

# Proof Nets for Unit-free Multiplicative-Additive Linear Logic

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A cornerstone of the theory of proof nets for unit-free multiplicative linear logic (MLL) is the abstract representation of cut-free proofs modulo inessential commutations of rules. The only known extension to additives, based on monomial weights, fails to preserve this key feature: a host of cut-free monomial proof nets can correspond to the same cut-free proof. Thus the problem of finding a satisfactory notion of proof net for unit-free multiplicative-additive linear logic (MALL) has remained open since the inception of linear logic in 1986. We present a new definition of MALL proof net which remains faithful to the cornerstone of the MLL theory.

*Key Words and Phrases:* Linear logic, proof nets, additives, cut elimination

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## 1 Introduction

The beautiful theory of proof nets for unit-free multiplicative linear logic (MLL) appeared alongside the introduction of linear logic [Girard 1987]. A proof net is an abstract representation of a proof: the translation of cut-free proofs into proof nets identifies proofs modulo inessential commutations of rules. The identifications have since been verified as canonical from a semantic perspective, with numerous full completeness results for MLL, e.g. [Abramsky and Jagadeesan 1994; Hyland and Ong 1993; Loader 1994; Tan 1997; Blute and Scott 1996; Devarajan et al. 1999]. Furthermore, the identifications correspond to coherences of free star-autonomous categories [Blute et al. 1996].

The problem of finding a satisfactory extension of the theory of proof nets to unit-free multiplicative-additive linear logic (MALL) has remained open since the inception of linear logic [Girard 1987]. Progress towards a solution was made by Girard [1996] with a notion of MALL proof net based on *monomial weights*. Unfortunately, monomial proof nets failed to extend the MLL theory faithfully: a single cut-free proof may correspond to a host of monomial proof nets, and there is no natural translation of cut-free proofs into monomial proof nets. Indeed, to quote Girard, monomial proof nets are “far from being absolutely satisfactory” [1996]. We illustrate the problems in detail in Section 6.1.

In this paper we propose a new notion of MALL proof net (Section 4) which adheres faithfully to the original MLL theory: we provide a simple function from cut-free proofs to cut-free proof nets, yielding the sought-after abstract representations of cut-free proofs modulo inessential commutations of rules. We define a cut-free proof net on a sequent  $\Gamma$  as a set of linkings on  $\Gamma$  satisfying a geometric

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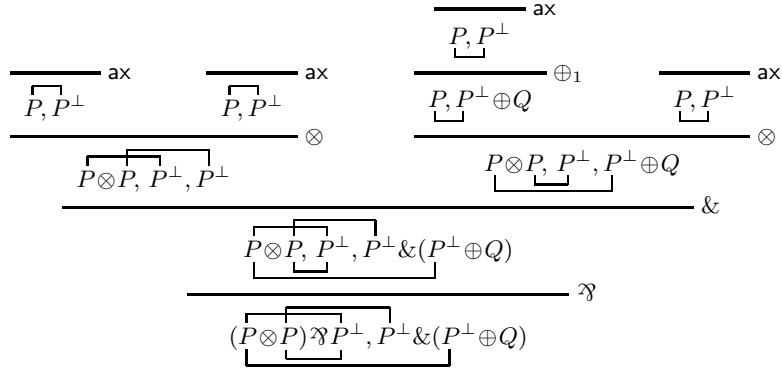


Fig. 1. Example of the inductive translation of a cut-free proof into one of our cut-free proof nets. The concluding proof net consists of two linkings, one drawn above the sequent, the other below. Each contains two axiom links. The proof nets further up in the derivation have one or two linkings, correspondingly above and/or below the sequent. Had we switched the order of the right-hand tensor rule and the plus rule, we would have obtained exactly the same pair of linkings; thus we identify cut-free proofs modulo a commutation of rules.

correctness criterion<sup>1</sup>, and prove that a set of linkings is the translation of a proof if and only if it is a proof net (Theorem 4.18, the cut-free *Sequentialisation Theorem*). Our definition of proof net is pleasingly succinct, taking only 12 lines (page 13). The reader can glean an impression of our approach by perusing Figure 1.

In Section 5 we extend our proof nets to include the cut rule, and present a notion of cut elimination. As with Girard’s monomial proof nets, in the presence of cuts multiple proof nets may correspond to the same proof. However, from a semantic point of view (viz. full completeness) the provision of abstract representations of MALL proofs modulo inessential rule commutations is crucial only in the cut-free setting. Moreover, our notion of cut elimination is simply defined, strongly normalising, and yields a category of cut-free proof nets which is unit-free star-autonomous, with products and coproducts. For an impressionistic overview see Figures 2 (cut), 3 (cut elimination), and 4 (composition). After extending to cut, our definition of proof net remains succinct: see the box on page 37.

A crisp notion of cut-free MALL proof net is fully motivated from a proof-theoretic perspective alone. However, just as MLL has blossomed through numerous fully complete semantics via cut-free MLL proof nets, we hope that the new definition of cut-free proof net presented here will lead to a similar blossoming of MALL. Since cut-free monomial proof nets for MALL are unsatisfactory for the reasons outlined earlier (detailed in Section 6.1), any MALL full completeness<sup>2</sup>

<sup>1</sup>Relaxing the criterion slightly yields a notion of proof net for MALL with mix (Section 4.8).

<sup>2</sup>On a more personal level, part of the first author’s motivation for finding a satisfactory notion of proof net came from a collaboration with Gordon Plotkin and Vaughan Pratt aimed at extending the Chu space full completeness result [Devarajan et al. 1999] to MALL: we were initially encumbered by the complexity of monomial proof nets. Ultimately we discovered that full completeness does not extend: the Gustave example (see Section 4.4.1) inhabits the model.

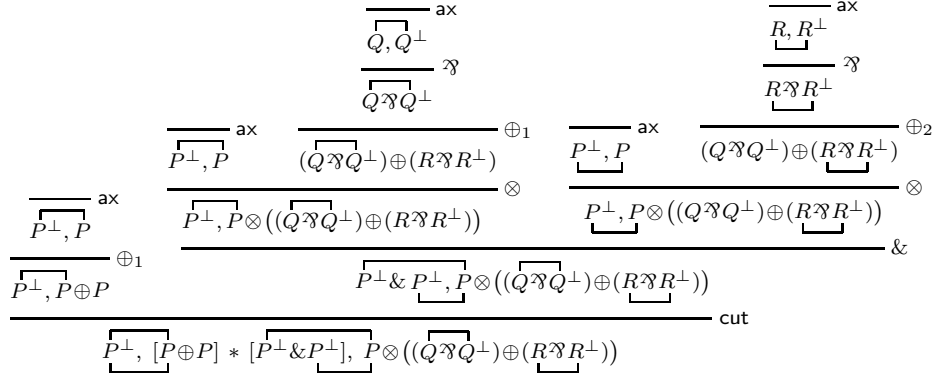


Fig. 2. Example of the translation of a proof with a cut into one of our proof nets. The concluding proof net is what we call a *cut sequent*: a MALL sequent (the formulas  $P^\perp$  and  $P \otimes ((Q \wp Q^\perp) \oplus (R \wp R^\perp))$ ) together with a *cut*, an unordered pair  $[P \oplus P] * [P^\perp \& P^\perp] \equiv \{P \oplus P, P^\perp \& P^\perp\}$  of complementary formulas. The concluding proof net comprises two linkings of three axiom links each, one linking drawn above the cut sequent, the other below. When transitioning through the cut rule, the axiom link on  $P^\perp, P \oplus P$  on the left becomes duplicated, so that a copy appears in each of the two final linkings; in general, when  $m$  linkings pass through the left of a cut rule, and  $n$  through the right, we construct all  $m \times n$  disjoint unions of the linkings on the conclusion. (Here  $m = 1$  and  $n = 2$ .)

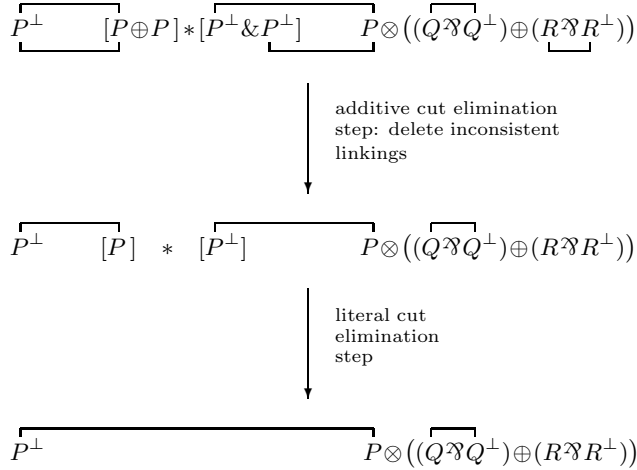


Fig. 3. Example of cut elimination, normalising in two steps. The top proof net, two linkings, was derived in Figure 2. The first elimination step, aside from eliminating the  $\oplus$  and  $\&$  to leave a literal cut  $[P] * [P^\perp]$ , deletes the underhanging linking: our rule for additive elimination is simply *delete inconsistent linkings*, where a linking is inconsistent if it chooses opposite arguments for the cut  $\oplus$  and  $\&$ . (Here the underhanging linking chooses  $\oplus$ -left and  $\&$ -right, and is therefore inconsistent, hence deleted in the cut elimination step.) Note that the end result is a cut-free proof net: it is the translation of the left branch of the  $\&$ -rule in Figure 2.

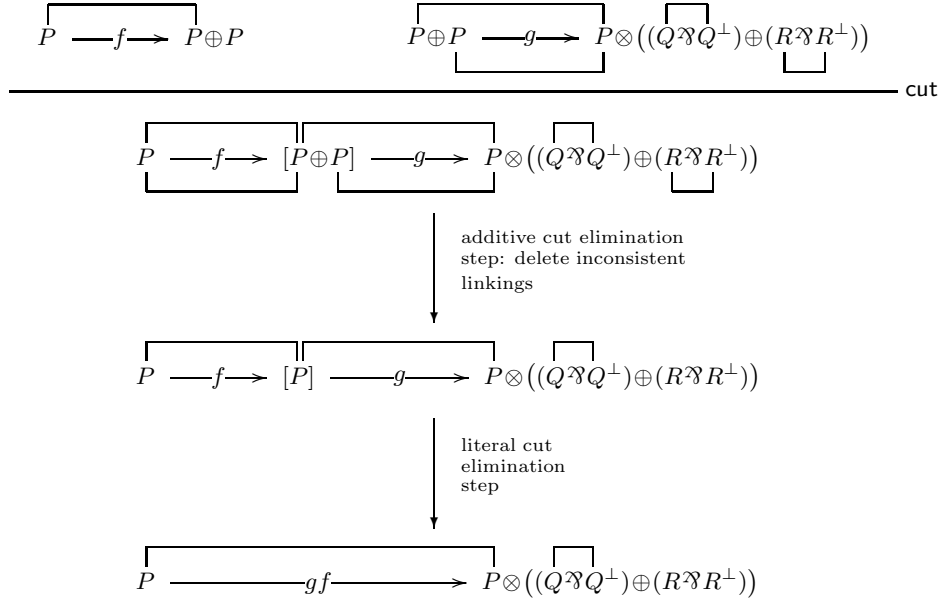


Fig. 4. Example of composition  $f, g \mapsto gf$  in our category of cut-free proof nets. Objects are MALL formulas, and a morphism  $h : A \rightarrow B$  is a cut-free proof net on the sequent  $A^\perp, B$ . The morphisms  $f$  (top-left) and  $g$  (top-right) are the left- and right hypotheses of the cut rule in Figure 2. The first step of composition is to cut the two morphisms; in doing so we are emulating precisely the cut rule of Figure 2. Having negated on the left of the arrow  $\rightarrow$ , the two cut formulas are no longer dual but identical; thus we are afforded the additional economy of superimposing them. The two ensuing computation steps are exactly those of Figure 3, modulo this superposition.

result based on them (e.g. [Abramsky and Melliès 1999], and the work in progress of Blute, Hamano and Scott on hypercoherence spaces) suffers accordingly, particularly with regard to faithfulness. Our new definition of MALL proof net should yield cleaner and more accessible MALL full completeness results.

### Liberation from monomials

The technical starting point for our definition of proof net was Girard’s definition of monomial proof net [1996], and indeed we employ variants of Girard’s ingenious notions of slice and jump. One of our contributions relative to [Girard 1996] is that we do not partition weights into monomials. In [1996] Girard remarks that he had been trying to circumvent this technical limitation since 1990, and lists three specific problems that must be solved in any attempt to eliminate it, i.e., to define what he calls “more liberal proof-nets”, such as ours:

*Weights must be monomials. However, weights of the form  $p \cup q$  will naturally occur if we want to allow more superimpositions. The present state of affairs is as follows:*

- (1) *in spite of years of efforts, I never succeeded in finding the right correctness criterion for these more liberal proof-nets;*

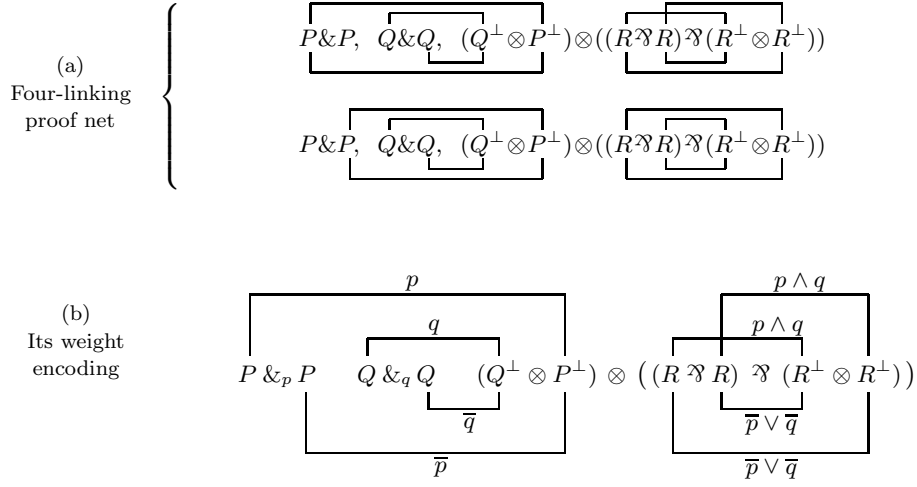


Fig. 5. (a) Four-linking example of one of our proof nets. Rather than draw all four linkings on one sequent, we have drawn two linkings (one above, one below) on each of two copies of the sequent. In Section 4.6 we show how our proof nets can be encoded as collections of axiom links labelled with predicates (‘weights’, c.f. [Girard 1996]). Subfigure (b) shows the weight encoding of (a). To distinguish the  $\&$ s, we have subscripted them. Every  $\&$ -assignment (assignment of ‘left’ or ‘right’ to each of  $\&_p$  and  $\&_q$ ) determines a linking by restricting to axiom links whose predicates hold, where we read the predicate  $p$  (resp.  $\bar{p}$ ) as “ $\&_p$  is assigned ‘left’ (resp. ‘right’)” (and  $q$  analogously),  $\wedge$  is “and” and  $\vee$  is “or”. We invite the reader to verify that taking each of the four possible  $\&$ -assignments in turn produces the four original linkings.

- (2) *general boolean coefficients might be delicate to represent (on the other hand, the case we consider has a natural presentation in terms of coherent spaces);*
- (3) *normalization in the full case might be messy.*

[Girard 1996, Appendix A.1.5]

An important stepping stone towards finding the right criterion to address (1) was to first settle the open problem of whether Girard’s criterion becomes insufficient without partitioning weights into monomials. We show that this is indeed the case: in Section 6.2 we present a non-monomial proof structure that does not correspond to any proof (it is not sequentialisable), yet satisfies Girard’s criterion. We address (2) by leaving weights implicit, defining a proof net on a sequent  $\Gamma$  as a set of axiom linkings on an extension  $\Gamma^+$  of  $\Gamma$  with complementary pairs of cut formula occurrences. (See Figure 5 for an example of extracting weights from a proof net.) Issue (3) is addressed by the fact that our definition of cut elimination is sufficiently simple that confluence and strong normalisation are immediate.

The proof that our correctness criterion captures proof translations (the *Sequentialisation Theorem*) hinges on an ordering on vertices called *domination*<sup>3</sup>. By introducing domination we avoid the use of empires [Girard 1987; 1996], thereby

<sup>3</sup>Unrelated to domination in flowgraphs.

sidestepping the problem of stability of maximal empires [Girard 1996, Section 1.5.3]—the main technical problem that led Girard to resort to monomials.

### Surjection collapsing Girard’s monomial proof nets to ours

Analogous with Girard [1996], and as in the standard approach for MLL, we define a *proof structure* as a stepping stone towards the definition of proof net. In Section 6.3 we define a surjection collapsing Girard’s proof structures to ours. There are more Girard proof structures than ours because of the redundancy issues related to monomials (see Section 6.1). This surjection respects sequentialisability, and hence collapses Girard’s proof nets to our own.

### Acknowledgements

Many thanks Vaughan Pratt, for invaluable feedback and support during the long development of this work. To Paul-André Melliès, for suggesting to the first author that the search for a satisfactory notion of MALL proof net was an interesting and potentially fruitful research topic. Thanks to Masahiro Hamano for technical comments.

## 2 MALL

By MALL we mean multiplicative-additive linear logic without units [Girard 1987]. Formulas are built from literals (propositional variables  $P, Q, \dots$  and their negations  $P^\perp, Q^\perp, \dots$ ) by the binary connectives *tensor*  $\otimes$ , *par*  $\wp$ , *with*  $\&$  and *plus*  $\oplus$ . Negation  $(-)^\perp$  extends to arbitrary formulas by de Morgan duality:  $(A \otimes B)^\perp = A^\perp \wp B^\perp$ ,  $(A \wp B)^\perp = A^\perp \otimes B^\perp$ ,  $(A \oplus B)^\perp = A^\perp \& B^\perp$ , and  $(A \& B)^\perp = A^\perp \oplus B^\perp$ . For technical convenience we take sequents to be unordered, i.e., a sequent is a non-empty set  $A_1, \dots, A_n$  of formula occurrences. Turnstiles are redundant since sequents are right-sided. Sequents are proved using the following rules:

$$\frac{}{P, P^\perp} \text{ax} \quad \frac{\Gamma, A \quad A^\perp, \Delta}{\Gamma, \Delta} \text{cut} \quad \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta} \otimes \quad \frac{\Gamma, A, B}{\Gamma, A \wp B} \wp$$

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B} \& \quad \frac{\Gamma, A}{\Gamma, A \oplus B} \oplus_1 \quad \frac{\Gamma, B}{\Gamma, A \oplus B} \oplus_2$$

Here, and throughout this document,  $P, Q, \dots$  range over propositional variables,  $A, B, \dots$  over formulas, and  $\Gamma, \Delta, \dots$  over sets of formula occurrences. In eliminating the permutation rule, we assume an implicit tracking of formula occurrences above the line of a rule to formula occurrences below the line. Without loss of generality (see [Girard 1987]) we restrict the axiom rule to literals.

## 3 Background: Cut-free MLL proof nets

For didactic purposes, we review the definition of cut-free MLL proof net [Girard 1987; Danos and Regnier 1989]. MLL is the subsystem of MALL obtained by removing the additive connectives,  $\oplus$  and  $\&$ . Our style of presentation anticipates the subsequent definition of MALL proof net.

An *axiom link* on an MLL sequent  $\Gamma$  is a pair of complementary literal occurrences of  $\Gamma$ . A *linking* on  $\Gamma$  is a partitioning of the set of literal occurrences of

$$\frac{\frac{\overline{P^\perp, P} \text{ ax} \quad \overline{P^\perp, P} \text{ ax}}{\overline{P^\perp, P \otimes P^\perp, P} \otimes} \otimes}{\overline{P^\perp, (P \otimes P^\perp) \wp P} \wp} \wp$$

Fig. 6. An example of the translation of a cut-free MLL proof into a linking, i.e., into a cut-free MLL proof structure.

$\Gamma$  into axiom links, i.e., a set of disjoint axiom links whose union contains every literal occurrence of  $\Gamma$ . A linking on an MLL sequent as also called a **cut-free MLL proof structure**.

*Example 3.1.* Two linkings are possible on the sequent  $P^\perp, (P \otimes P^\perp) \wp P$ :

$$\overline{P^\perp, (P \otimes P^\perp) \wp P} \quad \overline{P^\perp, (P \otimes P^\perp) \wp P}$$

### 3.1 A function from proofs to linkings

Let  $\Pi$  be a cut-free MLL proof of a sequent  $\Gamma$ . By downwards tracking of literal occurrences, the axiom rules of  $\Pi$  determine a linking  $\lambda_\Pi$  on  $\Gamma$ . Alternatively, one can define the same function from proofs to linkings by induction. The base case of an axiom rule  $\overline{P, P^\perp}$  defines the linking  $\{\{P, P^\perp\}\}$  comprising a single axiom link between  $P$  and  $P^\perp$ . Writing  $\lambda \triangleright \Gamma$  for the judgement “ $\lambda$  is a linking on  $\Gamma$ ”, the inductive translation is as follows:

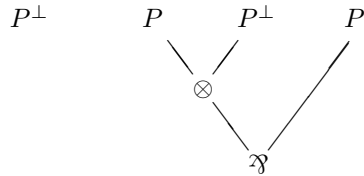
$$\frac{\lambda \triangleright \Gamma, A \quad \lambda' \triangleright B, \Delta}{\lambda \cup \lambda' : \Gamma, A \otimes B, \Delta} \otimes \quad \frac{\lambda \triangleright \Gamma, A, B}{\lambda \triangleright \Gamma, A \wp B} \wp$$

Here we use the implicit tracking of literal occurrences above the line of a rule to literal occurrences below the line. Figure 6 shows an example. Any linking  $\lambda$  which is the image of a proof is said to be **sequentialisable**, and any such proof is called a **sequentialisation** of  $\lambda$ . In general a linking has many distinct sequentialisations, corresponding to the fact that MLL proof nets are canonical abstract representations of MLL proofs modulo inessential rule commutations.

### 3.2 Geometric characterisation of sequentialisability

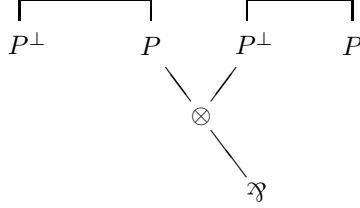
Henceforth we view an MLL sequent  $\Gamma$  as a graph: a disjoint union of parse trees, with literals above.

*Example 3.2.* We identify the sequent  $P^\perp, (P \otimes P^\perp) \wp P$  with the graph



Given a linking  $\lambda$  on  $\Gamma$ , define the **graph**  $\mathcal{G}_\lambda$  of  $\lambda$  as the result of adding each axiom link  $a$  of  $\lambda$  as a vertex above  $\Gamma$ , with edges from  $a$  down to its two literal occurrences. A  $\mathfrak{A}$ -**switching** of a linking  $\lambda$  on  $\Gamma$  is any subgraph of  $\mathcal{G}_\lambda$  obtained by deleting one of the two argument edges of each  $\mathfrak{A}$ -vertex.

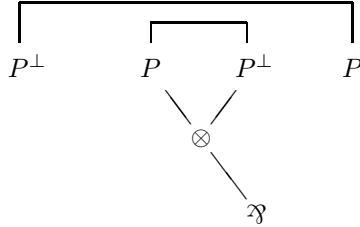
*Example 3.3.* One of two possible  $\mathfrak{A}$ -switchings of the second linking of Example 3.1:



In drawing an axiom link  $\sqcap$ , we view the horizontal section as a vertex, and the two verticals as edges.

*Definition 3.4.* A linking on an MLL sequent (i.e., a cut-free MLL proof structure) is a **cut-free MLL proof net** if each of its  $\mathfrak{A}$ -switchings is a tree (acyclic and connected).

*Example 3.5.* The first linking of Example 3.1 fails to be a cut-free MLL proof net. This  $\mathfrak{A}$ -switching is not a tree:



The second linking of Example 3.1 is a proof net: both  $\mathfrak{A}$ -switchings (one of which was depicted in Example 3.3) are trees.

**THEOREM 3.6 CUT-FREE MLL SEQUENTIALISATION.** *A linking on an MLL sequent is sequentialisable if and only if it is a proof net.*

This was proved by Girard, for a different geometric criterion, based on *long trips* [1987]. Danos and Regnier [1989] simplified the criterion to the elegant one above, showing it to be equivalent to Girard's.

*$\mathfrak{A}$ -Switching cycles.* The requirement that every  $\mathfrak{A}$ -switching of  $\mathcal{G}_\lambda$  be acyclic is clearly equivalent to demanding that  $\mathcal{G}_\lambda$  contain no  **$\mathfrak{A}$ -switching cycle**, that is, a cycle which contains at most one argument edge of each  $\mathfrak{A}$ . Here by cycle we mean any subgraph with vertex set  $\{x_1, \dots, x_n\}$  for  $n \geq 3$ , with all  $x_i$  distinct, and an edge  $x_i - x_{i+1}$  for each  $i \pmod n$ . We present various alternative formulations of cut-free MLL proof net as part of Section 4.5. By modularity of our forthcoming definition of MALL proof net over MLL, these yield corresponding alternative definitions of MALL proof net.

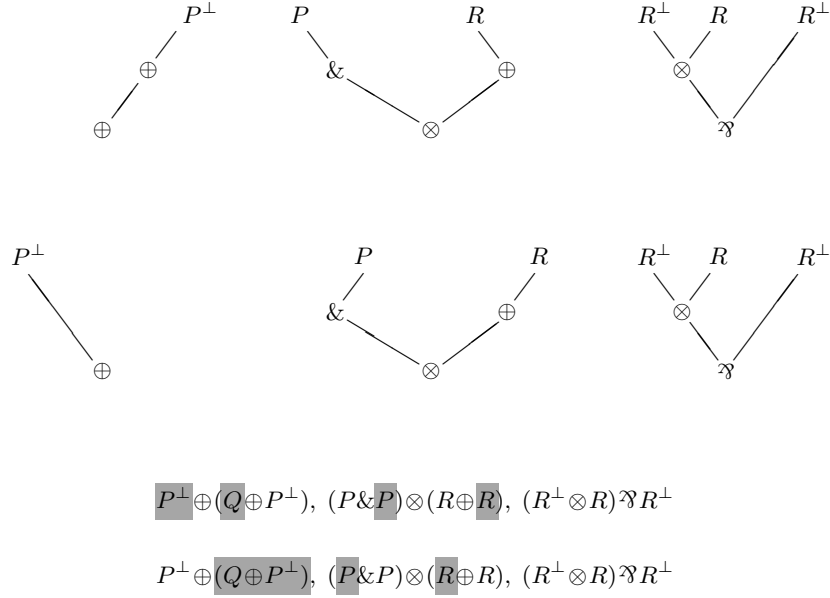


Fig. 7. Top: two additive resolutions of  $P^\perp \oplus (Q \oplus P^\perp)$ ,  $(P \& P) \otimes (R \oplus R)$ ,  $(R^\perp \otimes R) \wp R^\perp$ . Equivalent compact ‘in-line’ representations are shown underneath.

#### 4 Cut-free MALL proof nets

We begin by defining the notion of a *linking* on a MALL sequent, and define a simple function from cut-free MALL proofs to sets of linkings. With such a function in hand, it is natural to ask about its image and kernel:

- (I) *Image*. Can one characterise the sound sets of linkings, i.e., those that come from proofs?
- (K) *Kernel*. Does the kernel exactly characterise proof equivalence modulo inessential rule commutation?

We answer both in the affirmative. In Section 4.2 we present a geometric characterisation of those sets of linkings that arise as the translations of cut-free MALL proofs, and call them *proof nets*. In a sibling paper we show that any two cut-free MALL proofs differ by inessential rule commutations if and only if they map to the same proof net; see Section 4.7. Thus:

- *Our cut-free MALL proof nets provide canonical abstract representations of cut-free MALL proofs modulo inessential rule commutations.*

##### 4.1 A function from cut-free MALL proofs to sets of linkings

Define an *additive resolution* of a MALL sequent  $\Gamma$  to be any result of deleting one argument subtree of every additive connective ( $\&$  or  $\oplus$ ) of  $\Gamma$ . See Figure 7 for examples. An *axiom link* on  $\Gamma$  is a pair of complementary literal occurrences in  $\Gamma$ . A *linking* on  $\Gamma$  is a partitioning of the set of literal occurrences of an additive

resolution  $\Gamma \upharpoonright \lambda$  of  $\Gamma$  into axiom links.

*Example 4.1.* Let  $\Gamma$  be the sequent

$$P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp.$$

We denote an axiom link as an edge between its two complementary literal occurrences. The following set  $\lambda$  of three disjoint axiom links is an example of a linking on  $\Gamma$ :

$$P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp$$

For  $\lambda$  to be a linking, as opposed to merely an ad hoc collection of disjoint axiom links, it must be a partitioning of the literals of some additive resolution of  $\Gamma$ . This is indeed the case: the literals of (the axiom links of)  $\lambda$  are exactly those of the first of the two additive resolutions depicted in Figure 7:

$$P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp$$

Recall that we write  $\Gamma \upharpoonright \lambda$  for the additive resolution underlying a linking  $\lambda$ .

*Example 4.2.* Note that multiple linkings can have the same additive resolution. For example, the following linking  $\lambda'$ :

$$P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp$$

has the same additive resolution as the linking  $\lambda$  of Example 4.1, i.e.,  $\Gamma \upharpoonright \lambda = \Gamma \upharpoonright \lambda'$ :

$$P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp$$

Note that  $\lambda$  and  $\lambda'$  are the only two linkings possible on this additive resolution.

*Example 4.3.* This pair of disjoint axiom links fails to be a linking:

$$P \& Q, Q^\perp \otimes P^\perp.$$

It cannot be a linking because it contains a literal on each side of the  $\&$  connective.

Numerous other examples of linkings can be seen in Figures 1 (page 2) and 2 (page 3). One can easily verify that each of them takes the literals of an additive resolution. See also Figure 5 (page 5).

**4.1.1 Every linking induces an MLL proof structure.** Every additive connective ( $\oplus/\&$ ) remaining in an additive resolution is, by construction, unary (i.e., has one remaining argument). For example, one can observe this in the parse trees in Figure 7. Thus any additive resolution  $R$  of a MALL sequent  $\Gamma$  induces an MLL sequent  $R^-$  by collapsing its additive connectives. A linking  $\lambda$  on  $\Gamma$ , viewed as being on  $(\Gamma \upharpoonright \lambda)^-$ , is a cut-free MLL proof structure (as defined in Section 3), which we call the **MLL proof structure induced by  $\lambda$** .

*Example 4.4.* The MLL proof structure induced by the linking  $\lambda$  of Example 4.1 is:

$$P^\perp, P \otimes R, (R^\perp \otimes R) \wp R^\perp$$

$$\begin{array}{c}
 \frac{}{\{\{P, P^\perp\}\} \triangleright P, P^\perp} \text{ax} \quad \frac{\theta \triangleright \Gamma, A, B}{\theta \triangleright \Gamma, A \wp B} \wp \quad \frac{\theta \triangleright \Gamma, A \quad \theta' \triangleright \Gamma, B}{\theta \cup \theta' \triangleright \Gamma, A \& B} \& \\
 \\
 \frac{\theta \triangleright \Gamma, A \quad \theta' \triangleright B, \Delta}{\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\} \triangleright \Gamma, A \otimes B, \Delta} \otimes \quad \frac{\theta \triangleright \Gamma, A}{\theta \triangleright \Gamma, A \oplus B} \oplus_1 \quad \frac{\theta \triangleright \Gamma, B}{\theta \triangleright \Gamma, A \oplus B} \oplus_2
 \end{array}$$

Table I. Inductive definition of the function from cut-free MALL proofs to sets of linkings. Here  $\theta \triangleright \Gamma$  is the judgement “ $\theta$  is a set of linkings on  $\Gamma$ ”. We use the implicit tracking of literal occurrences downwards through rules. The base case **ax** is a singleton set of linkings whose only linking comprises a single axiom link, between  $P$  and  $P^\perp$ .

4.1.2 *The function from cut-free MALL proofs to sets of linkings.* Every cut-free MALL proof  $\Pi$  of  $\Gamma$  defines a set  $\theta_\Pi$  of linkings on  $\Gamma$  as follows. Define a **&-resolution**  $S$  of  $\Pi$  to be any result of deleting one branch above each  $\&$ -rule of  $\Pi$ . By downwards tracking of literal occurrences, the axiom rules of  $S$  determine a linking  $\lambda_S$  on  $\Gamma$ . Define  $\theta_\Pi = \{\lambda_S : S \text{ is a } \&\text{-resolution of } \Pi\}$ . See Figure 8 (page 12) for an example. Alternatively, Table I defines the same function by induction; see Figure 1 (page 2) for an example.

By structural induction, each linking is well-defined (i.e., takes the literals of an additive resolution); thus the translation is well-defined. The fact that both the above procedures yield the same set of linkings follows from a simple structural induction on proofs. A set of linkings  $\Lambda$  is **sequentialisable** if it is the translation of a proof; any such proof is called a **sequentialisation** of  $\Lambda$ .

## 4.2 Geometric characterisation of sequentialisability

In this section we define a *proof net* as a set of linkings satisfying three conditions. These conditions characterise the image of the function from cut-free proofs to sets of linkings defined in Section 4.1.2: in Theorem 4.18 (the cut-free *Sequentialisation Theorem*) we prove that a set of linkings is the translation of a proof if and only if it is a proof net. Our definition of proof net is pleasingly succinct, and is given in the box on page 13. In the remainder of this section we clarify the definition and work through examples. Analogous with Girard [1996], and as in the standard approach for MLL, we define a *proof structure* as a stepping stone towards the definition of proof net.

4.2.1 *Resolution condition.* Similar to the definition of additive resolution in Section 4.1, define a **&-resolution**  $\Gamma^*$  of a sequent  $\Gamma$  to be any result of deleting one argument subtree of every  $\&$  of  $\Gamma$ .

*Example 4.5.* The two possible  $\&$ -resolutions of the sequent

$$P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp$$

featured in Examples 4.1 and 4.2 are:

$$\begin{array}{l}
 \Gamma_1^* : \quad P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp \\
 \Gamma_2^* : \quad P^\perp \oplus (Q \oplus P^\perp), (P \& P) \otimes (R \oplus R), (R^\perp \otimes R) \wp R^\perp
 \end{array}$$

$$(II) \quad \frac{\frac{\frac{\overline{P^\perp, P}^{\text{ax}}}{P^\perp \oplus Q^\perp, P} \oplus_1}{(P^\perp \oplus Q^\perp) \oplus R^\perp, P} \oplus_1 \quad \frac{\frac{\frac{\overline{Q^\perp, Q}^{\text{ax}}}{P^\perp \oplus Q^\perp, Q} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, Q} \oplus_1 \quad \frac{\frac{\overline{R^\perp, R}^{\text{ax}}}{(P^\perp \oplus Q^\perp) \oplus R^\perp, R} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, Q \& R} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)} \&$$

$$(S_1) \quad \frac{\frac{\frac{\overline{P^\perp, P}^{\text{ax}}}{P^\perp \oplus Q^\perp, P} \oplus_1}{(P^\perp \oplus Q^\perp) \oplus R^\perp, P} \oplus_1}{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)} \&$$

$$(S_2) \quad \frac{\frac{\frac{\frac{\overline{Q^\perp, Q}^{\text{ax}}}{P^\perp \oplus Q^\perp, Q} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, Q} \oplus_1}{(P^\perp \oplus Q^\perp) \oplus R^\perp, Q \& R} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)} \&$$

$$(S_3) \quad \frac{\frac{\frac{\overline{R^\perp, R}^{\text{ax}}}{(P^\perp \oplus Q^\perp) \oplus R^\perp, R} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, Q \& R} \oplus_2}{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)} \&$$

$$\lambda_1 : \quad \overbrace{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)}$$

$$\lambda_2 : \quad \overbrace{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)}$$

$$\lambda_3 : \quad \overbrace{(P^\perp \oplus Q^\perp) \oplus R^\perp, P \& (Q \& R)}$$

Fig. 8. Example of the mapping of a cut-free MALL proof into a set of linkings. At the top is a proof II, followed by its three possible  $\&$ -resolutions  $S_1$ ,  $S_2$ ,  $S_3$ , followed by the corresponding linkings  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ . Each linking comprises a single axiom link. Categorically, this example expresses associativity  $(P \times Q) \times R \rightarrow P \times (Q \times R)$ . Note the compactness of the representation as a set of linkings relative to the size of the proof.

— **Definition: cut-free MALL proof net on a sequent  $\Gamma$**  —

**Additive resolution:** deletion of one argument subtree of each  $\oplus/\&$ ; **&-resolution** analogous.

**Linking**  $\lambda$  on  $\Gamma$ : partitioning of the literal occurrences of an additive resolution  $\Gamma \upharpoonright \lambda$  of  $\Gamma$  into complementary pairs (*axiom links*).

A set  $\Lambda$  of linkings on  $\Gamma$  **toggles** a  $\&$   $w$  if both arguments of  $w$  are in  $\Gamma \upharpoonright \Lambda \equiv \bigcup_{\lambda \in \Lambda} \Gamma \upharpoonright \lambda$ .

**Graph**  $\mathcal{G}_\Lambda$ :  $\Gamma \upharpoonright \Lambda$  + each axiom link  $a = \{l, l'\}$  of  $\Lambda$ , with edges  $l-a-l'$  and a **jump** edge  $a-w$  whenever  $a \in \lambda \setminus \lambda'$  and  $w$  is the unique  $\&$  toggled by  $\{\lambda, \lambda'\} \subseteq \Lambda$ .

**$\mathfrak{A}$ -switching** of  $\lambda$ : any subgraph of  $\mathcal{G}_{\{\lambda\}}$  obtained by deleting one argument edge of each  $\mathfrak{A}$ .

**Switching cycle:** cycle with  $\leq 1$  **switch edge** (= jump or argument edge) of each  $\mathfrak{A}/\&$ .

A set  $\theta$  of linkings on  $\Gamma$  is a **proof net** if it satisfies:

RESOLUTION: Exactly one linking of  $\theta$  is on any given  $\&$ -resolution of  $\Gamma$ .

MLL: Every  $\mathfrak{A}$ -switching of every  $\lambda \in \theta$  is a tree (i.e., each  $\lambda \in \theta$  induces an MLL proof net).<sup>4</sup>

TOGLING: Every set  $\Lambda$  of  $\geq 2$  linkings of  $\theta$  toggles a  $\&$  that is in no switching cycle in  $\mathcal{G}_\Lambda$ .<sup>5</sup>

A linking  $\lambda$  on  $\Gamma$  is **on** a  $\&$ -resolution  $\Gamma^*$  of  $\Gamma$  if every literal occurrence of  $\lambda$  is in  $\Gamma^*$ . A set of linkings  $\theta$  on  $\Gamma$  is a **cut-free proof structure** if it satisfies

(P1) RESOLUTION. For any  $\&$ -resolution  $\Gamma^*$  of  $\Gamma$ , exactly one linking of  $\theta$  is on  $\Gamma^*$ .

*Example 4.6.* Here is a two-linking proof structure  $\theta = \{\lambda_1, \lambda_2\}$  on the sequent of Example 4.5, with  $\lambda_1$  drawn above the sequent and  $\lambda_2$  drawn below:

$$\begin{array}{l} \lambda_1 : \\ \lambda_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\lambda_1}, \underbrace{(P \& P) \otimes (R \oplus R)}_{\lambda_2}, \underbrace{(R^\perp \otimes R) \wp R^\perp}_{\lambda_1}$$

To verify RESOLUTION, we must check that exactly one of the linkings fits on each of the two  $\&$ -resolutions of  $\Gamma$ , depicted in Example 4.5. Taking the  $\&$ -resolution  $\Gamma_1^*$ ,

$$\begin{array}{l} \lambda_1 : \\ \lambda_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\lambda_1}, \underbrace{(P \& P) \otimes (R \oplus R)}_{\lambda_2}, \underbrace{(R^\perp \otimes R) \wp R^\perp}_{\lambda_1}$$

we see that  $\lambda_1$  is on  $\Gamma_1^*$  (all six of its literals are in  $\Gamma_1^*$ ), but  $\lambda_2$  is not (its  $P$  literal is not in  $\Gamma_1^*$ ). Similarly, taking the second  $\&$ -resolution  $\Gamma_2^*$ ,

$$\begin{array}{l} \lambda_1 : \\ \lambda_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\lambda_1}, \underbrace{(P \& P) \otimes (R \oplus R)}_{\lambda_2}, \underbrace{(R^\perp \otimes R) \wp R^\perp}_{\lambda_1}$$

we see that  $\lambda_2$  is on  $\Gamma_2^*$  (all six of its literals are in  $\Gamma_2^*$ ), but  $\lambda_1$  is not (its  $P$  literal is not in  $\Gamma_2^*$ ). Hence RESOLUTION is satisfied.

<sup>4</sup>Tree = acyclic + connected. By dropping the connectedness requirement in the MLL condition one obtains a cut-free proof net for MALL augmented with the mix rule. See Section 4.8.

<sup>5</sup>In fact, it suffices to verify TOGLING merely for *saturated* sets of linkings  $\Lambda$ , namely, such that any strictly larger subset of  $\theta$  toggles more  $\&$ s than  $\Lambda$ . There is exactly one saturated set of linkings in  $\theta$  for each *partial &-resolution* of  $\Gamma$ , the latter being any result of deleting at most one argument subtree of each  $\&$  of  $\Gamma$ .

*Example 4.7.* The pair of linkings

$$\overbrace{P^\perp, P \otimes Q^\perp, Q \oplus Q}$$

fails RESOLUTION: any  $\&$ -free sequent is its own unique  $\&$ -resolution, and therefore RESOLUTION will hold if and only if there is a single linking.

*Example 4.8.* The singleton set of linkings

$$\overbrace{P \oplus (Q \& R), P^\perp}$$

(comprising just one link) satisfies RESOLUTION. Note that the sequent has two distinct  $\&$ -resolutions, but there is only one linking.

*Remark 4.9.* In the restricted case of an MLL sequent  $\Gamma$ , since there are no  $\&$ s, a set of linkings satisfies the resolution condition iff it comprises a single MLL linking on  $\Gamma$  (in the sense of Section 3). Thus our cut-free MALL proof structures generalise cut-free MLL proof structures.

*Example 4.10.* We invite the reader to verify the resolution condition for the sets of linkings in Figures 1, 2 and 5 (pages 2, 3, and 5).

In Section 4.3.1 we provide some intuition for the resolution condition. In Section 4.6 we show how any proof structure can be encoded using *weights* (c.f. [Girard 1996]), as illustrated by the example in Figure 5. In Section 6.3 we define a surjection collapsing Girard’s proof structures to our own, and in Section 6.4 we investigate the relationship between the two notions of proof structure in considerable detail.

**4.2.2 MLL condition.** The second requirement for a set of linkings  $\theta$  to be a proof net is “pointwise MLL correctness”:

(P2) MLL. *Every linking of  $\theta$  induces an MLL proof net.*

In other words, for each linking  $\lambda \in \theta$ , the MLL proof structure induced by  $\lambda$  (as defined in Section 4.1.1), is an MLL proof net (as defined in Section 3).

*Example 4.11.* See Figure 9, subfigures (a)–(d).

*Example 4.12.* The proof structure  $\theta = \{\lambda_1, \lambda_2\}$  in Example 4.6 (page 13) satisfies the MLL condition. Both  $\lambda_1$  and  $\lambda_2$  induce the same MLL proof net, whose graph is Figure 9(c).

Naturally one need not collapse to an MLL proof structure to check the MLL condition for a linking  $\lambda$ : one can simply leave the unary  $\oplus/\&$ s of the additive resolution in place, and verify that every  $\mathfrak{Y}$ -switching is a tree. For self-containedness of our definition of cut-free MALL proof net, without reference to MLL proof nets, we describe this formally.

Henceforth view a sequent  $\Gamma$  as a graph: a disjoint union of parse trees, with literals above. Construct the **graph  $\mathcal{G}_\lambda$  of  $\lambda$**  from the additive resolution  $\Gamma \upharpoonright \lambda$  (a subgraph of  $\Gamma$ ) by adding each axiom link  $a$  of  $\lambda$  as a vertex above  $\Gamma \upharpoonright \lambda$ , with edges from  $a$  down to its two literal occurrences. For example, Figure 9(e) shows

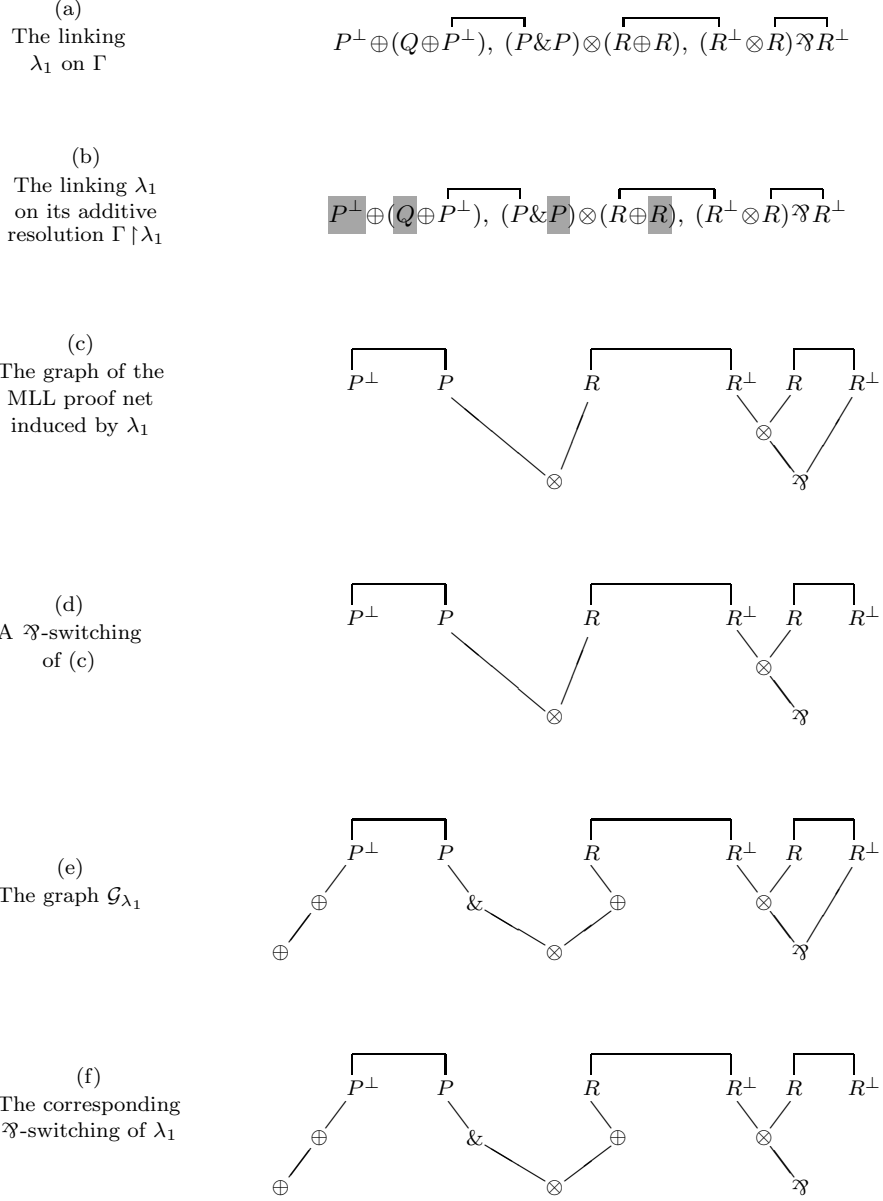


Fig. 9. Subfigure (a) shows a linking  $\lambda_1$  on a MALL sequent  $\Gamma$ , which is shown on its additive resolution in (b). Subfigure (c) is the MLL proof structure induced by  $\lambda_1$ , which is an MLL proof net since each of its  $\wp$ -switchings is a tree. Subfigure (d) shows one of its two  $\wp$ -switchings. Subfigure (e) is the graph  $\mathcal{G}_{\lambda_1}$  of  $\lambda_1$  on  $\Gamma$ , and (f) is the  $\wp$ -switching of  $\lambda_1$  in  $\mathcal{G}_{\lambda_1}$  corresponding to the  $\wp$ -switching (d) of the induced MLL proof net (c). In graphs (c)–(f), when drawing an axiom link  $\square$ , we view the horizontal section as a vertex, and the two verticals as edges.

the graph  $\mathcal{G}_{\lambda_1}$  of the linking  $\lambda_1$  of Figure 9(a). A  $\mathfrak{A}$ -*switching* of a linking  $\lambda$  on  $\Gamma$  is any subgraph of  $\mathcal{G}_\lambda$  obtained by deleting one of the two argument edges of each  $\mathfrak{A}$ -vertex. See Figure 9(f) for an example. Clearly, the induced MLL proof structure of a linking  $\lambda$  is an MLL proof net if and only if every  $\mathfrak{A}$ -switching of  $\lambda$  (in  $\mathcal{G}_\lambda$ ) is a tree. Thus we can reformulate the MLL condition on a set of linkings  $\theta$ , without reference to MLL proof nets, as follows:

(P2) MLL. *Every  $\mathfrak{A}$ -switching of every linking of  $\theta$  is a tree (acyclic and connected).*

By relaxing the connectedness requirement one obtains a cut-free proof net for MALL augmented with the mix rule. See Section 4.8.

**4.2.3 Toggling condition.** We require some auxiliary concepts to state our third and last proof net condition. A set of linkings  $\Lambda$  *toggles* a  $\&$ -occurrence  $w$  of  $\Gamma$  if both arguments of  $w$  are present in  $\bigcup_{\lambda \in \Lambda} \Gamma \upharpoonright \lambda$ , i.e., there exist  $\lambda_l, \lambda_r \in \Lambda$  such that the left argument of  $w$  is present in the additive resolution  $\Gamma \upharpoonright \lambda_l$  and the right argument of  $w$  is present in the additive resolution  $\Gamma \upharpoonright \lambda_r$ .

*Example 4.13.* Recall our running example,

$$\begin{array}{l} \lambda_1 : \\ \lambda_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\text{left}}, \underbrace{(P \& P) \otimes (R \oplus R)}_{\text{middle}}, \underbrace{(R^\perp \otimes R) \mathfrak{A} R^\perp}_{\text{right}}$$

The pair of linkings  $\theta = \{\lambda_1, \lambda_2\}$  toggles the  $\&$  of the underlying sequent  $\Gamma$  because its left argument (the left occurrence of  $P$ ) is present in the additive resolution  $\Gamma \upharpoonright \lambda_1$ , and its right argument (the right occurrence of  $P$ ) is in the additive resolution  $\Gamma \upharpoonright \lambda_2$ . Neither  $\{\lambda_1\}$  nor  $\{\lambda_2\}$  toggles the  $\&$ : a single linking can never toggle a  $\&$  because all additives are unary in an additive resolution.

Let  $\Lambda$  be a set of linkings. An axiom link  $a$  *depends on  $w$  in  $\Lambda$*  if, inside  $\Lambda$ ,  $a$  can be made to vanish by toggling  $w$  alone: there are  $\lambda, \lambda' \in \Lambda$  such that  $a \in \lambda$ ,  $a \notin \lambda'$ , and  $w$  is the only  $\&$  toggled by  $\{\lambda, \lambda'\}$ .

*Example 4.14.* In

$$\begin{array}{l} \lambda_1 : \\ \lambda_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\text{left}}, \underbrace{(P \& P) \otimes (R \oplus R)}_{\text{middle}}, \underbrace{(R^\perp \otimes R) \mathfrak{A} R^\perp}_{\text{right}}$$

let  $w$  be the  $\&$  of the sequent. The axiom link between the left-most  $R$  and the left-most  $R^\perp$  depends on  $w$  in  $\Lambda = \{\lambda_1, \lambda_2\}$ : it is present in  $\lambda_1 \in \Lambda$  but not in  $\lambda_2 \in \Lambda$ , and  $w$  is the only  $\&$  toggled by  $\{\lambda_1, \lambda_2\}$ . The axiom link between the right-most  $R$  and  $R^\perp$  does not depend on  $w$  in  $\Lambda$ , since it is present in both  $\lambda_1$  and  $\lambda_2$ . It is the only one of the 5 axiom links in  $\Lambda$  (more precisely, in  $\bigcup \Lambda$ ) that does not depend on  $w$  in  $\Lambda$ .

We now extend the definition of the graph of a linking to the graph of a set  $\Lambda$  of linkings on  $\Gamma$ . Let  $\Gamma \upharpoonright \Lambda = \bigcup_{\lambda \in \Lambda} \Gamma \upharpoonright \lambda$ , the union (superposition) of the additive resolutions of the linkings of  $\Lambda$ . One can think of this as the *partial additive resolution* of  $\Lambda$ , as some additives may be unary, some binary. The *graph  $\mathcal{G}_\Lambda$  of  $\Lambda$*  is the extension of  $\Gamma \upharpoonright \Lambda$  obtained as follows: (1) add every axiom link  $a$  of (a linking of)  $\Lambda$  as a vertex above  $\Gamma \upharpoonright \Lambda$ , with edges from  $a$  down to its two literal occurrences in  $\Gamma \upharpoonright \Lambda$  (equivalently, thus far we have constructed the superposition

$\bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda$ ); (2) add a **jump** edge from  $a$  to any  $\&$ -vertex on which  $a$  depends in  $\Lambda$ . For example, in Figure 10 (page 18), subfigure (b) shows the graph of the pair of linkings in subfigure (a), and subfigure (e) shows the graph of the pair of linkings in subfigure (d). Note that  $\Lambda \subseteq \Lambda'$  implies  $\mathcal{G}_\Lambda \subseteq \mathcal{G}_{\Lambda'}$ , and that for any linking  $\lambda$ ,  $\mathcal{G}_{\{\lambda\}} = \mathcal{G}_\lambda$  (the graph of a single linking, defined in Section 4.2.2), because  $\mathcal{G}_{\{\lambda\}}$  has no jumps (a single linking toggles no  $\&$ s).

A **switch edge** of a  $\&$ - or  $\wp$ -vertex  $x$  of  $\mathcal{G}_\Lambda$  is an edge between  $x$  and one of its arguments, or a jump of  $x$  (if  $x$  is a  $\&$ ). For example, Figure 9(e) on page 15 has three switch edges, the argument edge of the  $\&$ , and both argument edges of the  $\wp$ . Figure 10(b) has 8 switch edges, the two argument edges of the  $\wp$ , the two argument edges of the  $\&$ , and the four jumps.

A **switching cycle** in  $\mathcal{G}_\Lambda$  is a cycle containing at most one switch edge of each  $\&$  and  $\wp$ . (We use a standard notion cycle in a simple graph: any subgraph with vertex set  $\{x_1, \dots, x_n\}$  for  $n \geq 3$ , with all  $x_i$  distinct, and an edge  $x_i - x_{i+1}$  for all  $i \pmod n$ .) For example, the graph  $\mathcal{G}_\Lambda$  of Figure 10(b) contains the switching cycle  $C$  shown below it (subfigure (c)).

Our third and final proof net condition on a set of linkings  $\theta$  is:

(P3) TOGGLING. *Every set  $\Lambda$  of two or more linkings of  $\theta$  toggles a  $\&$  that is not in any switching cycle in  $\mathcal{G}_\Lambda$ .*

It is clear from the definition of the graph  $\mathcal{G}_\Lambda$  that it suffices to verify TOGGLING for **saturated** sets of linkings  $\Lambda$ , namely, such that any strictly larger subset of  $\theta$  toggles more  $\&$ s than  $\Lambda$ . Note that there is exactly one saturated set of linkings in  $\theta$  for each **partial  $\&$ -resolution** of  $\Gamma$ , the latter being any result of deleting at most one argument subtree each  $\&$  of  $\Gamma$ . We retain the more general quantification over  $\Lambda$  in the formulation of the toggling condition so that the definition of proof net is more succinct.

*Example 4.15.* The pair of linkings  $\Lambda$  in Figure 10(a) fails the toggling condition, because of the switching cycle  $C$  of subfigure (c), which traverses the  $\&$ .

More generally, whenever every  $\&$  is in a switching cycle (in the case of Example 4.15, just one  $\&$ ), the toggling condition fails.

*Example 4.16.* The pair of linkings  $\theta$  in Figure 10(d) satisfies the toggling condition. Any switching cycle in the graph  $\mathcal{G}_\theta$  (Figure 10(e)) is only permitted to use one switch edge of the  $\&$ , and therefore to traverse the  $\&$  it must go via the  $\otimes$  immediately below it. Since there is no cycle containing the  $\otimes$ , there is no switching cycle through the  $\&$ .

In the box on page 13 we defined a cut-free MALL proof net to be a set of linkings on a MALL sequent satisfying all three conditions introduced above: (P1) RESOLUTION, (P2) MLL, and (P3) TOGGLING. (In other words, a cut-free MALL proof net is a cut-free MALL proof structure satisfying the MLL and toggling conditions.) In the example below, we go through the full process of verifying all three conditions.

*Example 4.17.* Consider the pair of linkings on the sequent  $\Gamma \equiv P^\perp \& P^\perp, P \oplus P$  obtained as follows:

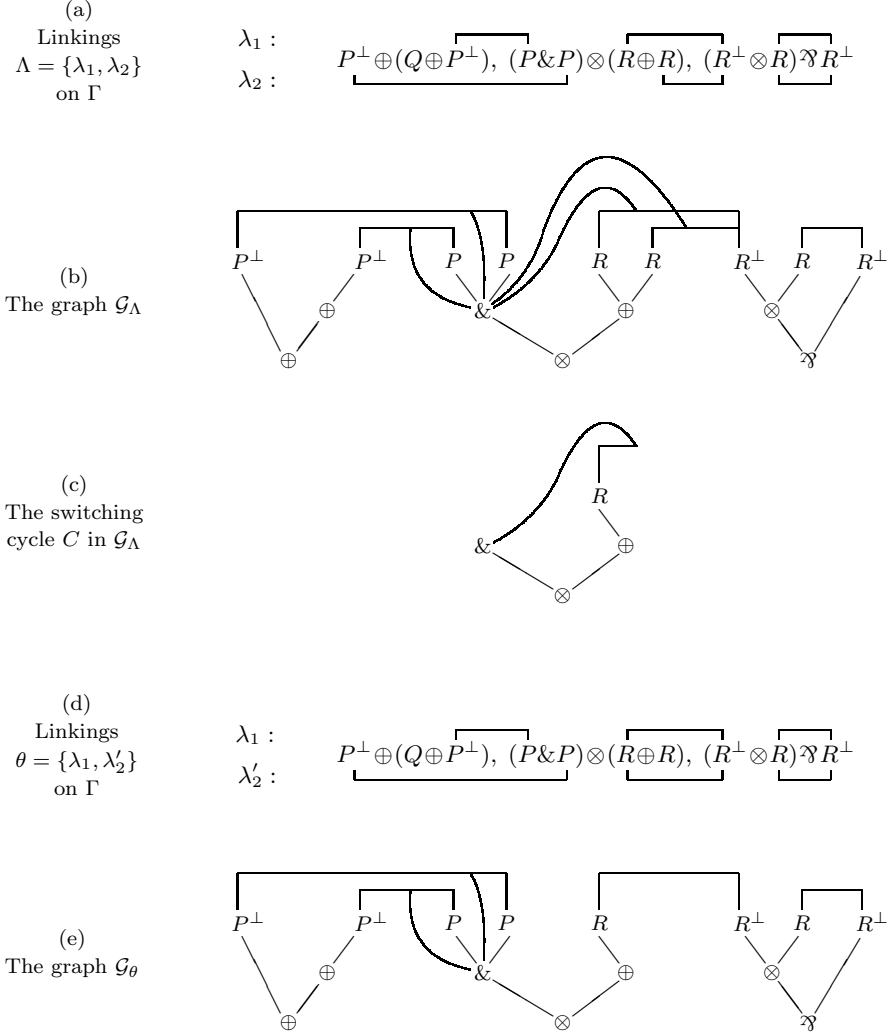
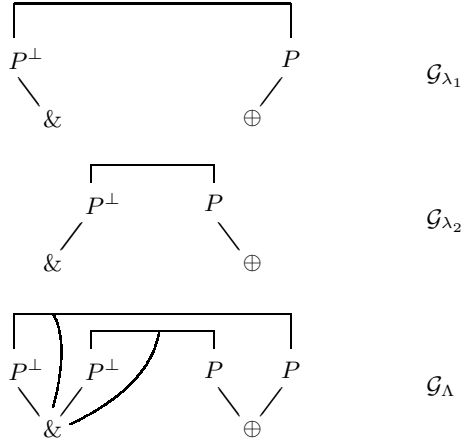


Fig. 10. Subfigure (a) is a pair of linkings  $\Lambda = \{\lambda_1, \lambda_2\}$  whose graph  $\mathcal{G}_\Lambda$  is depicted in subfigure (b). We overlap edges from axiom links entering the same literal occurrence ( $\wp$  means  $\wp$ ), and to distinguish jumps, we draw them as curved edges. There is no jump to the right-most axiom link, since it does not depend on the  $\&$  in  $\Lambda$ . (This was explained in detail in Example 4.14, page 16.) Subfigure (c) shows a switching cycle  $C$  in  $\mathcal{G}_\Lambda$ . Subfigure (d) is a pair of linkings  $\theta = \{\lambda_1, \lambda'_2\}$ , whose graph  $\mathcal{G}_\theta$  is (e). This pair of linkings satisfies the toggling condition: the only subset of two or more linkings is  $\theta$  itself, so to verify the condition we need only confirm that  $\mathcal{G}_\theta$  contains no switching cycle; this is apparent from the depiction of  $\mathcal{G}_\theta$  in subfigure (e).

$$\begin{array}{c}
 \overline{\overbrace{P^\perp, P}^{\text{ax}}} \quad \overline{\overbrace{P^\perp, P}^{\text{ax}}} \\
 \overline{\overbrace{P^\perp, P \oplus P}^{\oplus_2}} \quad \overline{\overbrace{P^\perp, P \oplus P}^{\oplus_1}} \\
 \overline{\overbrace{P^\perp \& P^\perp, P \oplus P}^{\&}}
 \end{array}$$

Let  $\lambda_1$  and  $\lambda_2$  be the upper- and lower linking of the concluding sequent respectively (each having just one axiom link). We shall verify that  $\Lambda = \{\lambda_1, \lambda_2\}$  is a cut-free proof net.  $\Gamma$  has two  $\&$ -resolutions,  $\Gamma_1^* \equiv P^\perp \& P^\perp, P \oplus P$  and  $\Gamma_2^* \equiv P^\perp \& P^\perp, P \oplus P$ . The resolution condition holds, since  $\Lambda$  contains exactly one linking on  $\Gamma_i^*$ , namely  $\lambda_i$ . Here are the graphs  $\mathcal{G}_{\lambda_1}$ ,  $\mathcal{G}_{\lambda_2}$ , and  $\mathcal{G}_\Lambda$ :



Each  $\lambda_i$  has just one  $\mathfrak{A}$ -switching, namely  $\mathcal{G}_{\lambda_i}$ ; since each  $\mathcal{G}_{\lambda_i}$  is a tree, the MLL condition holds. Finally, the toggling condition holds since  $\Lambda$  toggles the  $\&$ , which is not in any switching cycle in  $\mathcal{G}_\Lambda$ . (An outermost  $\&$  (i.e., one that is not an argument of any other connective) can never be in a switching cycle.) More generally, there are  $n^m$  proof nets on the sequent  $\&^m P^\perp, \oplus^n P$  (above  $m = n = 2$ ), in bijection with natural transformations  $\prod^m X \rightarrow \prod^n X$  on sets, or equivalently,  $\prod^n X \rightarrow \prod^m X$ .

In Section 4.4 we provide proof-theoretic intuition for the toggling condition.

**THEOREM 4.18 CUT-FREE SEQUENTIALISATION.** *A set of linkings is the translation of a cut-free proof iff it is a cut-free proof net.*

By a simple induction, the translation of a cut-free proof is a cut-free proof net. The proof of the converse reduces to a simple induction on the number of  $\mathfrak{A}$ s and  $\&$ s (Section 4.10) once we prove (Section 4.9):

**LEMMA 4.19 SEPARATION LEMMA.** *For any cut-free proof net  $\theta$ , if  $\mathcal{G}_\theta$  has a  $\mathfrak{A}$  or  $\&$ , then it has a  $\mathfrak{A}$  or  $\&$  that separates.*

Here a  $\mathfrak{A}$ - or  $\&$ -vertex  $x$  **separates** if it is not an argument (i.e., is an outermost connective), or it is the argument of  $y$  and deleting the edge between  $x$  and  $y$  disconnects  $\mathcal{G}_\theta$ . We shall prove the Separation Lemma via an ordering on  $\&$ s and

$\wp$ s which we call *domination*<sup>6</sup>, a concept reminiscent of the ordering induced by the notion of an *empire* of Girard [1996], but different in an essential way.

The remainder of this section is structured as follows. In Section 4.3 we provide some intuition for the resolution and MLL conditions, and in Section 4.4 we do the same for the toggling condition. In Section 4.5 we consider some alternative formulations of the definition of proof net. Section 4.6 describes how to encode a proof structure using weights. In Section 4.7 we observe that our cut-free proof nets exactly capture cut-free MALL proofs modulo commutations of rules. In Section 4.8 we define a *mix net* as the analogue of our proof nets in the case of MALL augmented with the mix rule. We conclude by proving the cut-free sequentialisation theorem in Sections 4.9 and 4.10 (the Separation Lemma and the main induction, respectively).

### 4.3 Intuition for the resolution and MLL conditions

**4.3.1 Intuition for the resolution condition.** Recall from Section 4.1.2 that the set  $\theta_\Pi$  of linkings obtained from a cut-free MALL proof comprises one link  $\lambda_S \in \theta_\Pi$  per  $\&$ -resolution  $S$  of the proof  $\Pi$ . This correspondence between proof  $\&$ -resolutions and linkings is what is captured in the resolution condition. (One can observe this correspondence in Figures 8 (page 12) and 11 (page 21).)

Define a  $\&$ -*assignment* of a sequent  $\Gamma$  to be a choice of left or right for each of its  $\&$ s, i.e., a function from the set of  $\&$ -occurrences of  $\Gamma$  to  $\{l, r\}$  ( $l=left, r=right$ ). Every  $\&$ -assignment  $\varphi$  defines a  $\&$ -resolution  $\Gamma^\varphi$  in the obvious way, by restricting each  $\&$  to the argument dictated by its assignment (i.e., delete the right (resp. left) argument subtree of  $w$  iff  $\varphi(w) = l$  (resp.  $r$ )). In turn, every  $\&$ -resolution  $\Gamma^*$  of a sequent  $\Gamma$  induces a  $\&$ -resolution  $\Pi \upharpoonright \Gamma^*$  of a proof  $\Pi$  of  $\Gamma$ : work upwards from the concluding rule of  $\Pi$  and delete branches of  $\&$ -rules according to which branch of the corresponding  $\&$ -occurrence is deleted in  $\Gamma^*$ . Note that more than one  $\&$ -assignment can give rise to the same  $\&$ -resolution of the sequent  $\Gamma$ , and that more than one  $\&$ -resolution of  $\Gamma$  can give rise to the same  $\&$ -resolution of a proof  $\Pi$  of  $\Gamma$ : see Figure 11.

**4.3.2 Intuition for the MLL condition.** Every  $\&$ -resolution  $S$  of a proof  $\Pi$  has all additive rules unary. (The  $\oplus$  rules are unary at the outset, and the  $\&$  rules become unary upon taking the  $\&$ -resolution.) Collapsing the unary additive rules of  $S$  (and the now-unary connectives in the corresponding formula parse trees) yields an MLL proof. Since every linking of  $\theta_\Pi$  comes from a  $\&$ -resolution of  $\Pi$ , i.e., from a disguised MLL proof, we demand that every linking of a MALL proof net be MLL correct.

### 4.4 Intuition for the toggling condition

In the preceding subsection we saw how a cut-free MALL proof  $\Pi$  determines a set of cut-free MLL proofs, one per  $\&$ -resolution of  $\Pi$ . However,  $\Pi$  is more than just a set of non-interacting MLL proofs, as each of them is implicitly embedded inside the tree of  $\Pi$ . Correspondingly, a set of linkings merely satisfying the resolution and MLL conditions need not be sequentialisable, as one must capture the constraint associated with the superposition of branches of the  $\&$ -resolutions of  $\Pi$  inside the

<sup>6</sup>Unrelated to domination in flowgraphs.

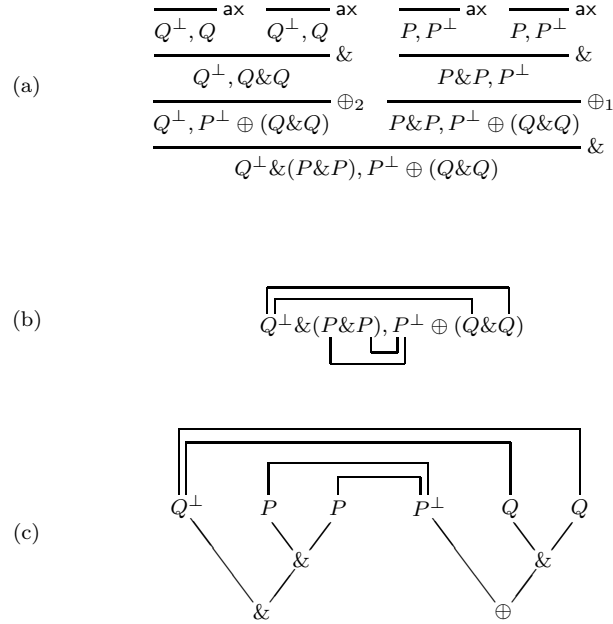


Fig. 11. (a) A proof  $\Pi$  of a sequent  $\Gamma$  illustrating a collapse from  $\&$ -assignments to  $\&$ -resolutions of  $\Gamma$  to  $\&$ -resolutions of  $\Pi$ .  $\Gamma$  has  $2^3 = 8$   $\&$ -assignments, more than its  $3 \times 2 = 6$   $\&$ -resolutions, more than the 4  $\&$ -resolutions of  $\Pi$ . (b) The set of linkings associated with  $\Pi$ , one from each of its  $\&$ -resolutions. It is convenient to show all four linkings on the same copy of the sequent; no ambiguity arises because every linking has only one link. (c) For additional clarity, we show the same set of four singleton linkings displayed on the parse trees of the two formulas (i.e., we show the union of the graphs  $\mathcal{G}_\lambda$  for each of the four linkings  $\lambda$ ).

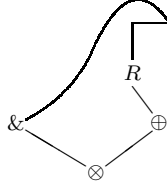
tree structure of  $\Pi$ . We have already seen an example: the pair of linkings  $\Lambda = \{\lambda_1, \lambda_2\}$

$$\begin{array}{l} \lambda_1 : \\ \lambda_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\&}, \underbrace{(P \& P)}_{\otimes} \underbrace{(R \oplus R)}_{\&}, \underbrace{(R^\perp \otimes R)}_{\&} \underbrace{R^\perp}_{\&}$$

of Figure 10(a) (page 18) satisfies the resolution condition (we verified this in Example 4.6, page 13) and the MLL condition ( $\lambda_1$  and  $\lambda_2$  induce the same MLL proof net, whose graph is Figure 9(c), page 15), but  $\Lambda$  is not sequentialisable. It fails to sequentialise because we will never be able to write down a rule to introduce the central tensor: its left argument  $P \& P$  must go in the left hypothesis of the rule, and its right argument  $R \oplus R$  in the right hypothesis; but then the  $\&$  will not be available in the right branch to superimpose a left- $\oplus$  and right- $\oplus$  rule as would be required to obtain  $\lambda_1$  with the left  $R$  of  $R \oplus R$  and  $\lambda_2$  with the right  $R$ .

In summary, there is a conflict between the central  $\otimes$  and the  $\&$ : the tensor wishes to separate its  $\&$  argument from its  $\oplus$  argument, into distinct non-interacting proofs; meanwhile the  $\&$  argument interacts with the  $\oplus$  argument since in the  $\lambda_i$  the  $\oplus$  goes left iff the  $\&$  goes left, a direct dependency (interaction) across the tensor. Via jumps, the toggling condition captures this kind of dependency, and

rules out  $\Lambda$  as a proof net: the graph  $\mathcal{G}_\Lambda$  (Figure 10(b), page 18) of  $\Lambda$  contains the switching cycle



(copied from Figure 10(c)) traversing the only  $\&$ , and therefore breaking the toggling condition. The jump captures the communication between the  $\&$  and the  $\oplus$ .

In contrast, the pair of linkings  $\theta = \{\lambda_1, \lambda'_2\}$

$$\begin{array}{l} \lambda_1 : \\ \lambda'_2 : \end{array} \quad \underbrace{P^\perp \oplus (Q \oplus P^\perp)}_{\text{left}}, \underbrace{(P \& P) \otimes (R \oplus R)}_{\text{middle}}, \underbrace{(R^\perp \otimes R) \wp R^\perp}_{\text{right}}$$

of Figure 10(d) (page 18), formed from  $\Lambda$  by shifting the end of just one axiom link in  $\lambda_2$  to create  $\lambda'_2$ , is sequentialisable: after writing down the rule introducing the central tensor, we can choose the left- $\oplus$  rule for  $R \oplus R$ , since both linkings  $\lambda_1$  and  $\lambda'_2$  choose the left argument of  $R \oplus R$ . In contrast to the case  $\Lambda = \{\lambda_1, \lambda_2\}$  above, there is no communication across the tensor between the  $\&$  and the  $\oplus$ . Thus the graph  $\mathcal{G}_\theta$  of  $\theta$  (Figure 10(e), page 18) does not possess a jump across the tensor, hence  $\theta$  satisfies the toggling condition, and is a proof net. (We verified this in Example 4.16, page 17.)

**4.4.1 The Gustave example.** In the previous section we saw sequentialisation hampered by a dependency across a tensor, and how this was captured by the toggling condition. In this section we examine a case of dependency across the other unswitched connective, the plus, yielding additional intuition for the toggling condition.

The following remarkable non-sequential function  $G$  is due to Gustave, studied by G. Berry in the context of sequential algorithms:

$$\begin{aligned} G(x, 1, 0) &= u \\ G(0, y, 1) &= v \\ G(1, 0, z) &= w \end{aligned}$$

for all  $x, y$ , and  $z$ , possibly divergent/halting. The actual outputs  $u, v, w$  are of no concern, so long as they are distinct and non-divergent. This partial function cannot be implemented sequentially. For example, suppose our implementation inspects the argument  $x$  of  $G(x, y, z)$  first. If  $x$  diverges, while  $y = 1$  and  $z = 0$ , the equations for  $G$  dictate an output  $u$ ; however, our implementation would diverge, having become stuck on the divergent  $x$ . By symmetry, we cannot choose to inspect  $y$  or  $z$  first either, hence there is no sequential implementation of  $G$ .

Girard (e.g. [1999, sec. 5.5.4]) and Abramsky and Melliès [1999] have studied a corresponding example in the context of models of linear logic. Analogous to

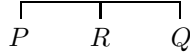
Girard/Abramsky/Melliès, in our setting one can capture the three equations specifying  $G$  as part of a set of five linkings on the sequent

$$(P \& Q) \oplus (P^\perp \otimes Q^\perp), \quad (Q \& (P^\perp \otimes Q^\perp)) \oplus P, \quad ((P^\perp \otimes Q^\perp) \& P) \oplus Q,$$

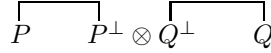
satisfying the resolution and toggling conditions. To emphasise the rotational symmetry, let us write  $R$  for  $P^\perp \otimes Q^\perp$ , so that the sequent becomes the more palatable

$$\Gamma = (P \& Q) \oplus R, \quad (Q \& R) \oplus P, \quad (R \& P) \oplus Q$$

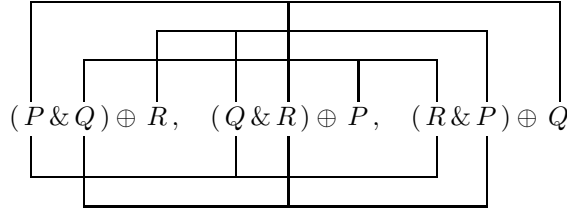
and let us write the “triplet” linking



to abbreviate a pair of axiom links

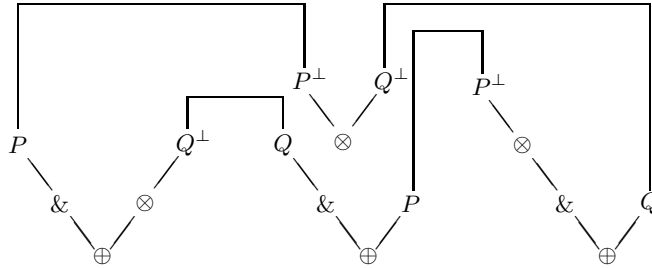


on a sequent. The **Gustave proof structure**  $G$  consists of the following five linkings on  $\Gamma$ , three shown above, and two below.



The top three linkings correspond to the three Gustave equations. The two underhanging linkings are added so that the resolution condition holds. (One can readily verify the resolution condition by working through each of the eight  $\&$ -resolutions and checking that exactly one linking fits in each case.) The MLL property holds since every linking induces the same MLL proof net, the pair of axiom links displayed immediately prior to the 5 Gustave linkings.

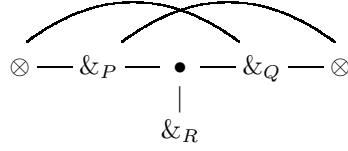
The Gustave proof structure is not the translation of any cut-free proof: any proof of  $\Gamma$  must end in a final  $\oplus$ -rule (a simple syntactic observation), hence any translation of a proof of  $\Gamma$  has at least one of the six  $\oplus$ -arguments uninhabited (corresponding to *softness* [Joyal 1995]);  $G$  touches all six arguments. Thus, by the sequentialisation theorem, we should be able to witness the failure of the toggling condition. This is indeed the case, since every  $\&$  is contained in the following switching cycle of the graph of  $G$ :



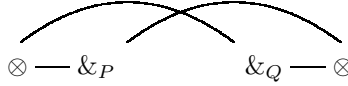
(Note that we did not require jumps to forge this switching cycle.)

4.4.2 *Strong—but not too strong.* We have seen from the previous two sections that the toggling condition captures unwanted dependencies across  $\otimes$ s and  $\oplus$ s by finding switching cycles through  $\&$ s. In this section we illustrate a subtle feature of the toggling condition: it is possible to have switching cycles through  $\&$ s *without* obstructing sequentialisability. One has to be extremely discerning of those switching cycles through  $\&$ s which are *essential*, in the sense that they represent intrinsic parallelism, versus those which are *harmless*, and do not impair sequentialisation.

We shall define a proof net  $\theta$  on a sequent  $\Gamma$  of three  $\&$ s, with  $2^3 = 8$  linkings (one per  $\&$ -assignment of three independent  $\&$ s). The graph  $\mathcal{G}_\theta$  of  $\theta$  will have the shape



with the portion



constituting a switching cycle in  $\mathcal{G}_\theta$  between the  $\&$ -occurrences  $\&_P$  and  $\&_Q$ .

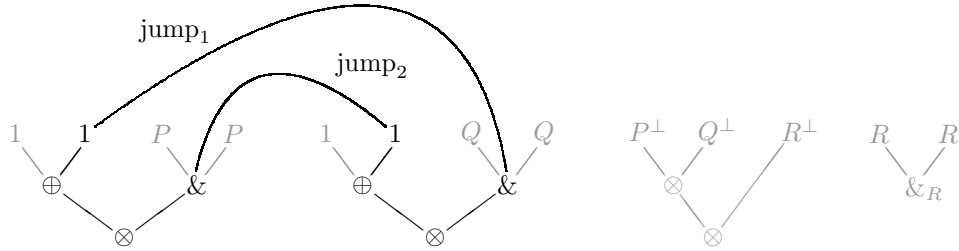
The underlying sequent will be

$$(1 \oplus 1) \otimes (P \& P), (1 \oplus 1) \otimes (Q \& Q), (P^\perp \otimes Q^\perp) \otimes R^\perp, R \& R$$

where 1 denotes the tensor unit. (As with MLL proof nets, our MALL proof nets extend trivially to the tensor unit 1: view each occurrence of 1 as  $(P \& P^\perp)$  for a fresh atom  $P$  each time.<sup>7</sup>) Links on the atoms  $P, Q, R$  are forced, so to determine the eight linkings of  $\theta$  it suffices to specify how the  $\oplus$ s choose their 1s:

- (a) in  $(1 \oplus 1) \otimes (P \& P)$  choose the right 1 iff  $R \& R$  is left and  $Q \& Q$  is right;
- (b) in  $(1 \oplus 1) \otimes (Q \& Q)$  choose the right 1 iff  $R \& R$  is right and  $P \& P$  is right.

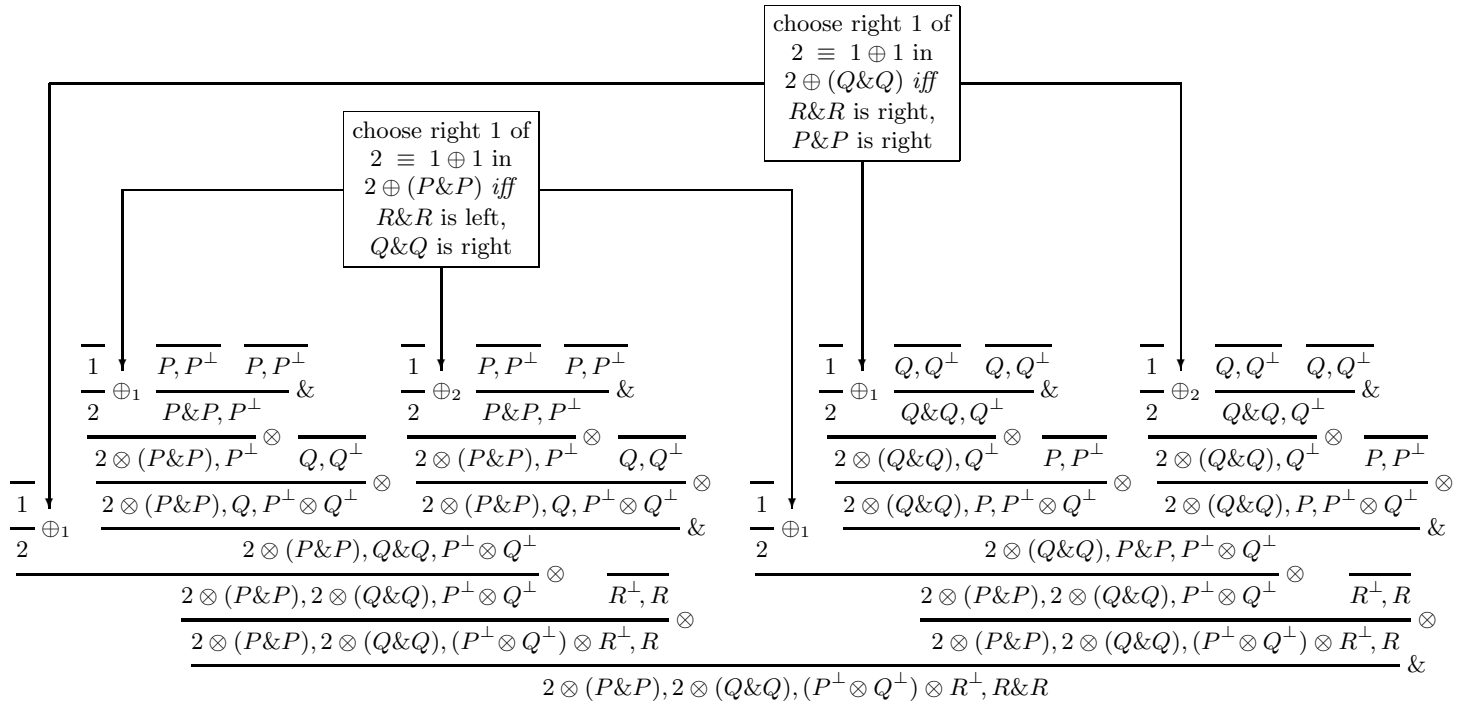
The set of linkings  $\theta$  is sequentialisable. (Figure 12 shows a sequentialisation.) The graph  $\mathcal{G}_\theta$  of  $\theta$  contains the following switching cycle  $C$ :



Due to the way we defined the linkings in clauses (a) and (b) above, given a set  $\Lambda$  of two or more linkings of  $\theta$ :

<sup>7</sup>Similarly, one can define the plus unit 0 as  $P^\perp \otimes P$  for a fresh atom  $P$  each time. Note the interesting complementarity between 0 and 1, units which are not dual in the logic.

Fig. 12. A sequentialisation of  $\theta$ .





- (A) every  $\mathfrak{A}$ -switching of  $\lambda$  is acyclic      (a) some  $\mathfrak{A}$ -switching of  $\lambda$  is acyclic  
 (B)  $\lambda$  is balanced  
 (C) every  $\mathfrak{A}$ -switching of  $\lambda$  is connected      (c) some  $\mathfrak{A}$ -switching of  $\lambda$  is connected

The clause of MLL condition (P2) holds for a linking  $\lambda$  precisely when  $\lambda$  satisfies  $(A)\wedge(C)$ .

PROPOSITION 4.20. *The following conditions are all equivalent to the clause of the MLL condition (P2) for a linking  $\lambda$ :  $(A)\wedge(C)$ ,  $(A)\wedge(c)$ ,  $(a)\wedge(C)$ ,  $(A)\wedge(B)$  and  $(B)\wedge(C)$ .*

The proof is essentially due to the simple combinatorial relationship between the number of vertices and the number of edges of a tree. See Appendix A.

4.5.2 *Switching acyclicity, switching path connectedness.* It is immediately clear (and we saw in Section 3.2) that (A) above is equivalent to  $\mathcal{G}_\lambda$  containing no switching cycle, i.e., being “switching acyclic”. Similarly, in the presence of (A), (c) is equivalent to  $\mathcal{G}_\lambda$  being “switching connected”, that is, any two vertices of  $\mathcal{G}_\lambda$  are connected by a *switching path*, i.e., a path that does not traverse two switch edges of any given  $\mathfrak{A}$ . The simplest way to prove the equivalence with (c) is to observe that one can carry out sequentialisation (specifically, the MLL restriction of the proof of the sequentialisation theorem) with this condition in place of (c).

4.5.3 *Illegal unions of switching cycles.* We provide an alternative formulation of the toggling condition (P3), assuming the MLL condition (P2). Call a union  $S$  of switching cycles in  $\mathcal{G}_\theta$  *illegal* if it is non-empty and for some  $\Lambda \subseteq \theta$  with  $S \subseteq \mathcal{G}_\Lambda$ , every  $\&$  toggled by  $\Lambda$  is in  $S$ .

(P3<sup>!</sup>)  $\mathcal{G}_\theta$  contains no illegal union of switching cycles.

Note that this condition implies condition (A) for each linking (every switching is acyclic). The proof of equivalence with (P3) follows from simple manipulation using Proposition 4.20 above. (Details are in Appendix B.)

CONJECTURE 4.21 SINGLE SWITCHING CYCLE CONJECTURE. *Property (P3<sup>!</sup>) is equivalent to:*

(P3<sup>!-</sup>)  $\mathcal{G}_\theta$  contains no illegal switching cycle.

*In other words, the original toggling condition (P3) is equivalent to:*

(P3<sup>-</sup>) *For any  $\Lambda \subseteq \theta$  and any switching cycle  $C$  in  $\mathcal{G}_\Lambda$ ,  $\Lambda$  toggles a  $\&$  that is not in  $C$ .*

The authors would be interested to hear of any progress on this conjecture.

4.5.4 *Additional jumps.* We shall use the following variation of (P2) to define a collapse from Girard’s proof structures to ours in Section 6.3. Given a set of linkings  $\theta$  on a sequent  $\Gamma$  and a subset  $\Lambda \subseteq \theta$ , let  $\mathcal{G}_\Lambda^\theta$  be defined as  $\mathcal{G}_\Lambda$  but with jump edges between every  $\&$ -vertex  $w \in \mathcal{G}_\Lambda$  and every axiom link  $a \in \mathcal{G}_\Lambda$  depending on  $w$  in  $\theta$ . Note that  $\mathcal{G}_\Lambda = \mathcal{G}_\Lambda^\Lambda$ . Define the variant (P2\*) of (P2) by using  $\mathcal{G}_{\{\lambda\}}^\theta$  in place of  $\mathcal{G}_\lambda$  in the definition of a  $\mathfrak{A}$ -switching of  $\lambda$ , and in taking the switching delete in addition all but one switch edge of each  $\&$  (i.e., we move from  $\mathfrak{A}$ -switchings to “ $\mathfrak{A}/\&$ -switchings”). Clearly (P2\*) implies (P2), since it involves more switchings.

In fact, (P2\*) is strictly stronger than (P2): for  $\theta = \{\lambda_1, \lambda_2\}$  of Example 4.6, the graph  $\mathcal{G}_{\lambda_1}^\theta$  has a switching cycle (the one presented in Figure 10), whereas  $\mathcal{G}_{\lambda_1}$  does not. However, it is not hard to check that (P2\*) is implied by (P2) and (P3) together.

4.5.5 *Other variations.* In Section 6.4 we develop a correspondence between (P1) and Girard’s so-called ‘technical condition’ [1996], and note in Section 6.5 that, without monomials, the Abramsky-Melliès reformulation [1999] of the technical condition is no longer valid. Section 6.4 includes intermediary conditions corresponding to (P1) and the technical condition.

## 4.6 Weights

In this section we describe how to encode any proof structure (hence any one of our proof nets) as a single set of axiom links labelled with predicates, called *weights* (c.f. [Girard 1996]). Figure 5 (page 5) conveys the idea informally with an example.

Recall from Section 4.3.1 that a  $\&$ -assignment of a sequent  $\Gamma$  is a function from its  $\&$ -occurrences to  $\{l, r\}$  ( $l=left, r=right$ ), and that every  $\&$ -assignment  $\varphi$  defines a  $\&$ -resolution  $\Gamma^\varphi$  by restricting each  $\&$  to the argument dictated by its assignment. Multiple  $\&$ -assignments can determine the same  $\&$ -resolution. For example, if  $\Gamma = (P\&_1Q)\&_2R$ , then the assignments “ $\&_1 \mapsto l, \&_2 \mapsto r$ ” and “ $\&_1 \mapsto r, \&_2 \mapsto r$ ” both determine the  $\&$ -resolution  $(P\&_1Q)\&_2R$  retaining only  $R$ . See also Figure 11 (page 21).

Let  $\theta$  be a proof structure on  $\Gamma$ . Given a  $\&$ -assignment  $\varphi$  of  $\Gamma$ , write  $\lambda_\varphi$  for the unique linking of  $\theta$  on the  $\&$ -resolution  $\Gamma^\varphi$  of  $\varphi$  (existence and uniqueness due to the resolution condition). Every axiom link  $a$  of  $\theta$  determines a predicate on  $\&$ -assignments, its *weight*  $\mu(a)$ , by  $\varphi \in \mu(a)$  iff  $a \in \lambda_\varphi$ . One can then represent  $\theta$  by its axiom links labelled with weights, as in Figure 5 (page 5), for example.

Weights can be expressed succinctly as follows. First, mark each  $\&$ -occurrence with a distinct subscript tag,  $x, y, \dots$ . Write  $x$  as shorthand for  $\{\varphi : \varphi(\&_x) = l\}$  (all  $\&$ -assignments that take  $\&_x$  to the left) and  $\bar{x}$  as shorthand for  $\{\varphi : \varphi(\&_x) = r\}$  (all  $\&$ -assignments that take  $\&_x$  to the right);  $\vee$  and  $\wedge$  are union and intersection, respectively. Again, see Figure 5 for an example.

The set of linkings of a proof structure is recoverable from its weight presentation as follows. Every  $\&$ -assignment  $\varphi$  determines a linking  $\lambda_\varphi$  by deleting each axiom link  $a$  whose predicate does not hold, i.e.  $\lambda_\varphi = \{a : \varphi \in \mu(a)\}$ . Taking each  $\&$ -assignment in turn produces the full set of linkings.

## 4.7 Representation of cut-free proofs modulo rule commutations

The kernel of our function from cut-free MALL proofs to sets of linkings coincides precisely with equivalence modulo rule commutation:

**THEOREM 4.22.** *Two cut-free MALL proofs translate to the same proof net if and only if they can be converted into each other by a series of rule commutations.*

A rule commutation is a local conversion on a proof that retains the subproofs of its hypotheses, with possible duplication/identification. Figure 13 shows two examples of rule commutations. Given two consecutive rules of a Gentzen sequent calculus, there is a well-defined notion of commutability: see [Troelstra and Schwichtenberg

$$\begin{array}{c}
 \frac{\frac{\frac{\Pi_1}{\Gamma, A} \quad \frac{\frac{\Pi_2}{B, \Delta, X} \quad \frac{\Pi_3}{B, \Delta, Y}}{\&}}{\Gamma, A \otimes B, \Delta, X \& Y} \otimes}{\Gamma, A \otimes B, \Delta, X \& Y} \otimes \\
 \longleftrightarrow \\
 \frac{\frac{\frac{\Pi_1}{\Gamma, A} \quad \frac{\Pi_2}{B, \Delta, X}}{\Gamma, A \otimes B, \Delta, X} \otimes \quad \frac{\frac{\Pi_1}{\Gamma, A} \quad \frac{\Pi_3}{B, \Delta, Y}}{\Gamma, A \otimes B, \Delta, Y} \otimes}{\Gamma, A \otimes B, \Delta, X \& Y} \otimes \\
 \\
 \frac{\frac{\frac{\Pi_1}{\Gamma, A, X} \quad \frac{\Pi_2}{\Gamma, B, X}}{\&} \quad \frac{\frac{\Pi_3}{\Gamma, A, Y} \quad \frac{\Pi_4}{\Gamma, B, Y}}{\&}}{\Gamma, A \& B, X \& Y} \& \\
 \longleftrightarrow \\
 \frac{\frac{\frac{\Pi_1}{\Gamma, A, X} \quad \frac{\Pi_3}{\Gamma, A, Y}}{\&} \quad \frac{\frac{\Pi_2}{\Gamma, B, X} \quad \frac{\Pi_4}{\Gamma, B, Y}}{\&}}{\Gamma, A \& B, X \& Y} \&
 \end{array}$$

Fig. 13. Two examples of rule commutations. The commutations can be read in either direction.

1996, Def. 5.2.1]. This theorem, together with a number of general theorems related to rule commutation in MALL (with and without the mix rule, and including the cut rule) are the subject of a sibling paper.

#### 4.8 Mix nets

Let  $\text{MALL}^{\text{mix}}$  denote the extension [Girard 1987] of MALL with the additional rule

$$\frac{\Gamma \quad \Delta}{\Gamma, \Delta} \text{mix}$$

and define the following variant of the MLL condition on a set of linkings  $\theta$  by relaxing connectedness:

(P2<sup>mix</sup>)  $\text{MLL}^{\text{mix}}$ . Every  $\mathfrak{A}$ -switching of every linking of  $\theta$  is acyclic.

Define a **cut-free mix net** to be a cut-free MALL proof net but for the relaxation of connectedness of  $\mathfrak{A}$ -switchings, i.e., a set of linkings satisfying RESOLUTION,  $\text{MLL}^{\text{mix}}$ , and TOGGING.

**THEOREM 4.23 CUT-FREE MIX SEQUENTIALISATION.** *A set of linkings is the translation of a cut-free  $\text{MALL}^{\text{mix}}$  proof iff it is a cut-free mix net.*

We prove this theorem concurrently with the main sequentialisation theorem. Only very minor modifications are necessary.

#### 4.9 Proof of the Separation Lemma

Throughout this section  $\theta$  is a cut-free proof net on a sequent  $\Gamma$ . For vertices  $x$  and  $y$  of the graph  $\mathcal{G}_\theta$ , write  $x \rightarrow y$  iff  $x$  is an argument of  $y$ , or  $x$  is an axiom link and  $y$  is either a  $\&$  on which  $x$  depends in  $\theta$  or one of the two literals linked by  $x$ , thus directing the edges of  $\mathcal{G}_\theta$  downwards. Property (P2) implies  $\mathcal{G}_\theta$  is connected.

A subset  $\Lambda \subseteq \theta$  is **saturated** if any strictly larger subset of  $\theta$  toggles more  $\&$ s than  $\Lambda$ . Clearly  $\theta$  itself is saturated. For  $\Lambda$  a set of linkings and  $w$  a  $\&$  of  $\Gamma$  let  $\Lambda^w$  denote the set of all linkings in  $\Lambda$  whose additive resolution does not contain the

right argument of  $w$ . Write  $\lambda \stackrel{w}{=} \lambda'$  if linkings  $\lambda, \lambda' \in \theta$  are either equal or  $w$  is the only  $\&$  toggled between  $\Gamma \upharpoonright \lambda$  and  $\Gamma \upharpoonright \lambda'$ . It is straightforward to check that:

- (S1) If  $\Lambda$  is saturated and toggles  $w$  then also  $\Lambda^w$  is saturated.
- (S2) If  $\Lambda$  is saturated and toggles  $w$  and  $\lambda \in \Lambda$  then  $\lambda \stackrel{w}{=} \lambda_w$  for some  $\lambda_w \in \Lambda^w$ .
- (S3) If  $\Lambda$  is saturated and toggles  $w$  and  $\lambda \stackrel{x}{=} \lambda'$  for  $\lambda, \lambda' \in \Lambda$  then  $\lambda \stackrel{w}{=} \lambda_w \stackrel{x}{=} \lambda'_w \stackrel{w}{=} \lambda'$  for some  $\lambda_w, \lambda'_w \in \Lambda^w$ .

LEMMA 4.24. *Let  $w$  be a  $\&$  toggled by a saturated set  $\Lambda \subseteq \theta$ , and let  $e$  be an edge in  $\mathcal{G}_\Lambda$  originating from an axiom link  $a$ , such that  $e \notin \mathcal{G}_{\Lambda^w}$ . Then the jump  $a \rightarrow w$  is in  $\mathcal{G}_\Lambda$ .*

PROOF. Let  $e$  be  $a \rightarrow x$ . If  $e$  is not a jump,  $e \notin \mathcal{G}_{\Lambda^w}$  implies  $a \notin \mathcal{G}_{\Lambda^w}$ . Choose  $\lambda \in \Lambda$  with  $a \in \lambda$ . By (S2)  $\lambda \stackrel{w}{=} \lambda_w$  for some  $\lambda_w \in \Lambda^w$ . Since  $a \notin \lambda_w$  (for  $a \notin \mathcal{G}_{\Lambda^w}$ ), the jump  $a \rightarrow w$  is in  $\mathcal{G}_\Lambda$ .

If  $e$  is a jump, we have  $\lambda, \lambda' \in \Lambda$  with  $a \in \lambda$ ,  $a \notin \lambda'$  and  $\lambda \stackrel{x}{=} \lambda'$ . By (S3)  $\lambda \stackrel{w}{=} \lambda_w \stackrel{x}{=} \lambda'_w \stackrel{w}{=} \lambda'$  for  $\lambda_w, \lambda'_w \in \Lambda^w$ . Either  $a \notin \lambda_w$  or  $a \in \lambda'_w$ , else  $e \in \mathcal{G}_{\Lambda^w}$ ; either way, the jump  $a \rightarrow w$  is in  $\mathcal{G}_\Lambda$ .  $\square$

LEMMA 4.25. *Every non-empty union  $S$  of switching cycles in  $\mathcal{G}_\theta$  has a jump out of it: for some axiom link  $a \in S$  and  $\&$ -vertex  $w \notin S$ , there is a jump  $a \rightarrow w$  in  $\mathcal{G}_\theta$ .*

PROOF. Let  $\Lambda$  be a minimal saturated subset of  $\theta$  with  $\mathcal{G}_\Lambda$  containing  $S$ . By (P2),  $\mathfrak{S}$ -switchings of singleton subsets of  $\theta$  are acyclic, so  $\Lambda$  contains at least two linkings. Let  $w$  be a  $\&$  toggled by  $\Lambda$  that is not in any switching cycle in  $\mathcal{G}_\Lambda$  (existing by (P3)), so  $w \notin S$ . Since  $\Lambda$  is minimal,  $S \not\subseteq \mathcal{G}_{\Lambda^w}$  (using (S1)), so some edge  $e$  of  $S$  is in  $\mathcal{G}_\Lambda$  but not in  $\mathcal{G}_{\Lambda^w}$ . Without loss of generality  $e$  is an edge from an axiom link  $a$ , because for any other edge  $y \rightarrow x$  in  $S$  we have  $a \rightarrow z_1 \rightarrow \dots \rightarrow z_n = y \rightarrow x$  in  $S$  for some axiom link  $a$ , and  $y \rightarrow x$  is in  $\mathcal{G}_{\Lambda^w}$  whenever  $a \rightarrow z_1$  is in  $\mathcal{G}_{\Lambda^w}$ . By Lemma 4.24 the jump  $a \rightarrow w$  is in  $\mathcal{G}_\Lambda$ , hence also in  $\mathcal{G}_\theta$ .  $\square$

Henceforth “ $\mathfrak{S}/\&$ ” abbreviates “ $\mathfrak{S}$  or  $\&$ ”. A **path** from  $x_0$  to  $x_n$  in  $\mathcal{G}_\theta$  is a sequence of distinct vertices  $x_0 x_1 \dots x_n$  such that for each  $i$  either  $x_i \rightarrow x_{i+1}$  or  $x_i \leftarrow x_{i+1}$  (note that we do not allow cycles). A **switching path** is a path in  $\mathcal{G}_\theta$  that does not traverse two switch edges of any  $\mathfrak{S}$  or  $\&$ . A switching path  $x_0 \dots x_n$  is **strong** if it does not end by entering a  $\mathfrak{S}/\&$  along one of its switch edges (i.e.,  $x_{n-1} \rightarrow x_n$  only if  $x_n$  is not a  $\mathfrak{S}/\&$ ). Suppose paths  $\pi = x_0 \dots x_n$  and  $\pi' = y_0 \dots y_m$  are disjoint but for  $x_n = y_0$ . If  $\pi$  and  $\pi'$  are switching paths, then their composite  $\pi; \pi' = x_0 \dots x_n y_1 \dots y_m$  is not necessarily a switching path (namely if  $x_n = y_0$  is a  $\&/\mathfrak{S}$  and  $x_{n-1} \rightarrow x_n = y_0 \leftarrow y_1$ ), whereas if  $\pi$  and  $\pi'$  are strong switching paths then  $\pi; \pi'$  is also a strong switching path.

Write  $x \leftarrow_G y$  if the subgraph  $G$  of  $\mathcal{G}_\theta$  contains a strong switching path from  $x$  to  $y$ . An  **$x$ -zone**  $X$  is a subgraph of  $\mathcal{G}_\theta$  such that for all  $x' \in X$ , there exists  $y$  such that  $x \leftarrow y \leftarrow_X x'$ .<sup>8</sup> Given a  $\mathfrak{S}/\&$ -vertex  $x$  and a vertex  $y$ , define  $x$  **dominates**  $y$ , denoted  $x \sqsupset y$ , if  $y$  is in an  $x$ -zone. If  $x$  is not dominated, it is **free**.

<sup>8</sup>The union of all  $x$ -zones is itself an  $x$ -zone, which could be called the **realm** of  $x$ , a concept reminiscent of the notion of **empire** of [Girard 1996], but different in an essential way.

LEMMA 4.26. *Domination  $\sqsupset$  is transitive.*

PROOF. We show that if  $X$  is an  $x$ -zone,  $y \in X$  and  $Y$  is a  $y$ -zone, then  $X \cup Y$  is an  $x$ -zone. Take  $z \in Y \setminus X$ . We have  $x \leftarrow x' \leftarrow_X y \leftarrow y' \leftarrow_Y z$  for some  $x' \in X$  and  $y' \in Y$ . If the strong switching path  $y' \leftarrow_Y z$  does not intersect  $X$ , then  $x' \leftarrow_X y \leftarrow y' \leftarrow_Y z$  constitutes a strong switching path, so we are done. Otherwise let  $y''$  be the last vertex along  $y' \leftarrow_Y z$  that is in  $X$ . Since  $y'' \in X$  we have  $x \leftarrow x'' \leftarrow_X y''$ , and the sub-path of  $y' \leftarrow_Y z$  from  $y''$  to  $z$  is a strong switching path  $y'' \leftarrow_Y z$ ; the composition of these paths yields  $x \leftarrow x'' \leftarrow_{X \cup Y} z$ , since the only common vertex is  $y''$ .  $\square$

LEMMA 4.27. *Let  $C$  be a switching cycle in  $\mathcal{G}_\theta$  containing an axiom link with a jump to a  $\&$ -vertex  $w \notin C$ . Then  $w$  dominates every vertex of  $C$ .*

PROOF.  $C$  is a  $w$ -zone.  $\square$

LEMMA 4.28. *If  $x$  is in a switching cycle in  $\mathcal{G}_\theta$  then  $w \sqsupset x$  for some  $\&$ -vertex  $w$  in no switching cycle in  $\mathcal{G}_\theta$ .*

PROOF. Apply Lemma 4.25 repeatedly, growing a collection of switching cycles one cycle at a time, until jumping to a  $\&$ -vertex  $w$  that is not in a switching cycle. This must happen eventually, since (P3) implies  $\mathcal{G}_\theta$  contains a  $\&$  that is not in any switching cycle. The result follows by Lemma 4.27 and the transitivity of  $\sqsupset$ .  $\square$

LEMMA 4.29. *If  $x \sqsupset x$  then  $x$  is in a switching cycle in  $\mathcal{G}_\theta$ .*

PROOF. If  $x \sqsupset x$  then  $x \leftarrow y \leftarrow_X x$  for some  $x$ -zone  $X$ , hence  $x$  is in a switching cycle in  $\mathcal{G}_\theta$ .  $\square$

LEMMA 4.30. *Every  $\mathfrak{A}/\&$  of  $\mathcal{G}_\theta$  is either free or is dominated by a free  $\mathfrak{A}/\&$ .*

PROOF. If  $x_0$  is neither free nor dominated by a free  $\mathfrak{A}/\&$ -vertex, then we can build an infinite chain  $x_0 \sqsupset x_1 \sqsupset \dots$  of distinct vertices with the same property. If  $x_i$  is in a switching cycle then take  $x_{i+1}$  to be the vertex given by Lemma 4.28, which is not in a switching cycle;  $x_{i+1}$  is fresh otherwise  $x_{i+1} \sqsupset x_{i+1}$  whence  $x_{i+1}$  is in a switching cycle (contradiction). If  $x_i$  is in no switching cycle then  $x_{i+1}$  exists since  $x_i$  is not free;  $x_{i+1}$  is fresh otherwise  $x_i \sqsupset x_i$ , whence  $x_i$  is in a switching cycle (contradiction).  $\square$

COROLLARY 4.31. *If  $\mathcal{G}_\theta$  has a  $\mathfrak{A}/\&$  then it has a free  $\mathfrak{A}/\&$ .*

LEMMA 4.32. *If  $x \sqsupset y_0$  and there is a path  $y_0 y_1 \dots y_n$  in  $\mathcal{G}_\theta$  which never enters a  $\mathfrak{A}/\&$  from above (i.e.,  $y_{i-1} \rightarrow y_i$  only if  $y_i$  is not a  $\mathfrak{A}/\&$ ), then  $x \sqsupset y_n$ .*

PROOF. Let  $X$  be an  $x$ -zone containing  $y_0$ , and let  $y_i$  be the last vertex on  $y_0 y_1 \dots y_n$  that is in  $X$ . Then  $x \leftarrow z \leftarrow_X y_i$  for some  $z$ . Now  $Y = X \cup \{y_i, \dots, y_n\}$  is an  $x$ -zone, since  $x \leftarrow z \leftarrow_Y y_i \leftarrow_Y y_j$  for  $j = i, \dots, n$ .  $\square$

Distinct  $\mathfrak{A}/\&$ -vertices  $x$  and  $y$  or  $\mathcal{G}_\theta$  are **back-to-back**, denoted  $x \longleftrightarrow y$ , if there is a switching path  $x z_0 \dots z_n y$  in  $\mathcal{G}_\theta$  such that  $x \leftarrow z_0$  and  $z_n \rightarrow y$ , and are **face-to-face**, denoted  $x \rightarrow \leftarrow y$ , if there exists a path  $x z_0 \dots z_n y$  in  $\mathcal{G}_\theta$  such that  $x \rightarrow z_0$  and  $z_n \leftarrow y$ , and none of the  $z_i$  are  $\mathfrak{A}/\&$ -vertices (so in particular  $x z_0 \dots z_n y$  is a strong switching path).

LEMMA 4.33. *If  $x \sqsupset z$  and  $y \sqsupset z$  for distinct free  $\mathfrak{A}/\&$ -vertices  $x$  and  $y$ , then  $x \longleftrightarrow y$ .*

PROOF. Let  $X$  be an  $x$ -zone containing  $z$ , so that for some  $x'$  we have  $x \leftarrow x'$  and a strong switching path  $\pi = x' \dots z$  in  $X$ . Let  $z'$  be the first vertex of  $x\pi$  with  $y \sqsupset z'$ . By Lemma 4.32 the predecessor  $z''$  of  $z'$  in  $x\pi$  is a  $\mathfrak{A}/\&$  and  $z'' \leftarrow z'$ , so the prefix  $\pi'$  of  $\pi$  up to  $z'$  is strong. Since  $y \sqsupset z'$  there is a strong switching path  $\pi'' = y_1 \dots y_n z'$  in a  $y$ -zone, with  $y \leftarrow y_1$ . The concatenation  $x\pi' y_n \dots y_1 y$  is a switching path (since none of the  $y_i$  is in  $\pi'$ ) witnessing  $x \longleftrightarrow y$ .  $\square$

A  $\mathfrak{A}/\&$ -vertex  $x$  of  $\mathcal{G}_\theta$  **separates** if it is not an argument (i.e., is an outermost connective), or it is the argument of  $y$  and deleting the edge between  $x$  and  $y$  disconnects  $\mathcal{G}_\theta$ .

LEMMA 4.34. *If a  $\mathfrak{A}/\&$ -vertex  $x$  is free and does not separate, then  $x \rightarrow\leftarrow y$  and  $x \longleftrightarrow z$  for free  $y$  and  $z$ .*

PROOF. Since  $x$  does not separate, it is in a cycle  $C$  (say clockwise) whose first (resp. last) edge is oriented out of (resp. into)  $x$ . Take  $y$  to be the first  $\mathfrak{A}/\&$  reached clockwise along  $C$  from  $x$ . Then  $x \rightarrow\leftarrow y$  (otherwise  $y \sqsupset x$  by Lemma 4.32) and  $y$  is free since  $y' \sqsupset y$  implies  $y' \sqsupset x$  by Lemma 4.32, contradicting the freedom of  $x$ .

Let  $z$  be the first vertex reached anti-clockwise from  $x$  that is not dominated by  $x$ , and let  $z'$  be its predecessor. By Lemma 4.32,  $z$  is a  $\mathfrak{A}/\&$ , and  $x \sqsupset z' \rightarrow z$ , therefore  $x \longleftrightarrow z$ . If  $z$  is not free, replace it by a free  $\mathfrak{A}/\&$  dominating  $z$  (hence also  $z'$ ) provided by Lemma 4.30, and appeal to Lemma 4.33.  $\square$

LEMMA 4.35. *Let  $x$  be a  $\mathfrak{A}/\&$  and let  $z_0 \dots z_n$  be a switching path in  $\mathcal{G}_\theta$  such that  $z_0 \rightarrow x$  and  $z_n \rightarrow x$ . Then  $x$  dominates every vertex of  $\{z_0, \dots, z_n\}$ .*

PROOF.  $\{z_0, \dots, z_n\}$  is a  $x$ -zone.  $\square$

All the auxiliary material is in place for us to prove the Separation Lemma (if  $\mathcal{G}_\theta$  has a  $\mathfrak{A}/\&$  then it has a separating  $\mathfrak{A}/\&$ ).

PROOF OF LEMMA 4.19 (SEPARATION LEMMA). Had  $\mathcal{G}_\theta$  no separating  $\mathfrak{A}/\&$  then  $x_0 \rightarrow\leftarrow x_1 \longleftrightarrow x_2 \rightarrow\leftarrow x_3 \longleftrightarrow \dots$  for free  $\mathfrak{A}/\&$ -vertices  $x_i$  with  $x_{i+1} \neq x_i$  by Lemma 4.34 ( $x_0$  exists by Corollary 4.31). The composition  $\pi$  of the paths witnessing the  $\rightarrow\leftarrow$  and  $\longleftrightarrow$  relations eventually intersects itself at a vertex  $x$ , yielding a path  $\pi' = x z_0 \dots z_n$  such that  $\{x, z_0, \dots, z_n\}$  is a cycle. Since each witness is a switching path,  $\pi'$  is a switching path (by design, composition at each  $x_i$  avoids introducing consecutive switch edges of  $x_i$ ). Furthermore, one of the  $x_i$  must be among the  $z_j$ . Using Lemma 4.28 if  $\{x, z_0, \dots, z_n\}$  is a switching cycle, and Lemma 4.35 otherwise, this  $x_i$  is dominated, a contradiction (since  $x_i$  is free).  $\square$

#### 4.10 Main induction in the proof of the cut-free sequentialisation theorem

With the Separation Lemma in hand, the proof that every cut-free proof net is the translation of a cut-free proof reduces to simple induction.

Let  $\theta$  be a proof net on  $\Gamma$ . We proceed by induction on the sum the number of  $\mathfrak{A}$ s and  $\&$ s of  $\mathcal{G}_\theta$ .

**Base case (primary induction)**  $\Gamma$  is  $\wp/\&$ -free, hence  $\theta$  comprises a single linking  $\lambda$  on  $\Gamma$ . We proceed by induction on the number of connectives of  $\Gamma$ .

- *Base case (secondary induction).*  $\Gamma$  contains no connectives, so  $\Gamma$  has the form  $P_1, P_1^\perp, \dots, P_n, P_n^\perp$  for  $n \geq 0$  and propositional variables  $P_1, \dots, P_n$ , and  $\lambda$  links the complementary literal occurrences  $P_i$  and  $P_i^\perp$  for  $i = 1, \dots, n$ . By (P2)  $n = 1$ . The axiom rule with conclusion  $P_1, P_1^\perp$  is a sequentialisation of  $\theta$ .
- *Induction step (secondary induction).* With no  $\wp$ s,  $\mathcal{G}_\lambda$  is the only  $\wp$ -switching of  $\lambda$ , so by (P2)  $\mathcal{G}_\lambda$  is a tree.
  - Suppose  $\Gamma = \Delta, A \oplus B$ , with  $\oplus$ -vertex  $x \in \mathcal{G}_\lambda$  corresponding to  $A \oplus B$ . Since  $\Gamma \upharpoonright \lambda$  is an additive resolution,  $x$  is unary in  $\mathcal{G}_\lambda$ , i.e., there is a unique  $y \in \mathcal{G}_\lambda$  with  $y \rightarrow x$ . Depending on whether  $y$  is the left/right argument of  $x$ , let  $\rho$  be a left/right  $\oplus$ -rule, with conclusion  $\Delta, A \oplus B$  and hypothesis  $\Gamma' = \Delta, A$  or  $\Delta, B$ , correspondingly. The linking  $\lambda$  on  $\Gamma$  also constitutes a linking  $\lambda'$  on  $\Gamma'$ , since no literals of the deleted  $\oplus$ -argument were incident with an axiom link of  $\lambda$ . The graph  $\mathcal{G}_{\lambda'}$  is a tree, because  $\mathcal{G}_\lambda$  is a tree. Hence  $\theta' = \{\lambda'\}$  is a proof net on  $\Gamma'$ . By induction,  $\theta'$  is the translation of a cut-free MALL proof of  $\Gamma'$ , which when followed by  $\rho$  constitutes a cut-free MALL proof of  $\Gamma$  whose translation is  $\theta$ .
  - Suppose  $\Gamma = \Delta, A_0 \otimes A_1$ , with  $\otimes$ -vertex  $x \in \mathcal{G}_\lambda$  corresponding to  $A_0 \otimes A_1$ . Deleting  $x$  separates the tree  $\mathcal{G}_\lambda$  into a left tree  $T_0$  and right tree  $T_1$  whose respective conclusions define sequents  $\Delta_0$  and  $\Delta_1$ , a partitioning of  $\Delta$ . Let  $\rho$  be a  $\otimes$ -rule with conclusion  $\Gamma$  and hypotheses  $\Delta_0, A_0$  and  $\Delta_1, A_1$ . Since  $\mathcal{G}_\lambda$  is a tree, no axiom link of  $\lambda$  goes between  $\Delta_0, A_0$  and  $\Delta_1, A_1$ , hence  $\lambda$  partitions to form linkings  $\lambda_0$  and  $\lambda_1$  on  $\Delta_0, A_0$  and  $\Delta_1, A_1$ , respectively. Each  $\theta_i = \{\lambda_i\}$  is a proof net on  $\Delta_i, A_i$  since each  $\mathcal{G}_{\lambda_i} = T_i$  is a tree. Appeal to the induction hypothesis with  $\theta_0$  and  $\theta_1$ , in the manner of the  $\oplus$  case above.

**Induction step (primary induction)**  $\Gamma$  has at least one  $\wp$  or  $\&$ . By (P2)  $\mathcal{G}_\theta$  is connected.

- Suppose  $\Gamma = \Delta, A \wp B$ , with  $\wp$ -vertex  $x \in \mathcal{G}_\theta$  corresponding to  $A \wp B$ . Let  $\rho$  be a  $\wp$ -rule with conclusion  $\Gamma$  and hypothesis  $\Gamma' = \Delta, A, B$ . The sequents  $\Gamma$  and  $\Gamma'$  have the same literal occurrences and essentially the same  $\&$ - and additive resolutions, so  $\theta$  constitutes a proof structure  $\theta'$  on  $\Gamma'$ . The  $\wp$ -switchings of the linkings of  $\theta'$  are trees, since they are obtained from those of  $\theta$  by deleting  $x$ . Moreover,  $\theta$  and  $\theta'$  have the same subsets  $\Lambda$  of linkings, toggling the same  $\&$ s and having the same switching cycles (because the vertex  $x$  cannot be in a switching cycle of any  $\Lambda \subseteq \theta$ ). Therefore  $\theta'$  is a proof net on  $\Gamma'$ . Appeal to the induction hypothesis with  $\theta'$ .
- Suppose  $\Gamma = \Delta, A_0 \& A_1$ , with vertex  $w \in \mathcal{G}_\theta$  corresponding to  $A_0 \& A_1$ . Let  $\rho$  be a  $\&$ -rule with conclusion  $\Gamma$  and left and right hypotheses  $\Gamma_0 = \Delta, A_0$  and  $\Gamma_1 = \Delta, A_1$ , respectively. Define the sets of linkings  $\theta_i$  on  $\Gamma_i$  to comprise those linkings of  $\theta$  which are on  $\Gamma_i \subseteq \Gamma$ . Each  $\theta_i$  is a proof net since any switching cycle of  $\theta_i$  is a switching cycle of  $\mathcal{G}_\theta$ . Appeal to the induction hypothesis with each  $\theta_i$ .
- Suppose  $\mathcal{G}_\theta$  has no  $\rightarrow$ -terminal (i.e. concluding)  $\wp$  or  $\&$ . By the Separation

Lemma  $\mathcal{G}_\theta$  has a  $\mathfrak{N}/\&$ -vertex  $x$  such that the deletion of the edge  $x \rightarrow y$  disconnects  $\mathcal{G}_\theta$  into  $G_0$  and  $G_1$ .

Let  $G_0$  be the component containing  $x$ , and let  $\Gamma_0$  comprise the formula-occurrences corresponding to the  $\rightarrow$ -terminal vertices of  $G_0$  (some formulas of  $\Gamma$  together with the subformula occurrence  $A\&B$  corresponding to  $x$ ). Define  $\theta_0 = \{\lambda \upharpoonright \Gamma_0 : \lambda \in \theta\}$  on  $\Gamma_0$  (each  $\lambda \upharpoonright \Gamma_0$  is well-defined since no  $a \in \lambda$  goes between  $G_0$  and  $G_1$ ).

Let  $\Gamma_1$  be the subsequent of  $\Gamma$  containing the formulas corresponding to the  $\rightarrow$ -terminal vertices of  $G_1$ . In  $G_1$ ,  $y$  is  $\rightarrow$ -initial. Form  $G_1^+$  by adding literals  $P$  and  $P^\perp$ , the axiom link  $a = \{P, P^\perp\}$ , and edges  $y \leftarrow P \leftarrow a \rightarrow P^\perp$ . Let  $\widehat{\Gamma}_1$  be  $\Gamma_1$  with  $P$  substituted for the subformula occurrence  $A\&B$  corresponding to  $x$ , and let  $\Gamma_1^+ = \widehat{\Gamma}_1, P^\perp$ . Define  $\theta_1 = \{\lambda \upharpoonright \widehat{\Gamma}_1 \cup \{a\} : \lambda \in \theta\}$  on  $\Gamma_1^+$ .

*Claim:*  $x \in \Gamma \upharpoonright \lambda$  for all  $\lambda \in \theta$ .

*Proof.* If not, there is  $\lambda \in \theta$  and a  $\&$ -vertex  $w$  with  $x$  in  $\Gamma \upharpoonright \lambda$  but not in  $\Gamma \upharpoonright \lambda_w$  for some  $\lambda_w \in \theta$  such that  $\lambda \stackrel{w}{=} \lambda_w$ . Thus there is a jump  $b \rightarrow w$  in  $\mathcal{G}_\theta$  for some  $b \in G_0$  with  $b \in \lambda \setminus \lambda_w$ . Since linkings are total on additive resolutions there exists an axiom link  $c \in \lambda_w \setminus \lambda$  connecting to the formula containing  $x$ , but not satisfying  $c \rightarrow \dots \rightarrow x$ , so there is a jump  $c \rightarrow w$  in  $\mathcal{G}_\theta$ . If  $w \in G_0$  then  $c \rightarrow w$  is a jump from  $G_1$  to  $G_0$ , and if  $w \in G_1$  then  $b \rightarrow w$  is a jump from  $G_0$  to  $G_1$ ; either case violates the disconnectedness of  $G_0$  from  $G_1$ .  $\blacksquare$

The claim implies that  $\theta_0$  and  $\theta_1$  are sets of linkings on  $\Gamma_0$  and  $\Gamma_1^+$ , respectively. Moreover,  $\mathcal{G}_{\theta_0} = G_0$  and  $\mathcal{G}_{\theta_1} = G_1^+$ . We now check that  $\theta_0$  and  $\theta_1$  are proof nets, i.e., satisfy (P1)–(P3). Since  $\theta$  satisfies (P1),  $\theta_0$  (resp.  $\theta_1$ ) has at least one linking on every  $\&$ -resolution of  $\Gamma_0$  (resp.  $\Gamma_1^+$ ). Had  $\theta_i$  two distinct linkings on the same  $\&$ -resolution, there would be a jump from an axiom link in  $G_i$  to a  $\&$  in  $G_{1-i}$ , violating the disconnectedness of  $G_0$  from  $G_1$ . Thus  $\theta_i$  satisfies (P1). (P2) is trivially inherited from  $\theta$ . Finally, (P3) holds since any set of linkings  $\Lambda'$  in  $\theta_0$  or  $\theta_1$  corresponds to a set of linkings  $\Lambda$  in  $\theta$  toggling the same  $\&$ s, such that any switching cycle in  $\mathcal{G}_{\Lambda'}$  is a switching cycle in  $\mathcal{G}_\Lambda$ .

By induction  $\theta_0$  is the translation of a cut-free proof  $\Pi_0$  of  $\Gamma_0$  and likewise  $\theta_1$  is the translation of  $\Pi_1$ . Substituting  $\Pi_0$  for the axiom rule with conclusion  $P, P^\perp$  in  $\Pi_1$  yields a proof whose translation is  $\theta$ .  $\square$

In the case of  $\text{MALL}^{\text{mix}}$ , the connectedness requirement of (P2) does not apply. This condition is used three times in the above proof. To prove that a set of linkings is the translation of a cut-free  $\text{MALL}^{\text{mix}}$  proof if it is a cut-free mix net, in each part of the inductive proof above, the case that  $\mathcal{G}_\theta$  is not connected can be dealt with by partitioning  $\Gamma$  into a number of non-empty subsequents  $\Gamma_i$ , each harbouring a connected component of  $\mathcal{G}_\theta$ . The mix net  $\theta$  projects to mix nets  $\theta_i$  on  $\Gamma_i$ , which by induction are translations of cut-free  $\text{MALL}^{\text{mix}}$  proofs  $\Pi_i$ . By the mix rule these combine into a proof that translates to  $\theta$ .

## 5 Cut

A *cut* is an unordered pair  $\{A, A^\perp\}$  of complementary MALL formulas. We write  $A * A^\perp$  for  $\{A, A^\perp\}$ , and treat  $A * A^\perp$  akin to a MALL formula, referring to  $*$  as the *cut connective*. A *cut sequent* is a non-empty set of occurrences of MALL

$$\begin{array}{c}
 \frac{}{\{\{P, P^\perp\}\} \triangleright [] P, P^\perp} \text{ax} \qquad \frac{\theta \triangleright [\Omega] \Gamma, A \qquad \theta' \triangleright [\Omega'] A^\perp, \Delta}{\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\} \triangleright [\Omega, \Omega', A * A^\perp] \Gamma, \Delta} \text{cut} \\
 \\
 \frac{\theta \triangleright [\Sigma, \Omega] \Gamma, A \qquad \theta' \triangleright [\Sigma, \Omega'] \Gamma, B}{\theta \cup \theta' \triangleright [\Sigma, \Omega, \Omega'] \Gamma, A \& B} \& \qquad \frac{\theta \triangleright [\Omega] \Gamma, A \qquad \theta' \triangleright [\Omega'] B, \Delta}{\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\} \triangleright [\Omega, \Omega'] \Gamma, A \otimes B, \Delta} \otimes \\
 \\
 \frac{\theta \triangleright [\Omega] \Gamma, A}{\theta \triangleright [\Omega] \Gamma, A \oplus B} \oplus_1 \qquad \frac{\theta \triangleright [\Omega] \Gamma, B}{\theta \triangleright [\Omega] \Gamma, A \oplus B} \oplus_2 \qquad \frac{\theta \triangleright [\Omega] \Gamma, A, B}{\theta \triangleright [\Omega] \Gamma, A \wp B} \wp
 \end{array}$$

Table II. Rules for deriving sequentialisable sets of linkings on MALL cut sequents. Here  $\theta \triangleright [\Omega] \Gamma$  is the judgement “ $\theta$  is a sequentialisable set of linkings on the cut sequent  $[\Omega] \Gamma$ ”. We use the implicit tracking of literal occurrences downwards through rules. The base case **ax** is a single linking with a single axiom link and no cuts.

formulas and cuts. A **cut-additive resolution** of a cut sequent  $\Delta$  is any result of deleting some cuts from  $\Delta$  and one argument subtree of every remaining additive connective ( $\&$  or  $\oplus$ ). Thus every remaining  $\&$  and  $\oplus$  is unary.

*Example 5.1.* Here is a cut sequent followed by one of its cut-additive resolutions:

$$\begin{array}{c}
 P \otimes P, Q * Q^\perp, P^\perp \oplus Q, (R \oplus S) * (R^\perp \& S^\perp) \\
 P \otimes P, \boxed{Q * Q^\perp}, P^\perp \oplus \boxed{Q}, \boxed{(R \oplus S) * (R^\perp \& S^\perp)}
 \end{array}$$

An **axiom link** on a cut sequent  $\Delta$  is a pair of complementary literal occurrences in  $\Delta$ . A **linking** on  $\Delta$  is a partitioning of the literal occurrences of a cut-additive resolution  $\Delta \upharpoonright \lambda$  of  $\Delta$  into axiom links.

Write  $[\Omega] \Gamma$  for the cut sequent obtained by taking the disjoint union of a set  $\Omega$  of cut occurrences and a MALL sequent  $\Gamma$ . A set of linkings on  $[\Omega] \Gamma$  is **sequentialisable** if it can be derived from the rules in Table II. Figures 2 (page 3) and 14 (page 36) show examples.

Any derivation of a set of linkings using the rules of Table II projects in an obvious way to a MALL proof, namely, by restricting to the underlying sequents (viz., read  $\Gamma$  for  $\theta \triangleright [\Omega] \Gamma$ ). For example, the two derivations of Figure 14 each yield the same MALL proof of  $P, P^\perp \& P^\perp$ .

Write  $\Pi \rightsquigarrow \theta$  if  $\Pi$  is the MALL proof obtained from a derivation of a set of linkings  $\theta$ , and say that  $\Pi$  is a **sequentialisation** of  $\theta$ . If a MALL proof  $\Pi'$  can be obtained from  $\Pi$  by a series of rule commutations in which no  $\&$ -rules are moved upwards, then  $\Pi$  and  $\Pi'$  are sequentialisations of the same set of linkings. In the cut-free case,  $\rightsquigarrow$  is a function from proofs to sets of linkings, exactly the translation defined in Table I. In the presence of cuts, more than one set of linkings may correspond to the same proof. For example, since the two derivations in Figure 14 have the same underlying MALL proof (say  $\Pi$ ), the concluding sets of linkings (say  $\theta$  and  $\theta'$ ) have a common sequentialisation:  $\Pi \rightsquigarrow \theta$ ,  $\Pi \rightsquigarrow \theta'$ , and  $\theta \neq \theta'$ .

We can of course extend the cut-free translation of proofs by always choosing  $\Sigma$  to be empty in the  $\&$ -rule (i.e., “never superimpose cuts”). However, our notion of proof net defined below, which characterises sequentialisability, does not characterise the image of this translation, since there would exist sequentialisable sets of linkings that are not proof translations, such as

$$\begin{array}{c}
\frac{\frac{\overline{\text{ax}}}{\overline{P, P^\perp}} \quad \frac{\overline{\text{ax}}}{\overline{P, P^\perp}}}{\overline{P, P^\perp * P, P^\perp}} \text{ cut} \quad \frac{\frac{\overline{\text{ax}}}{\overline{P, P^\perp}} \quad \frac{\overline{\text{ax}}}{\overline{P, P^\perp}}}{\overline{P, P^\perp * P, P^\perp}} \text{ cut}}{\overline{P, P^\perp * P, P^\perp * P, P^\perp \& P^\perp}} \& \\
\\
\frac{\frac{\overline{\text{ax}}}{\overline{P, P^\perp}} \quad \frac{\overline{\text{ax}}}{\overline{P, P^\perp}}}{\overline{P, P^\perp * P, P^\perp}} \text{ cut} \quad \frac{\frac{\overline{\text{ax}}}{\overline{P, P^\perp}} \quad \frac{\overline{\text{ax}}}{\overline{P, P^\perp}}}{\overline{P, P^\perp * P, P^\perp}} \text{ cut}}{\overline{P, P^\perp * P, P^\perp \& P^\perp}} \&
\end{array}$$

Fig. 14. Examples of the translation of a proof with cuts. Each of the conclusions is a set of two linkings, one drawn above the cut sequent and one drawn below. The only difference between the derivations is the final  $\&$ -rule. The first application keeps the cuts in the hypotheses separate (an instance of the  $\&$ -rule taking  $\Sigma$  empty and  $\Omega = \Omega' = P^\perp * P$ ), whereas the second application superimposes the two cuts ( $\Sigma = P^\perp * P$  and each of  $\Omega$  and  $\Omega'$  empty).

$$\overline{P \oplus \overline{P, P^\perp * P, P^\perp} \& P^\perp}$$

Moreover, under this convention two proofs that differ only in a commutation of cut and  $\&$ -rules would be translated to different sets of linkings.

Note that the alternative of taking  $\Sigma$  maximal (i.e., “superimpose as many cuts as possible”) does not define a canonical function from proofs to sets of linkings, since there may be a choice of how to make the identifications. The following two  $\&$ -rules illustrate such a choice.

$$\begin{array}{c}
\frac{\overline{P, P^\perp * P, P^\perp * P, P^\perp} \quad \overline{P, P^\perp * P, P^\perp * P, P^\perp}}{\overline{P, P^\perp * P, P^\perp * P, P^\perp \& P^\perp}} \& \\
\\
\frac{\overline{P, P^\perp * P, P^\perp * P, P^\perp} \quad \overline{P, P^\perp * P, P^\perp * P, P^\perp}}{\overline{P, P^\perp * P, P^\perp * P, P^\perp \& P^\perp}} \&
\end{array}$$

Girard was aware of this issue in the context of monomial proof nets; see Appendix A.1.6 of [1996].

### 5.1 Geometric characterisation of sequentialisability

In the presence of cut, we update all the auxiliary definitions of Section 4 ( $\&$ -resolution,  $\mathcal{G}_\Delta$ , switching cycle, etc.) by substituting “cut sequent” for “sequent” and “cut-additive resolution” for “additive resolution” throughout.

**Definition: MALL proof net**

**cut:** unordered pair of complementary MALL formulas.  
**cut sequent:** non-empty set of occurrences of formulas and cuts.  
**&-resolution:** deletion of one argument subtree of each &.  
**Cut-additive resolution:** deletion of some cuts and one argument subtree of each  $\oplus/\&$ .  
**Linking**  $\lambda$  on a cut sequent  $\Delta$ : partitioning of the literal occurrences of a cut-additive resolution  $\Delta \upharpoonright \lambda$  of  $\Delta$  into complementary pairs (*axiom links*).  
 A set  $\Lambda$  of linkings on  $\Delta$  *toggles* a &  $w$  if both arguments of  $w$  are in  $\Delta \upharpoonright \Lambda \equiv \bigcup_{\lambda \in \Lambda} \Delta \upharpoonright \lambda$ .  
**Graph**  $\mathcal{G}_\Lambda$ :  $\Delta \upharpoonright \Lambda$  + each axiom link  $a = \{l, l'\}$  of  $\Lambda$ , with edges  $l-a-l'$  and a **jump** edge  $a-w$  whenever  $a \in \lambda \setminus \lambda'$  and  $w$  is the unique & toggled by  $\{\lambda, \lambda'\} \subseteq \Lambda$ .  
**Switching cycle:** cycle with  $\leq 1$  **switch edge** (= jump or argument edge) of each  $\wp/\&$ .  
 A set  $\theta$  of linkings on  $\Delta$  is a **proof net** if it satisfies:  
 CUT: Every cut has a literal in some axiom link of  $\theta$ .  
 RESOLUTION: Exactly one linking of  $\theta$  is on any given &-resolution of  $\Delta$ .  
 MLL: Every  $\wp$ -switching of every linking of  $\theta$  is a tree (acyclic and connected<sup>9</sup>).  
 TOGGLING: Every set  $\Lambda$  of  $\geq 2$  linkings of  $\theta$  toggles a & that is in no switching cycle in  $\mathcal{G}_\Lambda$ .<sup>10</sup>

*Definition 5.2.* A set  $\theta$  of linkings on a cut sequent  $\Delta$  is a **proof net** if:

- (P0) CUT. At least one literal occurrence of every cut is in  $\theta$  (i.e., in some axiom link of some linking of  $\theta$ ).
- (P1) RESOLUTION. For any &-resolution  $\Delta^*$  of  $\Delta$ , exactly one linking of  $\theta$  is on  $\Delta^*$ .
- (P2) MLL. Every  $\wp$ -switching of every linking of  $\theta$  is a tree (acyclic and connected<sup>9</sup>).
- (P3) TOGGLING. Every set  $\Lambda$  of two or more linkings of  $\theta$  toggles a & that is not in any switching cycle in  $\mathcal{G}_\Lambda$ .<sup>10</sup>

We say that  $\theta$  is a **proof structure** if it satisfies (P0) and (P1).

Note that (P1)–(P3) are inherited from the cut-free case.

*Alternative definitions of proof net.* The material in Section 4.5 still applies now that we have extended proof nets with cut, with one small change: in the equation defining balance, add the number of cuts to the number of tensors. The equivalence proofs (appendices A and B) extend verbatim once a cut vertex is viewed as a tensor.

**THEOREM 5.3 SEQUENTIALISATION.** *A set of linkings on a cut sequent is sequentialisable iff it is a proof net.*

**PROOF.** The contents of Section 4.9 apply verbatim to the case that  $\theta$  is a proof net on a cut sequent  $\Gamma$ . Moreover, the proof of the separation lemma shows that if  $\mathcal{G}_\theta$  has a  $\wp/\&$  then it has a separating  $\wp/\&$  that is free. Now the proof of Theorem 4.18 needs to be adapted in three places to apply to the case with cut. First of all, in the base case of the primary induction, a cut is treated exactly as an outermost tensor. Secondly, in the case  $\Gamma = \Delta, A_0 \& A_1$  some garbage collection needs to be done in order to ensure that  $\theta_1$  and  $\theta_2$  satisfy (P0): cuts for which no literal occurrence is in  $\theta_i$  need to be deleted from  $\Gamma_i$ . Finally, when separating a sequent by applying

<sup>9</sup>By dropping connectedness, we obtain a proof net for MALL augmented by the mix rule.

<sup>10</sup>See footnote 5, page 13.

the separation lemma, care must be taken that the complementarity between the two MALL formulas in a cut is not violated by the separation.

*Claim:* If a cut  $A * A^\perp$  contains a free  $\wp/\&$ -vertex  $x$ , then every linking in  $\theta$  visits literal occurrences in  $A * A^\perp$ .

*Proof.* If not, there is  $\lambda \in \theta$  and a  $\&$ -vertex  $w$  with the cut-vertex  $c$  in  $\Gamma \upharpoonright \lambda$  but not in  $\Gamma \upharpoonright \lambda_w$  for some  $\lambda_w \in \theta$  such that  $\lambda \stackrel{w}{=} \lambda_w$ . Thus there are jumps  $a \rightarrow w$  and  $b \rightarrow w$  in  $\mathcal{G}_\theta$  for axiom links  $a$  and  $b$  in each of  $A$  and  $A^\perp$ . By Lemma 4.35  $w \sqsupset c$  and hence, by Lemma 4.32,  $w \sqsupset x$ , contradicting the freeness of  $x$ . ■

Thus we may assume that when a separating  $\wp/\&$  lays in a cut  $A * A^\perp$ , every linking in  $\theta$  visits that cut. Therefore,  $\theta$  remains a proof net if we replace  $A * A^\perp$  by  $A \otimes A^\perp$ , and then the argument in the proof of Theorem 4.18 yields a sequentialisation. This sequentialisation remains a MALL proof upon replacing the tensor occurrence by the cut connective, and that MALL proof is a sequentialisation of  $\theta$ . □

## 5.2 Cut elimination.

Let  $\theta$  be a set of linkings on the cut sequent  $\Delta$ , and let  $A * A^\perp$  be a cut of  $\Delta$ . Define the *elimination* of  $A * A^\perp$  as follows.

- If  $A$  is a literal, delete  $A * A^\perp$  from  $\Delta$ , and replace any pair of axiom links  $\{l, A\}, \{A^\perp, l'\}$  in a linking of  $\theta$  ( $l$  and  $l'$  being other occurrences of  $A^\perp$  and  $A$  respectively) with the axiom link  $\{l, l'\}$ .
- If  $A = A_1 \otimes A_2$  and  $A^\perp = A_1^\perp \wp A_2^\perp$  (or vice versa), replace  $A * A^\perp$  with two cuts  $A_1 * A_1^\perp$  and  $A_2 * A_2^\perp$ . Retain all the original linkings.
- If  $A = A_1 \& A_2$  and  $A^\perp = A_1^\perp \oplus A_2^\perp$  (or vice versa) replace  $A * A^\perp$  with two cuts  $A_1 * A_1^\perp$  and  $A_2 * A_2^\perp$ . Retain precisely the *inconsistent* linkings, namely those  $\lambda \in \theta$  such that in  $\Delta \upharpoonright \lambda$  the (unary)  $\&$  and  $\oplus$  of the cut take opposite arguments (i.e., such that the right argument of the  $\&$  is in  $\Delta \upharpoonright \lambda$  and the left argument of the  $\oplus$  is in  $\Delta \upharpoonright \lambda$ , or vice versa). Finally, ‘garbage collect’ by deleting  $A_i * A_i^\perp$  if no literal occurrence of  $A_i * A_i^\perp$  is in any of the remaining linkings.

An example of cut elimination was presented in Figure 3.

PROPOSITION 5.4. *Eliminating a cut from a proof net yields a proof net.*

PROOF. Section 5.4. □

THEOREM 5.5. *Cut elimination of proof nets is strongly normalising.*

PROOF. Confluence is immediate from the definition; cut elimination reduces the size of the cut sequent, and is therefore strongly normalising. □

## 5.3 A category of proof nets

Cut elimination gives rise to a category  $\mathcal{N}$  of MALL proof nets. Objects are MALL formulas, and a morphism  $A \rightarrow B$  is a cut-free proof net on the sequent  $A^\perp, B$ . The composition of  $\theta : A \rightarrow B$  and  $\theta' : B \rightarrow C$  is the normal form of the proof net  $\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\}$  on  $A^\perp, B * B^\perp, C$ . See Figure 4 (page 4). Composition is associative, since cut elimination is strongly normalising. The identity  $A \rightarrow A$

contains a linking  $\lambda$  on  $A^\perp$ ,  $A$  iff  $\lambda$  matches the  $i^{\text{th}}$  literal of  $A^\perp$  with the  $i^{\text{th}}$  literal of  $A$ .

This category has a very simple structure as a unit-free star-autonomous category (i.e., all the natural transformations, isomorphisms and coherence diagrams that do not involve units; see Appendix C), and has products and coproducts.

Tensor  $- \otimes - : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$  acts symbolically on objects (i.e., the tensor of formulas  $A$  and  $B$  is the formula  $A \otimes B$ ), and the tensor  $\theta \otimes \theta' : A \otimes C \rightarrow B \otimes D$  of  $\theta : A \rightarrow B$  and  $\theta' : C \rightarrow D$  is obtained as follows, using the notation of Figure I:

$$\frac{\frac{\theta \triangleright A^\perp, B \quad \theta' \triangleright D, C^\perp}{\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\} \triangleright A^\perp, B \otimes D, C^\perp} \otimes}{\theta \otimes \theta' \triangleright A^\perp \wp C^\perp, B \otimes D} \wp$$

Like tensor, duality  $(-)^{\perp} : \mathcal{N}^{\text{op}} \rightarrow \mathcal{N}$  acts symbolically on objects. On morphisms it is trivial, since a proof net on  $A, B$  can be read equally well as a morphism  $A^\perp \rightarrow B$  or  $B^\perp \rightarrow A$ . Associativity is immediate since the formula graphs  $A \otimes (B \otimes C)$  and  $(A \otimes B) \otimes C$  are topologically equivalent, in particular with the same leaves/literal occurrences. (See Figure 8 (page 12) for a simple example of associativity.) Commutativity, and the natural isomorphism  $\mathcal{N}(A \otimes B, C^\perp) \cong \mathcal{N}(A, (B \otimes C)^\perp)$ , are similarly trivial. Finally,  $\&$  is product, and  $\oplus$  is coproduct, each acting syntactically on objects and defined on morphisms in a manner analogous to tensor above. The universal property of  $\&$  follows immediately from the fact that  $\&$  takes the disjoint union of non-empty sets of linkings;  $\oplus$  is dual.

Finally, we note that the category  $\mathcal{N}$  of proof nets is soft [Joyal 1995], that is, any morphism  $\otimes_{1 \leq i \leq m} (A_i \& A'_i) \rightarrow \wp_{1 \leq j \leq n} (B_j \oplus B'_j)$  factorises through either a product projection on the left or a coproduct injection on the right. This is immediate, via sequentialisation, from the corresponding observation at the level of proofs. (Alternatively, it is straightforward to verify softness directly.)

#### 5.4 Proof that eliminating a cut from a proof net yields a proof net

In this section we establish that cut elimination preserves (P0)–(P3). Preservation of (P0) is trivial. Preservation of (P1) for a literal or multiplicative cut is also trivial; for an additive cut it is an immediate consequence of the following lemma.

LEMMA 5.6. *Let  $A * A^\perp$  be an additive cut in a cut sequent  $\Gamma$  with  $A = A_0 \& A_1$  and  $A^\perp = A_0^\perp \oplus A_1^\perp$  (or vice versa), and let  $\lambda, \lambda'$  be linkings of a proof net on  $\Gamma$  such that the cut  $\&$  is the only  $\&$  toggled between  $\Gamma \upharpoonright \lambda$  and  $\Gamma \upharpoonright \lambda'$ . Then  $\lambda$  and  $\lambda'$  take the same argument of  $A^\perp$ , i.e., exactly one of  $A_0^\perp$  and  $A_1^\perp$  occurs in both  $\Gamma \upharpoonright \lambda$  and  $\Gamma \upharpoonright \lambda'$ .*

PROOF. If  $\lambda$  and  $\lambda'$  took opposite arguments of  $A^\perp$ , an axiom link above  $A^\perp$  would depend on the cut  $\&$ . The resulting jump yields a switching cycle of  $\{\lambda, \lambda'\}$  containing the only  $\&$  toggled by  $\{\lambda, \lambda'\}$ , in violation of (P3).  $\square$

Preservation of (P2) is straightforward for a literal or additive cut, since  $\wp$ -switchings correspond before and after the elimination. Preservation of (P2) for a multiplicative cut is a corollary of the well-definedness of cut elimination for MLL proof nets, since the elimination of a multiplicative cut from one of our proof nets corre-

sponds precisely to the parallel elimination of copies of the cut in the induced MLL proof nets. (To be self-contained, we give a direct proof in Appendix D.)

The remainder of this section is devoted to the proof that cut elimination preserves (P3).

Fix a proof net  $\theta$  on a cut sequent  $\Gamma$ . We localise the notion of domination of Section 4.9 from  $\theta$  to any saturated set of linkings  $\Lambda \subseteq \theta$ . Write  $x \rightarrow_\Lambda y$  if the edge  $x \rightarrow y$  of  $\mathcal{G}_\theta$  is in  $\mathcal{G}_\Lambda$ . A subgraph  $X$  of  $\mathcal{G}_\Lambda$  is an ***x-zone under  $\Lambda$***  if for all  $x' \in X$  there exists  $y$  with  $x \leftarrow_\Lambda y \leftarrow_X x'$ ; given a  $\mathfrak{A}/\&$ -vertex  $x \in \mathcal{G}_\Lambda$  and a vertex  $y \in \mathcal{G}_\Lambda$ , define  $x$  ***dominates  $y$  in  $\Lambda$*** , denoted  $x \sqsupset_\Lambda y$ , if  $y \in X$  for some  $x$ -zone  $X$  under  $\Lambda$ . Lemmas 4.25, 4.26, 4.27, 4.28, 4.29, 4.32, and 4.35 of Section 4.9 localise from  $\theta$  to any saturated set of linkings  $\Lambda \subseteq \theta$ , as follows:

LOCALISED LEMMA 4.25. *For every non-empty union  $S$  of switching cycles in  $\mathcal{G}_\Lambda$  there is a jump  $a \rightarrow w$  in  $\mathcal{G}_\Lambda$  between an axiom link  $a \in S$  and a  $\&$ -vertex  $w \notin S$  which is toggled by  $\Lambda$ .*

LOCALISED LEMMA 4.26. *Localised domination  $\sqsupset_\Lambda$  is transitive.*

LOCALISED LEMMA 4.27. *Let  $C$  be a switching cycle in  $\mathcal{G}_\Lambda$  containing an axiom link with a jump to a  $\&$ -vertex  $w \notin C$ . Then  $w \sqsupset_\Lambda x$  for all vertices  $x \in C$ .*

LOCALISED LEMMA 4.28. *If  $x$  is in a switching cycle in  $\mathcal{G}_\Lambda$  then  $w \sqsupset_\Lambda x$  for some  $\&$ -vertex  $w$  toggled by  $\Lambda$  that is in no switching cycle in  $\mathcal{G}_\Lambda$ .*

LOCALISED LEMMA 4.29. *If  $x \sqsupset_\Lambda x$  then  $x$  is in a switching cycle in  $\mathcal{G}_\Lambda$ .*

LOCALISED LEMMA 4.32. *If  $x \sqsupset_\Lambda y_0$  and there is a path  $y_0 y_1 \dots y_n$  in  $\mathcal{G}_\Lambda$  which never enters a  $\mathfrak{A}/\&$  from above, then  $x \sqsupset_\Lambda y_n$ .*

LOCALISED LEMMA 4.35. *Let  $x$  be a  $\mathfrak{A}/\&$  and let  $z_0 \dots z_n$  be a switching path in  $\mathcal{G}_\Lambda$  such that  $z_0 \rightarrow_\Lambda x$  and  $z_n \rightarrow_\Lambda x$ . Then  $x \sqsupset_\Lambda z_i$ , for each  $0 \leq i \leq n$ .*

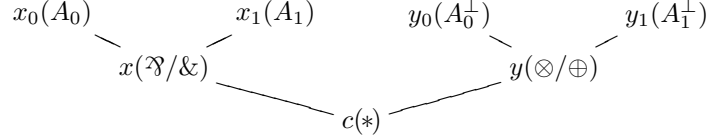
The proof of Localised Lemma 4.25 is a relatively straightforward adaptation of the proof of Lemma 4.25; we present it in full below. The proofs of the remaining localised lemmas are obtained by making the following substitutions in the proofs of the originals in Section 4.9:  $\Lambda$  for  $\theta$ ,  $\sqsupset_\Lambda$  for  $\sqsupset$ ,  $\rightarrow_\Lambda$  for  $\rightarrow$ , and *zone under  $\Lambda$*  for *zone*.

*Proof of Localised Lemma 4.25.* Let  $\Lambda_m$  be a minimal saturated subset of  $\Lambda$  with  $\mathcal{G}_{\Lambda_m}$  containing  $S$ . Switchings of singleton sets of linkings are cycle-free by (P2), so  $\Lambda_m$  contains at least two linkings. Let  $w$  be a  $\&$  toggled by  $\Lambda_m$  that is not in any switching cycle in  $\mathcal{G}_{\Lambda_m}$  (existing by (P3)), so  $w \notin S$ . Since  $\Lambda_m \subseteq \Lambda$ ,  $w$  is certainly toggled by  $\Lambda$ . Since  $\Lambda_m$  is minimal,  $S \not\subseteq \mathcal{G}_{\Lambda_m^w}$  (using (S1)), so some edge  $e$  of  $S$  is in  $\mathcal{G}_{\Lambda_m}$  but not in  $\mathcal{G}_{\Lambda_m^w}$ . Without loss of generality  $e$  is an edge from an axiom link  $a$ , because for any other edge  $y \rightarrow x$  in  $S$  we have  $a \rightarrow z_1 \rightarrow \dots \rightarrow z_n = y \rightarrow x$  in  $S$  for some axiom link  $a$ , and  $y \rightarrow x$  is in  $\mathcal{G}_{\Lambda_m^w}$  whenever  $a \rightarrow z_1$  is in  $\mathcal{G}_{\Lambda_m^w}$ . By Lemma 4.24 the jump  $a \rightarrow w$  is in  $\mathcal{G}_{\Lambda_m}$ , hence also in  $\mathcal{G}_\Lambda$ .  $\square$

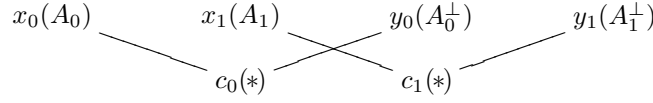
5.4.1 *Proof that cut elimination preserves the toggling condition (P3).* Preservation is immediate for the elimination of a literal cut  $P * P^\perp$ , since for every set  $\Lambda$  of linkings on  $\Gamma$ , the  $\&$ -vertices toggled by  $\Lambda$  and the switching cycles of  $\Lambda$  correspond

before and after the elimination. Thus consider the elimination of an additive cut  $(A_0 \& A_1) * (A_0^\perp \oplus A_1^\perp)$  or a multiplicative cut  $(A_0 \wp A_1) * (A_0^\perp \otimes A_1^\perp)$ .

Let  $\theta'$  on the cut sequent  $\Gamma'$  be the result of eliminating  $(A_0 \& A_1) * (A_0^\perp \oplus A_1^\perp)$  or  $(A_0 \wp A_1) * (A_0^\perp \otimes A_1^\perp)$  from the proof net  $\theta$  on  $\Gamma$ . Let  $x$  be the  $\&$  or  $\wp$  and  $y$  the  $\oplus$  or  $\otimes$  of the cut, let  $x_0, x_1$  and  $y_0, y_1$  be the arguments of  $x$  and  $y$  respectively, and let  $c$  be the cut vertex  $*$  between  $x$  and  $y$ .



Thus in  $\Gamma'$  each of  $c$ ,  $x$  and  $y$  have been deleted, and cut vertices  $c_0$  between  $x_0$  and  $y_0$  and  $c_1$  between  $x_1$  and  $y_1$  have been added,



unless one of  $A_0, A_0^\perp$  or  $A_1, A_1^\perp$  disappeared in the ‘garbage collection’ phase of additive elimination, in which case only one of  $c_0$  or  $c_1$  is present. Suppose  $\theta'$  fails (P3), i.e., there exists a set of two or more linkings  $\Lambda' \subseteq \theta'$  such that every  $\&$  in  $\Gamma'$  toggled by  $\Lambda'$  is in a switching cycle in  $\mathcal{G}_{\Lambda'}$ .

LEMMA 5.7. *There exists a saturated set of linkings  $\Lambda \subseteq \theta$  on  $\Gamma$  such that  $\Lambda$  on  $\Gamma$  toggles the same  $\&$ s as  $\Lambda'$  on  $\Gamma'$ , except perhaps  $x$  in addition (in the case of an additive cut).*

PROOF. Since cut elimination simply deletes linkings,  $\Lambda'$  can also be viewed as a set of linkings on  $\Gamma$ , and  $\Lambda' \subseteq \theta$ . Furthermore,  $\Lambda'$  on  $\Gamma$  toggles exactly the same  $\&$ s as  $\Lambda'$  on  $\Gamma'$ , except perhaps  $x$  in addition (in the case of an additive cut). Let  $\Lambda$  be a minimal saturated set of linkings of  $\theta$  on  $\Gamma$  containing  $\Lambda'$ . By minimality,  $\Lambda$  on  $\Gamma$  toggles the same  $\&$ s as  $\Lambda'$  on  $\Gamma$ .  $\square$

LEMMA 5.8. *The vertex  $y$  is not in a switching cycle in  $\mathcal{G}_\Lambda$ .*

PROOF. If  $y$  is in a switching cycle, then by Localised Lemma 4.28,  $\Lambda$  toggles a  $\&$ -vertex  $w \sqsubset_\Lambda y$  in no switching cycle of  $\Lambda$ . We have  $w \sqsubset_\Lambda x$  by Localised Lemma 4.32. Necessarily  $w \neq x$ , otherwise  $w \sqsubset_\Lambda w$  and by Localised Lemma 4.29  $w$  is in a switching cycle in  $\mathcal{G}_\Lambda$ , a contradiction. By Lemma 5.7,  $w$  is toggled by  $\Lambda'$  on  $\Gamma'$ , hence<sup>11</sup>  $w$  is in a switching cycle  $C$  in  $\mathcal{G}_{\Lambda'}$ .

Suppose  $C$  does not go through both  $c_0$  and  $c_1$ . Then  $C$  induces a switching cycle in  $\mathcal{G}_\Lambda$ , still containing  $w$ , obtained by re-routing a possible passage through  $c_0$  or  $c_1$  to go through  $c$  instead. This yields a contradiction.

Suppose  $C$  goes through both  $c_0$  and  $c_1$ . Re-routing both passages to go through  $c$  instead either yields two switching cycles through  $c$  with  $w$  in one of them, a contradiction, or yields a switching cycle  $C_y$  through  $y$  and a switching path  $P_x = z_0 \dots z_n$  in  $\mathcal{G}_\Lambda$  with  $z_0 \rightarrow_\Lambda x$  and  $z_n \rightarrow_\Lambda x$ , such that  $w$  is either in  $C_y$  or  $P_x$ . The first possibility immediately yields a contradiction, so assume  $w \in P_x$ . By Localised

Lemma 4.35,  $x \sqsupset_{\Lambda} w$ , so by transitivity (Localised Lemma 4.26),  $w \sqsupset_{\Lambda} w$ , hence by Localised Lemma 4.29,  $w$  is in a switching cycle in  $\mathcal{G}_{\Lambda}$ , a contradiction.  $\square$

LEMMA 5.9. *Every  $\&$ -vertex  $v \neq x$  toggled by  $\Lambda$  on  $\Gamma$  is in a switching cycle in  $\mathcal{G}_{\Lambda}$ .*

PROOF. By Lemma 5.7,  $v$  is toggled by  $\Lambda'$  on  $\Gamma'$ , hence<sup>11</sup>  $v$  is in a switching cycle  $C$  in  $\mathcal{G}_{\Lambda'}$ . Suppose  $C$  goes through  $c_0$  and/or  $c_1$ . By re-routing the passage(s) through  $c_0$  and/or  $c_1$  to go through  $c$  instead,  $C$  induces a switching cycle in  $\mathcal{G}_{\Lambda}$  that contains  $y$ , in contradiction with Lemma 5.8. Thus  $C$  does not go through  $c_0$  or  $c_1$ . Hence  $C$  is also a switching cycle in  $\mathcal{G}_{\Lambda}$ , containing  $v$ .  $\square$

COROLLARY 5.10. *If the cut is multiplicative, every  $\&$  toggled by  $\Lambda$  on  $\Gamma$  is in a switching cycle in  $\mathcal{G}_{\Lambda}$ .*

Thus if the cut is multiplicative,  $\theta$  fails to be a proof net, a contradiction. Henceforth we assume the cut is additive.

LEMMA 5.11. *The  $\&$ -vertex  $x$  is the unique  $\&$  toggled by  $\Lambda$  that is not in any switching cycle in  $\mathcal{G}_{\Lambda}$ .*

PROOF. Since  $\theta$  is a proof net,  $\Lambda$  toggles a  $\&$ -vertex  $v$  in no switching cycle in  $\mathcal{G}_{\Lambda}$ . By Lemma 5.9, necessarily  $v = x$ .  $\square$

There exist linkings  $\lambda_1, \lambda_2 \in \Lambda'$  such that  $y_0 \in \lambda_1 \upharpoonright \Gamma'$  but  $y_0 \notin \lambda_2 \upharpoonright \Gamma'$ , for otherwise  $\Lambda$  would not toggle  $x$ . Thus there is a jump  $a \rightarrow_{\Lambda} u$  in  $\mathcal{G}_{\Lambda}$  from an axiom link  $a$  above  $y_0$  to a  $\&$ -vertex  $u$  toggled by  $\Lambda$ . If  $u = x$ , we immediately obtain a switching cycle  $x, c, y, y_0, \dots, a, x$  in  $\mathcal{G}_{\Lambda}$ , a contradiction. Thus  $u \neq x$ , so by Lemma 5.9,  $u$  is in a switching cycle in  $\mathcal{G}_{\Lambda}$ , and by Localised Lemma 4.28 and Lemma 5.11,  $x \sqsupset_{\Lambda} u$ . Thus  $x$  is in a switching cycle in  $\mathcal{G}_{\Lambda}$  going through  $c, y, y_0, a$  and  $u$  (or a shortcut thereof, if the strong switching path from  $x$  to  $u$  intersects the path  $x, c, y, y_0, \dots, a$ ), a contradiction. Hence  $\theta'$  satisfies (P3).

This completes the proof that eliminating a cut preserves the toggling condition (P3), and hence the proof that cut elimination is well-defined on proof nets (Proposition 5.4).

## 6 Girard's monomial proof nets

In this section we relate our MALL proof nets to the monomial proof nets of Girard [1996]. We begin in Section 6.1 with a detailed explanation of why monomial proof nets are unsatisfactory. (The reader unfamiliar with monomial proof nets should be able to follow the general shape of the discussion.) In Section 6.2 we settle the open problem of whether Girard's criterion becomes insufficient without partitioning weights into monomials, by presenting a non-monomial proof structure that is not sequentialisable, yet satisfies Girard's criterion. In Section 6.3 we define a surjection collapsing Girard's proof structures to ours, and collapsing Girard's proof nets to ours. In Section 6.4 we show that our resolution condition (P1) corresponds to Girard's so-called 'technical condition', and finally in Section 6.5 we show that, without monomials, the reformulation of the technical condition by

<sup>11</sup>Recall that  $\Lambda'$  was chosen as a witness to the failure of (P3) for  $\theta'$ : any  $\&$  in  $\Gamma'$  toggled by  $\Lambda'$  is in a switching cycle in  $\mathcal{G}_{\Lambda'}$ .

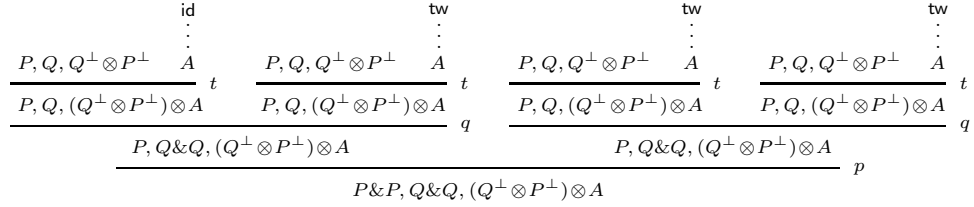


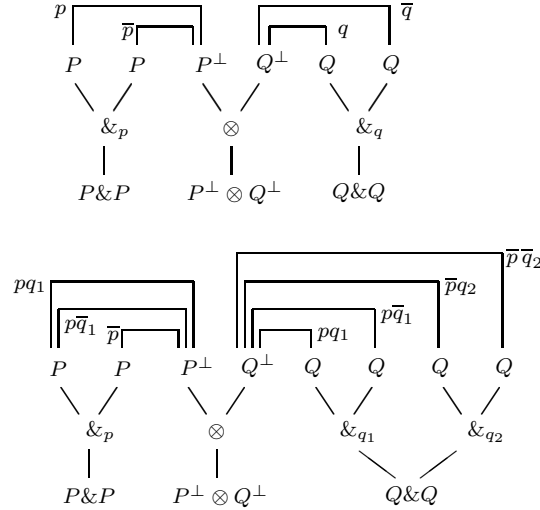
Fig. 15. The proof  $\Pi_{tqp}$ . (We omit the unique cut-free proof of  $P, Q, Q^\perp \otimes P^\perp$ .)

Abramsky and Mellies [1999] is no longer valid. The material in Sections 6.3–6.5 applies to cut-free proof structures only.

### 6.1 Why monomial proof nets are unsatisfactory

We give a detailed account of how monomial proof nets [Girard 1996] fail to provide abstract representations of cut-free MALL proofs modulo inessential commutations of rules. A single cut-free proof may correspond to a host of monomial proof nets, and there is no natural translation of cut-free MALL proofs into monomial proof nets.

Consider the following pair of cut-free monomial proof nets:



Eigenvariables associated with &s are shown as subscripts; we omit implied weights. These two monomial proof nets correspond to the same proof. The second monomial proof net has two forms of redundancy relative to the first: (i) the & with eigenweight  $q$  has been replaced by two similar ‘copies’, and (ii) the axiom link of weight  $p$  has been split into two.

Even if one attempts to fix a choice of representation (*e.g.* favouring the first monomial proof net above over the second), one still runs into difficulty. As a concrete illustration, we exhibit cut-free proofs  $\Pi_\alpha$  and monomial proof nets  $\theta_\beta$  for which the binary relation of sequentialisation is the “zig-zag” shown in Figure 16. Define  $\Pi_{tqp}$  to be the proof shown in Figure 15, where  $A = (R \wp R) \wp (R^\perp \otimes R^\perp)$ ,

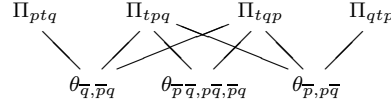
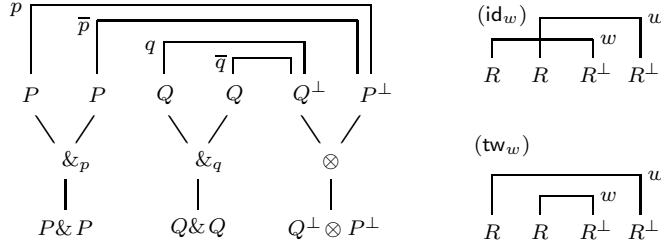


Fig. 16. The “zig-zag” sequentialisation relation between four proofs  $\Pi_\alpha$  and three monomial proof nets  $\theta_\beta$ , demonstrating that monomial proof nets fail to provide canonical representatives of proofs. The proofs and nets are defined in the main text. (By redundancies of type (i) and (ii) described at the beginning of Section 6.1, there are in addition a host of other monomial proof nets  $\theta_\beta$  which are parodies of the three above, and also sequentialise to the  $\Pi_\alpha$ .) By contrast, we represent all the  $\Pi_\alpha$  by the same proof net. Note that this implies that the surjection from Girard’s proof nets to ours (defined in Section 6.3) is not injective.

$\text{id}$  denotes the identity proof and  $\text{tw}$  denotes the twist proof. Let  $\Pi_{t pq}$  be the result of commuting rules  $q$  and  $p$  in  $\Pi_{t qp}$ , let  $\Pi_{qt p}$  be the result of commuting  $t$  and  $q$  in the right half of  $\Pi_{t qp}$ , and let  $\Pi_{pt q}$  be the result of commuting  $t$  and  $p$  in the right half of  $\Pi_{t pq}$ . Define the monomial proof nets  $\theta_\beta$  as follows. To specify  $\theta_\beta$  it suffices to present a configuration of weighted axiom links. On  $P$  and  $Q$  literals, fix the configuration as below-left:



We have taken as eigenweights the labels of the  $\&$ -rules of the  $\Pi_\alpha$ . The configuration of axiom links on  $A$  will be a disjoint union of axiom links in the identity and twist configurations:  $\text{id}_w$  and  $\text{tw}_w$  (above-right) denote a pair of axiom links of weight  $w$  in the identity and twist configurations, respectively. We specify the  $\theta_\beta$  by the following disjoint unions of weighted identity and twist configurations on  $A$ :

$$\begin{aligned} \theta_{\bar{p}, p\bar{q}} &: \text{id}_{pq} \sqcup \text{tw}_{\bar{p}} \sqcup \text{tw}_{p\bar{q}} \\ \theta_{\bar{q}, \bar{p}q} &: \text{id}_{pq} \sqcup \text{tw}_{\bar{q}} \sqcup \text{tw}_{\bar{p}q} \\ \theta_{\bar{p}\bar{q}, p\bar{q}, \bar{p}q} &: \text{id}_{pq} \sqcup \text{tw}_{\bar{p}\bar{q}} \sqcup \text{tw}_{p\bar{q}} \sqcup \text{tw}_{\bar{p}q} \end{aligned}$$

The sequentialisation relation between the  $\Pi_\alpha$  and the  $\theta_\beta$  is as indicated in Figure 16. Since the  $\Pi_\alpha$  are equivalent modulo inessential rule commutations, any satisfactory theory of proof nets should provide a canonical representation uniting all of them. With monomial proof nets one would have to close under the “zig-zag” sequentialisation relation between proofs and monomial proof nets depicted in Figure 16, thereby creating a matching between the set of proofs  $\Pi_\alpha$  and the set of monomial proof nets  $\theta_\beta$ , and then artificially choose a representative from amongst the  $\theta_\beta$ .

In contrast, in our setting each  $\Pi_\alpha$  maps to the same proof net: the four-linking proof net in Figure 5 (page 5). Thus we preserve the spirit of MLL proof nets by providing an abstract representation of all of the  $\Pi_\alpha$  in one.

### 6.2 Girard’s criterion is insufficient without monomials

A key stepping stone towards our formulation of a new definition of proof net was to first settle the open problem of whether Girard’s proof net correctness criterion [1996] becomes insufficient upon relaxing the dependency condition, which is the requirement that every weight be a monomial. The answer is yes: in Figure 17 we present a cut-free non-monomial Girard proof net  $\theta$  which is not sequentialisable. By *non-monomial Girard proof net* we mean a proof net as in [Girard 1996] but for the omission of the dependency condition.

Figure 17 also encodes one of our proof structures  $\theta$ , via the notion of weight described in Section 4.6. It is not a proof net, since (P3) fails:  $\mathcal{G}_\theta$  contains a switching cycle passing through all four  $\&$ s (follow the four jumps  $\&_{p_i}$  to the axiom link  $\{R_{i+1}^\perp, R_{i+1}\} \pmod{4}$ ).

### 6.3 Collapsing monomial proof structures to ours

Let a *non-monomial Girard proof structure* be a proof structure as in [Girard 1996] but for the omission of the dependency condition. Note that there are considerably more such proof structures than our own, since weights can be partitioned in a multitude of ways. Define a non-monomial Girard proof structure to be *compact* if (a) any non-literal formula occurrence is the conclusion of exactly one link, except that a formula  $A \oplus B$  may be the conclusion of both a  $\oplus_1$ - and a  $\oplus_2$ -link, and (b) any two literal occurrences constitute the conclusions of at most one axiom link. Each non-monomial Girard proof structure, and thus also each monomial one, can be collapsed into a compact non-monomial Girard proof structure by identifying, along with their premises, links of the same type with the same conclusion(s), and summing the weights of links and formulas so identified. Clearly this collapse does not preserve the dependency condition. Any compact non-monomial Girard proof structure can be obtained as the collapse of a (monomial) Girard proof structure.

Compact non-monomial Girard proof structures are in bijection with our proof structures. The counterpart of Girard’s ‘technical condition’ is implied by (P1) and our definition of a set of linkings in terms of additive resolutions. The surjective map from (monomial) Girard proof structures to our proof structures obtained by composing the collapse and the bijection preserves the property of being a sequentialisation of a particular MALL proof.

The bijection between compact non-monomial Girard proof structures and our proof structures can be further refined: it is not too hard to see that compact non-monomial Girard proof nets are in bijection with sets of linkings in our sense which satisfy (P1) and (P2\*), where (P2\*) is the strengthening of (P2) defined in Section 4.5.4 (page 27). (To obtain the correspondence, note that it suffices to jump to axiom links in Girard’s setting.) Obviously, by the sequentialisation theorem of [Girard 1996] and ours, those compact non-monomial Girard proof structures that arise by collapsing (monomial) Girard proof nets are in bijection with our proof nets.

### 6.4 (P1) is equivalent to Girard’s ‘technical condition’

In this section we work with compact non-monomial Girard proof structures, as defined above. Recall Girard’s ‘technical condition’ (G), and our condition (P1):

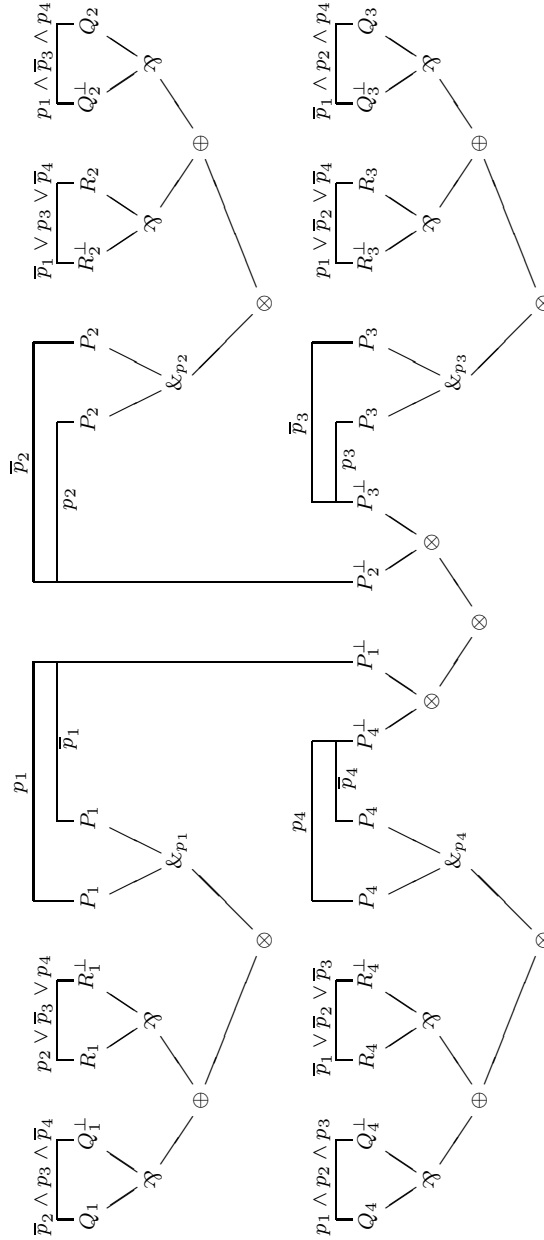


Fig. 17. Girard’s correctness criterion is insufficient without monomials: this abbreviated non-monomial Girard proof net is not sequentialisable. Expand the abbreviation as follows: view each  $p_i$  as an eigenvariable and split each  $\oplus$  into a separate  $\oplus_1$  and  $\oplus_2$ ; formulas and remaining weights are implied.

- (G) If  $v$  is a weight generated by the weights occurring in  $\Theta$ , and  $x$  is a  $\&$ -vertex, then  $v.\neg w(x)$  does not depend on  $p_x$ , where  $w(x)$  is the weight of  $x$  and  $p_x$  is the eigenvariable of  $x$ .
- (P1) For any  $\&$ -resolution  $\Theta^*$  of  $\Theta$ , exactly one linking of  $\Theta$  is on  $\Theta^*$ .

Here  $\Theta$ , cut-free, is a collection of slices  $\Theta(\phi)$  (in the sense of [Girard 1996]), one for each valuation  $\phi$ , such that  $\Theta(\phi)$  resolves  $\&$ s consistently with  $\phi$ , i.e., the additive resolution underlying  $\Theta(\phi)$  is a subgraph of the  $\&$ -resolution  $\Theta^*(\phi)$  determined by  $\phi$ . (Recall that a valuation is an assignment of 0 or 1 to each eigenvariable, i.e., a choice of right or left for each  $\&$ .)

To prove (G)  $\iff$  (P1) we shall use the following stepping-stone conditions. The first is the relaxation of (G) obtained by restricting to weights of axiom links.

- (G-AX) if  $v$  the a weight of an axiom link of  $\Theta$ , and  $x$  is a  $\&$ -vertex, then  $v.\neg w(x)$  does not depend on  $p_x$ .
- (SELF-INDEP) no  $\&$  depends on itself, i.e., for all  $\&$ -vertices  $x$  of  $\Theta$ ,  $w(x)$  does not depend on  $p_x$ .
- (PRESENCE) an axiom link  $z$  depends on a  $\&$ -vertex  $x$  in a slice  $S$  only if  $x$  is in  $S$ .  
(Recall that  $z$  depends on  $x$  in  $\Theta(\phi)$  iff  $z$  is in  $\Theta(\phi)$  but not in  $\Theta(\phi^x)$ , where  $\phi^x$  is the valuation obtained from  $\phi$  by toggling the 0/1 value of  $\phi(x)$ .)

PROPOSITION 6.1.

$$(G) \iff (G\text{-AX}) \text{ and } (SELF\text{-INDEP}) \iff (PRESENCE) \iff (P1)$$

For brevity, write  $u \curvearrowright p$  if the weight  $u$  depends on the eigenvariable  $p$ , and  $u \not\curvearrowright p$  if  $u$  does not depend on  $p$ .

PROOF OF (G)  $\iff$  (G-AX) AND (SELF-INDEP). (G)  $\implies$  (SELF-INDEP) follows by taking  $v = \neg w(L)$  in (G), and  $x \curvearrowright p$  iff  $\neg x \curvearrowright p$ ; (G)  $\implies$  (G-AX) is trivial.

Conversely, (G-AX) and (SELF-INDEP)  $\implies$  (G) because, for all  $u \not\curvearrowright p$ , and any weights  $w$  and  $w'$ :

- (1)  $[w.u \not\curvearrowright p]$  and  $[w'.u \not\curvearrowright p] \implies (w \vee w').u \not\curvearrowright p$ ;
- (2)  $[w.u \not\curvearrowright p]$  and  $[w'.u \not\curvearrowright p] \implies (w.w').u \not\curvearrowright p$ ;
- (3)  $w.u \not\curvearrowright p \implies (\neg w).u \not\curvearrowright p$ .

In turn, (1) and (2) follow from:

- (a)  $[x \not\curvearrowright p]$  and  $[y \not\curvearrowright p] \implies x \vee y \not\curvearrowright p$ ;
- (b)  $[x \not\curvearrowright p]$  and  $[y \not\curvearrowright p] \implies x.y \not\curvearrowright p$ ;

respectively, since  $(w.w').u = (w.u).(w'.u)$  and  $(w \vee w').u = (w.u) \vee (w'.u)$ . Finally, for (3), if  $\neg w.u \curvearrowright p$ , then  $(w.u) \vee (\neg w.u) = (w \vee \neg w).u = u$  depends on  $p$ , a contradiction.  $\square$

Before continuing with the proof of the proposition, we present an auxiliary lemma.

LEMMA 6.2. *If the  $\&$ -vertex  $x \in \Theta(\phi)$  but  $x \notin \Theta(\phi^x)$ , then some axiom link of  $\Theta(\phi^x)$  depends on  $x$  in  $\Theta(\phi^x)$ .*

PROOF. Let  $y$  be the first vertex below  $x$  which is in both  $\Theta(\phi)$  and  $\Theta(\phi^x)$ ; pick any axiom link in  $\Theta(\phi^x)$  above the other argument of  $y$ .  $\square$

Extend the notation “depends on”  $\curvearrowright$  and “does not depend on”  $\not\curvearrowright$  to vertices, e.g. “ $x \curvearrowright y$  in  $S$ ” means “the vertex  $x$  depends on the  $\&$ -vertex  $y$  in (the slice)  $S$ ”.

PROOF OF (PRESENCE)  $\implies$  (SELF-INDEP).

(SELF-INDEP) fails  $\implies x \in \Theta(\phi)$  but  $x \notin \Theta(\phi^x)$  for some  $\phi$   
 $\implies$  an axiom link depends on  $x$  in  $\Theta(\phi^x)$  (by Lemma 1)  
 $\implies x \in \Theta(\phi^x)$  (by (PRESENCE)).

This is a contradiction ( $x \notin \Theta(\phi^x)$  and  $x \in \Theta(\phi^x)$ ).  $\square$

PROOF OF (G-AX)  $\iff$  (PRESENCE), GIVEN (SELF-INDEP). Below  $a$  is an axiom link,  $x$  is a  $\&$ -vertex,  $p_x$  is the eigenvariable of  $x$ , and  $\phi$  is a valuation.

(G-AX)  $\iff (\forall a) w(a). \neg w(x) \not\curvearrowright p_x$   
 $\iff (\forall a) (\forall \phi) \phi(w(a). \neg w(x)) = \phi^x(w(a). \neg w(x))$   
 $\iff (\forall a) (\forall \phi) [a \in \Theta(\phi) \text{ and } x \notin \Theta(\phi)] \text{ iff } [a \in \Theta(\phi^x) \text{ and } x \in \Theta(\phi^x)]$   
 $\iff (\forall a) x \notin \Theta(\phi) \text{ implies } [a \in \Theta(\phi) \text{ iff } a \in \Theta(\phi^x)]$   
 $\iff (\forall a) x \notin \Theta(\phi) \text{ implies } a \not\curvearrowright x \text{ in } \Theta(\phi)$  (by (SELF-INDEP))  
 $\iff (\forall a) a \curvearrowright x \text{ in } \Theta(\phi) \text{ implies } x \in \Theta(\phi)$  (contrapositive)  
 $\iff$  (PRESENCE)

$\square$

PROOF OF (PRESENCE)  $\implies$  (P1). Existence follows immediately by definition of  $\Theta$  as a set of slices  $\Theta(\phi) \subseteq \Theta^*(\phi)$ , one for each valuation  $\phi$ . For uniqueness: suppose  $\lambda$  and  $\lambda'$  are distinct linkings of  $\Theta$ , say from slices  $\Theta(\phi)$  and  $\Theta(\phi')$ , with both  $\lambda$  and  $\lambda'$  on the same  $\&$ -resolution  $\Theta^*$ . Claim: there exists a  $\&$ -vertex  $x$  with  $\phi(x) \neq \phi'(x)$ , and  $x$  is not in at least one of  $\Theta(\phi)$  or  $\Theta(\phi')$ . Proof of Claim: pick any  $x$  such that  $\phi(x) \neq \phi'(x)$  ( $x$  exists, since  $\phi \neq \phi'$ ); if  $x$  is not in  $\Theta^*$ , then it is not in  $\Theta(\phi) \subseteq \Theta^*$ , as sought; if  $x$  is in  $\Theta^*$ , then it cannot be in both  $\Theta(\phi)$  and  $\Theta(\phi')$ , since  $\Theta^*$  chooses (say) left for  $x$ , and one of  $\Theta(\phi)$  and  $\Theta(\phi')$  chooses right (since  $\phi(x) \neq \phi'(x)$ ); QED Claim.

Without loss of generality,  $x$  is not in  $\Theta(\phi)$ . Thus by (PRESENCE),  $\Theta(\phi^x) = \Theta(\phi)$ . Repeating this argument for the remaining  $\&$ -vertices on which  $\phi$  and  $\phi'$  differ, we obtain  $\Theta(\phi') = \Theta(\phi)$ , a contradiction.  $\square$

PROOF OF (P1)  $\implies$  (PRESENCE). Suppose the axiom link  $z \curvearrowright x$  in  $\Theta(\phi)$ , and  $x$  is not in  $\Theta(\phi)$ . Let  $\lambda$  be the linking of  $\Theta(\phi)$ , and let  $\lambda'$  be the linking of  $\Theta(\phi^x)$  (distinct from  $\lambda$ , since  $z$  depended on  $x$ ). Since  $\Theta(\phi) \subset \Theta^*(\phi)$  (by definition), and  $x$  is absent from  $\Theta(\phi)$ , we have also  $\Theta(\phi) \subseteq \Theta^*(\phi')$ ; thus both  $\lambda$  and  $\lambda'$  fit on  $\Theta^*(\phi')$ , contradicting (P1).  $\square$

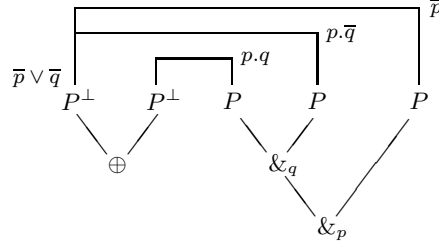
This completes the proof of Proposition 6.1. Thus Girard’s ‘technical condition’ is equivalent to our own (P1).

### 6.5 The Abramsky-Melliès reformulation of Girard’s technical condition is valid only with monomials

Recall Girard’s ‘technical condition’ [1996], and its reformulation by Abramsky and Melliès [1999]:

- (G) If  $v$  is a weight generated by the weights occurring in  $\Theta$ , and  $x$  is a  $\&$ -vertex, then  $v \cdot \neg w(x)$  does not depend on the eigenvariable  $p_x$ .
- (AM) If  $v$  is a weight occurring in  $\Theta$  and  $x$  is a  $\&$ -vertex, if  $v \sim p_x$  then  $v < w(L)$ .

The following is a simple non-monomial example which satisfies (G) but not (AM):



Thus (AM) become invalid without partitioning into monomials. Similarly, the Gustave example (page 23) fails (AM), but satisfies (G). (To see that the Gustave example satisfies (G), observe that it satisfies

(PRESENCE) an axiom link  $z$  depends on a  $\&$ -vertex  $x$  in a slice  $S$  only if  $x$  is in  $S$ .

shown equivalent to (G) in Proposition 6.1.) In particular, the technical condition implies softness only with monomials.

## Appendices

### A Reformulations of the MLL condition

In this section we prove Proposition 4.20 (page 27).

PROOF. To streamline the proof, without loss of generality we may consider  $\lambda$  in the statement of the Proposition as a linking on an MLL sequent (after collapsing the unary additives).

Define an **abstract switching** as the graph  $\mathcal{G}_\lambda$  of a linking  $\lambda$  on a sequent in the variant of MLL in which  $\otimes$  is a binary connective and  $\wp$  is unary. An abstract switching  $s$  is **balanced** if  $|\mathbf{ax}| = |\otimes| + 1$ , where  $|\mathbf{ax}|$  and  $|\otimes|$  are the numbers of axiom links and tensors in  $s$ , respectively. Clearly, any  $\wp$ -switching of a linking on an MLL sequent  $\Gamma$  induces an abstract switching, namely by abstracting from the underlying structure of  $\Gamma$ , regarding any subformula reached by a deleted argument edge of a  $\wp$ -vertex as a separate formula in the sequent. Moreover, a linking on an MLL sequent is balanced iff any of the induced abstract switchings is balanced. We now show the following:

- (i) If an abstract switching is acyclic and connected, it is balanced.
- (ii) If an abstract switching is acyclic and balanced, it is connected.
- (iii) If an abstract switching is balanced and connected, it is acyclic.

From this we obtain:

$(A) \wedge (c) \Rightarrow (B)$	by (i);
$(a) \wedge (C) \Rightarrow (B)$	by (i);
$(A) \wedge (B) \Rightarrow (C)$	by (ii);
$(C) \wedge (B) \Rightarrow (A)$	by (iii);
$(A) \Rightarrow (a)$	because every linking has at least one $\mathfrak{A}$ -switching;
$(C) \Rightarrow (c)$	because every linking has at least one $\mathfrak{A}$ -switching.

These six implications yield the statement of the proposition.

*Proof of (i)*, by structural induction on  $s$ :

*Induction base:* If  $s$  has no connectives, it must be the disjoint union of  $n$  pairs of complementary literals, each connected by a linking. As sequents are non-empty and  $s$  is connected,  $n = 1$ . Hence  $s$  is balanced.

*Induction step:* Suppose  $s$  has an outermost  $\mathfrak{A}$ -vertex  $p$ . Deleting  $p$  yields an abstract switching  $s'$  that is still a tree, and hence balanced by induction. This implies that  $s$  is balanced.

Suppose  $s$  has an outermost  $\otimes$ -vertex  $t$ . Deleting  $t$  yields an abstract switching  $s'$ , consisting of two disjoint abstract switchings  $s_0$  and  $s_1$ , each of which is a tree. By induction  $s_0$  and  $s_1$  are balanced, so  $s'$  satisfies  $|\mathbf{ax}| = |\otimes| + 2$ , and  $s$  is balanced.

*Proof of (ii)*, by structural induction on  $s$ :

*Induction base:* If  $s$  has no connectives,  $|\otimes| = 0$ , so  $|\mathbf{ax}| = 1$ , and  $s$  is connected.

*Induction step:* Suppose  $s$  has an outermost  $\mathfrak{A}$ -vertex  $p$ . Deleting  $p$  yields an abstract switching  $s'$  that is still acyclic and balanced, and hence connected by induction. This implies that  $s$  is connected.

Suppose  $s$  has an outermost  $\otimes$ -vertex  $t$  with argument vertices  $t_0$  and  $t_1$ . Deleting  $t$  yields an abstract switching  $s'$ , satisfying  $|\mathbf{ax}| = |\otimes| + 2$ . Because  $s$  is acyclic, in  $s'$  the vertices  $t_0$  and  $t_1$  are not connected. Let  $s_0$  be the part of  $s'$  that is connected to  $t_0$  and let  $s_1$  be the remainder of  $s'$ . As  $s_0$  is acyclic and connected, it must be balanced by (i). This implies that  $s_1$  must be balanced as well, as any surplus in axiom links in  $s_1$  would be compensated by a shortage in  $s_0$  and vice versa. Hence, by induction,  $s_1$  is connected too. This implies that  $s$  is connected.

*Proof of (iii):* In fact we prove the stronger statement that if an abstract switching is connected and satisfies  $|\mathbf{ax}| \geq |\otimes| + 1$ , it is acyclic. We proceed by structural induction on  $s$ .

*Induction base:* If  $s$  has no connectives,  $|\otimes| = 0$ , so  $|\mathbf{ax}| = 1$ , and  $s$  is acyclic.

*Induction step:* Suppose  $s$  has an outermost  $\mathfrak{A}$ -vertex  $p$ . Deleting  $p$  yields an abstract switching  $s'$  that is still connected and satisfies  $|\mathbf{ax}| \geq |\otimes| + 1$ . Hence it is acyclic by induction. This implies that  $s$  is acyclic.

Suppose  $s$  has an outermost  $\otimes$ -vertex  $t$  with argument vertices  $t_0$  and  $t_1$ . Deleting  $t$  yields an abstract switching  $s'$ , satisfying  $|\mathbf{ax}| \geq |\otimes| + 2$ . In  $s'$  the vertices  $t_0$  and  $t_1$  are not connected, for if they were,  $s'$  would be connected, therefore acyclic by induction, and hence balanced by (i), a contradiction. Let  $s_0$  be the part of  $s'$  that is connected to  $t_0$  and  $s_1$  the remainder, connected to  $t_1$ . As least one of  $s_0$  and  $s_1$  must satisfy  $|\mathbf{ax}| \geq |\otimes| + 1$ . Hence it is acyclic by induction, and thus balanced by (i). This implies that also the other satisfies  $|\mathbf{ax}| \geq |\otimes| + 1$ , and thus is acyclic by induction. Hence  $s$  is acyclic.  $\square$

### B Illegal unions of switching cycles

We prove that condition (P3<sup>l</sup>), defined in Section 4.5.3, is equivalent to the toggling condition (P3). Employing Proposition 4.20, condition (P2) in our definition of a MALL proof net  $\theta$  can be partitioned into

- (P2a) every  $\mathfrak{A}$ -switching of every linking  $\lambda \in \theta$  is acyclic, and
- (P2b) every linking  $\lambda \in \theta$  is balanced.

The former is clearly equivalent to

- (P2a') for no linking  $\lambda \in \theta$  does  $\mathcal{G}_\lambda$  contain a switching cycle.

Therefore, conditions (P2a) and (P3) can be combined into

- (P2a3) every set  $\Lambda \subseteq \theta$  with a switching cycle in  $\mathcal{G}_\Lambda$  toggles a  $\&$  that is not in any switching cycle in  $\mathcal{G}_\Lambda$ .

This condition is in turn equivalent to:

- (P2a3') for any  $\Lambda \subseteq \theta$  and any non-empty union of switching cycles  $S$  in  $\mathcal{G}_\Lambda$ ,  $\Lambda$  toggles a  $\&$  that is not in  $S$ .

Now (P2a3') can be reformulated as:

- (P3<sup>l</sup>)  $\mathcal{G}_\theta$  contains no illegal union of switching cycles.

### C Unit-free star-autonomous categories

Define a category  $\mathbb{C}$  to be *unit-free star-autonomous* (c.f. [Barr 1979]) if it is equipped with the following structure:

- Tensor. A functor  $- \otimes - : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ .
- Associativity. A natural isomorphism  $a_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$  natural in objects  $A, B, C \in \mathbb{C}$  such that the following pentagon commutes:

$$\begin{array}{ccccc}
 ((A \otimes B) \otimes C) \otimes D & \xrightarrow{a} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a} & A \otimes (B \otimes (C \otimes D)) \\
 \downarrow a \otimes \text{id} & & & & \uparrow \text{id} \otimes a \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a} & & \xrightarrow{a} & A \otimes ((B \otimes C) \otimes D)
 \end{array}$$

- Symmetry. A natural isomorphism  $c_{A,B} : A \otimes B \rightarrow B \otimes A$  natural in objects  $A, B \in \mathbb{C}$  such that  $c_{B,A} \circ c_{A,B} = \text{id}_{A \otimes B}$  and the following hexagon commutes:

$$\begin{array}{ccccc}
 (A \otimes B) \otimes C & \xrightarrow{a} & A \otimes (B \otimes C) & \xrightarrow{c} & (B \otimes C) \otimes A \\
 \downarrow c \otimes \text{id} & & & & \downarrow a \\
 (B \otimes A) \otimes C & \xrightarrow{a} & B \otimes (A \otimes C) & \xrightarrow{\text{id} \otimes c} & B \otimes (C \otimes A)
 \end{array}$$

- A full and faithful functor  $(-)^{\perp} : \mathbb{C}^{\text{op}} \rightarrow \mathbb{C}$ .
- An isomorphism  $\mathbb{C}(A \otimes B, C^{\perp}) \rightarrow \mathbb{C}(A, (B \otimes C)^{\perp})$  natural in all objects  $A, B, C$  of  $\mathbb{C}$ .

The above structure is simply the result of taking the definition of star-autonomous category and removing any data involving the tensor unit.

## D Direct proof that (P2) is preserved upon eliminating a multiplicative cut

Let  $\Gamma$  (resp.  $\Gamma'$ ) be the cut sequent before (resp. after) the elimination of the multiplicative cut. By definition, the linkings remain the same. We prove the following stronger result: if every  $\mathfrak{A}$ -switching of a linking  $\lambda$  on  $\Gamma$  is a tree, then every  $\mathfrak{A}$ -switching of  $\lambda$  on  $\Gamma'$  is a tree. If the eliminated cut vertex  $c$  is absent from  $\lambda \upharpoonright \Gamma$ , every  $\mathfrak{A}$ -switching of  $\lambda$  on  $\Gamma'$  is a  $\mathfrak{A}$ -switching of  $\lambda$  on  $\Gamma$ , hence a tree; therefore assume  $c$  is present. Let  $x$  be the eliminated  $\mathfrak{A}$ , with arguments  $x_0, x_1$ , and let  $y$  be the eliminated  $\otimes$ , with arguments  $y_0, y_1$ . Thus  $x_i - x - c - y - y_i$  in  $\mathcal{G}_\Gamma$ . Let  $\hat{\Gamma}$  be the result of deleting  $x, c$  and  $y$  (and associated edges) from  $\Gamma$ . Claim: every  $\mathfrak{A}$ -switching  $\sigma$  of  $\lambda$  on  $\hat{\Gamma}$  is the disjoint union of three trees, one containing the  $x_i$ , one containing  $y_0$ , and one containing  $y_1$ . Proof: let  $\sigma_i$  be the  $\mathfrak{A}$ -switching of  $\lambda$  on  $\Gamma$  obtained from  $\sigma$  by adding  $x_i - x - c - y$  and  $y_0 - y - y_1$ , *a priori* a tree; were the  $y_i$  connected by a path  $\pi$  in  $\sigma$  then  $y\pi y$  would be a cycle in each  $\sigma_i$ ; were the  $x_j$  and  $y_k$  connected by a path  $\pi$  in  $\sigma$  then  $cx_j\pi y_k y c$  would be a cycle in  $\sigma_j$ ; were the  $x_i$  disconnected in  $\sigma$  then (given the disconnection of the  $x_i$  from the  $y_i$ ) they would be disconnected in each  $\sigma_i$ . A  $\mathfrak{A}$ -switching of  $\lambda$  on  $\Gamma'$  is a  $\mathfrak{A}$ -switching of  $\lambda$  on  $\hat{\Gamma}$  together with cuts  $x_0 - c_0 - y_0$  and  $x_1 - c_1 - y_1$ , and is therefore (by the claim) a tree.

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