatorial proposition or the empty graph. Assume  $Q$  is empty or false; otherwise by induction there is a combinatorial proof  $G \rightarrow Q$  which we can compose with inclusion  $Q \rightarrow P$  to obtain a combinatorial proof of  $P$ , and we are done. By Lemma 9,  $P_i \vee Q$  is true, so by induction

has a combinatorial proof  $h_i : G_i \rightarrow P_i \vee Q$ . Let  $G$  be the fusion of  $G_1$  and  $G_2$  obtained by joining the portions  $h_i^{-1}(P_i)$  of  $G_i$ . By Lemma 7,  $G$  is nicely coloured. Define  $h : G \rightarrow P$  by  $h(v) = h_i(v)$  iff  $v \in V(G_i)$ . This is a graph homomorphism: let  $vw \in E(G)$  with  $v \in V(G_i)$  and  $w \in V(G_j)$ ; if  $i = j$  then  $h(v)h(w) \in E(P)$  since  $h_i$  is a homomorphism; if  $i \neq j$  then  $vw$  arose from fusion, so  $h(v) \in P_i$  and  $h(w) \in P_j$ , hence  $h(v)h(w) \in E(P)$  since  $P_1 \wedge P_2 \subseteq P$  has all edges between the  $P_k$ .

The self-evident colour class property for  $h$  is inherited from the  $h_i$ , so it remains to show that  $h$  is a skew fibration. Let  $v \in V(G)$  and  $h(v)w \in E(P)$ . By symmetry, assume  $v \in V(G_1)$ . If  $h(v)w \in E(Q)$  we obtain the desired skew lifting since  $h_1$  is a skew fibration. Otherwise  $h(v)w \in E(P_1 \wedge P_2)$ . Since  $Q$  is empty or false, there is a vertex  $x$  in  $h_2^{-1}(P_2)$  (by soundness for  $P_2 \vee Q$ , if  $Q$  is non-empty), and  $vx \in E(G)$  (since fusion joined the  $h_i^{-1}(P_i)$ ). If  $h(x)w \notin E(P_2)$  we are done; otherwise since  $h_2$  is a skew fibration and  $h(x)w \in E(P_2)$  there exists  $xy \in E(G_2)$  with  $h(y)w \notin E(P_2)$ . Since  $vy \in E(G)$  (again by fusion), we have the desired skew lifting of  $h(v)w$  at  $v$ . (See figure below. Note:  $h(y) = w$  is possible.)



□

## Appendix: Proof of Lemma 1

A **01-form** is a proposition generated from 0 and 1 by  $\wedge$  and  $\vee$ . By induction on its number of  $\wedge$ 's and  $\vee$ 's, a 01-form  $\mu$  is true iff its graph  $G(\mu)$  contains a 1-clique; for  $\mu_1 \wedge \mu_2$  (resp.  $\mu_1 \vee \mu_2$ ) is true iff  $\mu_1$  is true *and* (resp. *or*)  $\mu_2$  is true, and for any graphs  $G_1$  and  $G_2$  a clique in  $G_1 \wedge G_2$  is a join of a clique in  $G_1$  and a clique in  $G_2$ , while a clique of  $G_1 \vee G_2$  is either a clique in  $G_1$  or a clique in  $G_2$ . Thus Lemma 1 holds for 01-forms.

An  **$\wedge\vee$ -form** is a proposition generated from atoms by  $\vee$  and  $\wedge$ . For any  $\wedge\vee$ -form  $\tau$  and valuation  $f$ , let  $\tau^f$  be the 01-form obtained from  $\tau$  by replacing every atom  $a$  by  $\hat{f}(a) \in \{0, 1\}$ . By a simple induction,  $\tau$  is true iff  $\tau^f$  is true for all valuations  $f$ , and  $G(\tau^f) = G(\tau)^f$ . Thus Lemma 1 holds for all  $\wedge\vee$ -forms.

Define the  $\wedge\vee$ -form  $\phi'$  of a proposition  $\phi$  by the obvious recursion:  $a' = a$  for all atoms  $a$ ,  $(\rho \diamond \theta)' = \rho' \diamond \theta'$  for  $\diamond \in \{\wedge, \vee\}$ ,  $(\neg(\rho \wedge \theta))' = (\neg\rho)' \vee (\neg\theta)'$ ,  $(\neg(\rho \vee \theta))' = (\neg\rho)' \wedge (\neg\theta)'$ ,  $(\neg\neg\theta)' = \theta'$ ,  $(\rho \Rightarrow \theta)' = (\neg\rho)' \vee \theta'$ . By a simple induction,  $\phi'$  is true iff  $\phi$  is true, and  $G(\phi') = G(\phi)$ . Thus Lemma 1 holds for all propositions. □

## References

- [BV02] BOLDI, V., & S. VIGNA. *Fibrations of graphs*. *Discrete Math.* 243 2002.
- [Bol78] BOLLABÁS, B. *Extremal Graph Theory*. Academic Press 1978.
- [Bol02] BOLLABÁS, B. *Modern Graph Theory*. Springer 2002.
- [BLS99] BRANDSTÄDT, A., V.B. LE & J.P. SPINRAD. *Graph Classes: A Survey*. *SIAM monographs on Discrete Mathematics and Applications* 1999.
- [CLS81] CORNEIL, D.G., H LERCHS & L.K. STEWART-BURLINGHAM. *Complement reducible graphs*. *Disc. Appl. Math.* 3 1981 163–174.
- [Fr1879] FREGE, G. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. 1879. Translation, *Begriffsschrift, a formal language, modelled upon that of arithmetic, for pure thought*, in [vHe67, 5–82].
- [Gen35] GENTZEN, G. *Untersuchungen über das logische Schließen I, II*. *Mathematische Zeitschrift* 39 1935 176–210 405–431. Translation in [Gen69, 68–131].
- [Gen69] GENTZEN, G. *The Collected Papers of Gerhard Gentzen*. North-Holland. Edited and introduced by M.E. Szabo, 1969.
- [Gir87] GIRARD, J.-Y. *Linear logic*. *Th. Comp. Sci.* 50 1–102 1987.
- [Gir91] GIRARD, J.-Y. *A new constructive logic: classical logic*. *Math. Struc. Comp. Sci.* 1 1991 255–296.
- [Gra66] GRAY, J.W. *Fibred and cofibred categories*. In *Proceedings of the Conference on Categorical Algebra: La Jolla 1965* 21–83. Springer-Verlag 1966.
- [Gro59] GROTHENDIECK, A. *Technique de descente et théorèmes d'existence en géométrie algébrique, I. Généralités. Descente par morphismes fidèlement plats*. *Seminaire Bourbaki* 190 1959–60.
- [vHe67] VAN HEIJENOORT (ed.). *From Frege to Gödel. A Source Book in Mathematical Logic 1879–1931*. Harvard Univ. Press 1967. Reprinted 1970.
- [Hil28] HILBERT, D. *Die Grundlagen der Mathematik, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität* 6 1928 65–85. Translation, *The foundations of mathematics*, in [vHe67, 465–479].
- [Joh87] JOHNSTONE, P.T. *Notes on Logic and Set Theory*. Cambridge University Press 1987.
- [KM71] KELLY, G.M. & S. MAC LANE. *Coherence in closed categories*. *Journal of Pure and Applied Algebra* 1 1971 97–140.
- [KW84] KETONEN, J. & R. WEYHRAUCH. *A Decidable Fragment of Predicate Calculus*. *Th. Comp. Sci.* 32 1984 297–307.
- [Kot59] KOTZIG, A. *On the theory of finite graphs with a linear factor*. *Mat. Fiz. Casopis* 9 & 10 1959.
- [deM68] DE MORGAN, A. *Review of J.M. Wilson's 'Elementary Geometry'*. *The Athenaeum* 2 1868 71–73.
- [Pei33] PEIRCE, C.S. *Collected Papers*. Ed. Hartshorne & Weiss, in Moore & Robin (Ed.) *Studies in the Philosophy of Charles S. Peirce*, U. Mass. Press 1964.
- [Ret03] RETORÉ, C. *Handsome proof-nets: perfect matchings and cographs*. *Th. Comp. Sci.* 294 2003 473–488.
- [Tai68] TAIT, W.W. *A nonconstructive proof of Gentzen's Hauptstutz for second-order predicate logic*. *Bul. Amer. Math. Soc.* 72 1968 980–988.
- [Whi78] WHITEHEAD, G.W. *Elements of Homotopy Theory*. Springer-Verlag 1978.