Hypergame semantics:
Ten years later

Dominic J. D. Hughes

Stanford University

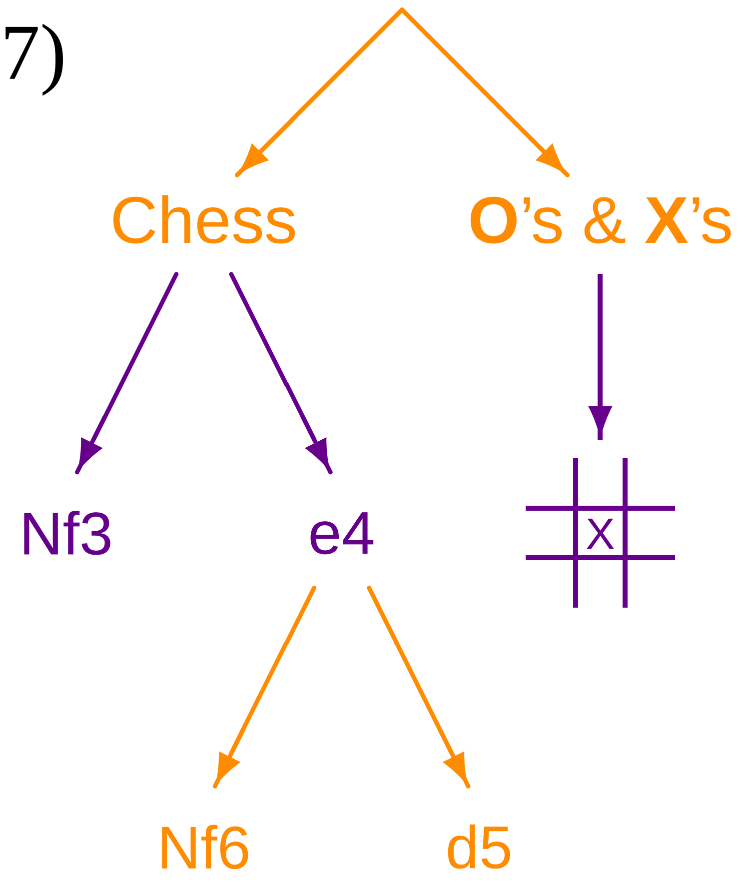
GaLoP'06, Seattle, 11 August 2006

Hypergames model/semantics

- For a 2nd-order logic / polymorphic progr. language
- Two principles:
  - (**GM**) **G**ames as **M**oves
  - (**UC**) **U**niformity by **C**opycat expansion
- *Imperial College Games Workshop* 1996
  LICS 97, PhD thesis (Luke Ong, Oxford)
  - System F (LICS 97, PhD)
  - affine linear (Murawski-Ong '01, Murawski PhD '01)
  - Type isos (de Lataillade, yesterday)
- Here cleaner notation: no $\times$, transition games, Felscher

(GM) Games as Moves

- Zwicker's Hypergame H (1987)
  - O chooses terminating game G; play G (P starts)
- " Is H terminating? " $\sim$ Russell's paradox (Mirimanoff 1917)



(GM) Games as Moves

- $\langle a, b \rangle \mapsto a \quad : \quad \forall X \, . \, X \times X \to X$

(GM) Games as Moves

- $\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X$
  - at $\mathbb{N}$: $\quad \langle 7, 3 \rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

(GM) Games as Moves

- $\langle a, b\rangle \mapsto a \quad : \quad \forall X \, . \, X \times X \to X$
  - at $\mathbb{N}$: $\langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false}\rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

(GM) Games as Moves

- $$\overbrace{\langle a, b\rangle \mapsto a}^{\pi_1} \quad : \quad \overbrace{\forall X \,.\, X \times X \to X}^{\mathsf{P}}$$

  - at $\mathbb{N}$: $\quad \langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

  - at $\mathbb{B}$: $\quad \langle \mathsf{true}, \mathsf{false}\rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

(GM) Games as Moves

- $\overbrace{\langle a, b\rangle \mapsto a}^{\pi_1} \quad : \quad \overbrace{\forall X\,.\,X \times X \to X}^{\mathsf{P}}$
  - at $\mathbb{N}$: $\langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false}\rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
  - at P : $\langle \pi_2, \pi_1\rangle \mapsto \pi_2 \quad : \quad \mathsf{P} \times \mathsf{P} \to \mathsf{P}$

(GM) Games as Moves

- $\overbrace{\langle a, b\rangle \mapsto a}^{\pi_1} \quad : \quad \overbrace{\forall X \,.\, X \times X \to X}^{\mathsf{P}}$
  - at $\mathbb{N}$: $\langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false}\rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
  - at P : $\langle \pi_2, \pi_1\rangle \mapsto \pi_2 \quad : \quad \mathsf{P} \times \mathsf{P} \to \mathsf{P}$

(UC) Uniformity by Copycat expansion

- $\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X$ Uniform

(UC) Uniformity by Copycat expansion

- $\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X$ Uniform
  - at $\mathbb{N}$: $\langle 7, 3 \rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false} \rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

(UC) Uniformity by Copycat expansion

- $\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \rightarrow X$ — Uniform
  - at $\mathbb{N}$: $\langle 7, 3 \rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false} \rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$

- $\left\{ \begin{array}{ll} \langle a, b \rangle \mapsto a & \text{if } X = \mathbb{N} \\ \langle a, b \rangle \mapsto b & \text{otherw.} \end{array} \right\} : \forall X \,.\, X \times X \rightarrow X$ — ~~Uniform~~ Ad hoc

(UC) Uniformity by Copycat expansion

- $\langle a, b\rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X$ Uniform
  - at $\mathbb{N}$: $\langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false}\rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

- $\left\{ \begin{array}{ll} \langle a, b\rangle \mapsto a & \text{if } X = \mathbb{N} \\ \langle a, b\rangle \mapsto b & \text{otherw.} \end{array} \right\} : \forall X \,.\, X \times X \to X$ ~~Uniform~~ Ad hoc
  - at $\mathbb{N}$: $\langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

(UC) Uniformity by Copycat expansion

- $\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X$ Uniform
  - at $\mathbb{N}$: $\langle 7, 3 \rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false} \rangle \mapsto \mathsf{true} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

- $\left\{ \begin{array}{ll} \langle a, b \rangle \mapsto a & \text{if } X = \mathbb{N} \\ \langle a, b \rangle \mapsto b & \text{otherw.} \end{array} \right\} : \forall X \,.\, X \times X \to X$ ~~Uniform~~ Ad hoc
  - at $\mathbb{N}$: $\langle 7, 3 \rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at $\mathbb{B}$: $\langle \mathsf{true}, \mathsf{false} \rangle \mapsto \mathsf{false} \quad : \quad \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

at $\circ$: $\quad \circ \times \circ \longrightarrow \circ$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

$$\bullet$$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

$$\bullet \qquad\qquad\qquad \bullet$$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

at $\circ$: $\quad \circ \times \circ \longrightarrow \circ$

at $\mathbb{N}$: $\quad \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n n

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n n

7

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n n

7 7

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n n

7 7

Convention:

O = Opponent = Orange

P = Player = Purple

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

$\bullet$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{B}$: $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \longrightarrow X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

$\bullet$ $\bullet$

at $\mathbb{B}$: $\mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$

b

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \rightarrow X$$

at $\circ$: $\circ \times \circ \rightarrow \circ$

$\bullet$ $\bullet$

at $\mathbb{B}$: $\mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$

b b

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{B}$: $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$

b b

true

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \to \circ$

$\bullet$ $\bullet$

at $\mathbb{B}$: $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$

b b

true true

(UC) Uniformity by Copycat expansion

$$\forall X.\, X \times X \to X$$

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n n

3 3

~~Uniform~~

Ad hoc

at $\mathbb{B}$: $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$

b b

true true

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \rightarrow \quad X$$

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \longrightarrow \quad X$$

at $\circ$: $\quad \circ \quad \times \quad \circ \quad \longrightarrow \quad \circ$

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \longrightarrow \quad X$$

at $\circ$: $\quad \circ \quad \times \quad \circ \quad \longrightarrow \quad \circ$

$$\bullet$$

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \longrightarrow \quad X$$

$$\text{at } \circ \ : \quad \circ \quad \times \quad \circ \quad \longrightarrow \quad \circ$$

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \longrightarrow \quad X$$

at $\circ$: $\qquad \circ \quad \times \quad \circ \quad \longrightarrow \quad \circ$

$\bullet \qquad \qquad \bullet$

at $\mathbb{N}\!\rightarrow\!\mathbb{B}$: $\qquad (\mathbb{N}\!\rightarrow\!\mathbb{B}) \times (\mathbb{N}\!\rightarrow\!\mathbb{B}) \quad \longrightarrow \quad (\mathbb{N}\!\rightarrow\!\mathbb{B})$

(UC) Uniformity by Copycat expansion

$$\forall X.\quad X \quad \times \quad X \quad \rightarrow \quad X$$

at $\circ$: $\quad \circ \quad \times \quad \circ \quad \rightarrow \quad \circ$

$\bullet \qquad\qquad\qquad \bullet$

at $\mathbb{N}\rightarrow\mathbb{B}$: $\quad (\mathbb{N}\rightarrow\mathbb{B}) \times (\mathbb{N}\rightarrow\mathbb{B}) \quad \rightarrow \quad (\mathbb{N}\rightarrow\mathbb{B})$

b

(UC) Uniformity by Copycat expansion

$\forall X.$	X	$\times$	X	$\to$	X
at $\circ$:	$\circ$	$\times$	$\circ$	$\to$	$\circ$
					$\bullet$
	$\bullet$				
at $\mathbb{N}\to\mathbb{B}$:	$(\mathbb{N}\to\mathbb{B})$	$\times$	$(\mathbb{N}\to\mathbb{B})$	$\to$	$(\mathbb{N}\to\mathbb{B})$
					b
	b				

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \rightarrow \quad X$$

at $\circ$: $\quad \circ \quad \times \quad \circ \quad \rightarrow \quad \circ$

$\bullet \qquad \bullet$

at $\mathbb{N}\rightarrow\mathbb{B}$: $\quad (\mathbb{N}\rightarrow\mathbb{B}) \times (\mathbb{N}\rightarrow\mathbb{B}) \quad \rightarrow \quad (\mathbb{N}\rightarrow\mathbb{B})$

$b \qquad b$

n

(UC) Uniformity by Copycat expansion

$$\forall X.\quad X \times X \to X$$

at $\circ$: $\quad \circ \times \circ \to \circ$

$\bullet \qquad \bullet$

at $\mathbb{N}\!\to\!\mathbb{B}$: $\quad (\mathbb{N}\!\to\!\mathbb{B}) \times (\mathbb{N}\!\to\!\mathbb{B}) \to (\mathbb{N}\!\to\!\mathbb{B})$

$n \quad b \qquad n \quad b$

(UC) Uniformity by Copycat expansion

$$\forall X. \quad X \quad \times \quad X \quad \to \quad X$$

at $\circ$: $\quad \circ \quad \times \quad \circ \quad \to \quad \circ$

$\bullet \qquad \bullet$

at $\mathbb{N}\to\mathbb{B}$: $\quad (\mathbb{N}\to\mathbb{B}) \times (\mathbb{N}\to\mathbb{B}) \quad \to \quad (\mathbb{N}\to\mathbb{B})$

b $\qquad$ b

n $\qquad$ n

5

(UC) Uniformity by Copycat expansion

$$\forall X.\quad X \quad \times \quad X \quad \longrightarrow \quad X$$

at ○ : $\circ \quad \times \quad \circ \quad \longrightarrow \quad \circ$

$\bullet$ $\bullet$

at $\mathbb{N}\to\mathbb{B}$: $(\mathbb{N}\to\mathbb{B}) \times (\mathbb{N}\to\mathbb{B}) \longrightarrow (\mathbb{N}\to\mathbb{B})$

b

b

n

n

5

5

(UC) Uniformity by Copycat expansion

$$\forall X.\quad X \times X \to X$$

at $\circ$: $\quad \circ \times \circ \to \circ$

$\bullet \qquad \bullet$

at $\mathbb{N}\to\mathbb{B}$: $\quad (\mathbb{N}\to\mathbb{B}) \times (\mathbb{N}\to\mathbb{B}) \to (\mathbb{N}\to\mathbb{B})$

$b \quad b$

$n \quad n$

$5 \quad 5$

true

(UC) Uniformity by Copycat expansion

	$\forall X.$	X	$\times$ X	$\rightarrow$	X
at $\circ$:		$\circ$	$\times$ $\circ$	$\rightarrow$	$\circ$
at $\mathbb{N}\rightarrow\mathbb{B}$:		$(\mathbb{N}\rightarrow\mathbb{B})$	$\times$ $(\mathbb{N}\rightarrow\mathbb{B})$	$\rightarrow$	$(\mathbb{N}\rightarrow\mathbb{B})$

b

n

b

n

5

5

true

true

(UC) Uniformity by Copycat expansion

$$\forall X.\quad X \quad \times \quad X \quad \to \quad X$$

at $\circ$: $\quad \circ \quad \times \quad \circ \quad \to \quad \circ$

$\bullet$ $\qquad\qquad\qquad\qquad \bullet$

at $\mathbb{N}{\to}\mathbb{B}$: $\quad (\mathbb{N}{\to}\mathbb{B}) \times (\mathbb{N}{\to}\mathbb{B}) \to (\mathbb{N}{\to}\mathbb{B})$

$b \qquad\qquad b$

$n \qquad\qquad n$

$5 \qquad\qquad 5$

true $\qquad\qquad$ true

$$\langle\, \textsf{prime?}\ ,\ \textsf{even?}\,\rangle \mapsto \textsf{prime?}$$

Related

- Hypergame semantics: (**GM**) & (**UC**)
  - System F (Imperial '96, LICS 97, PhD)
  - affine linear (Murawski-Ong '01, Murawski PhD'01)
  - Type isos (de Lataillade, yesterday)

  full compl.

- $\cap$-game semantics: $\neg$(**GM**) & (**U**$\cap$)
  - affine linear (Abramsky '97, Newton Inst. '95)

    $\cdots$ Abramsky & Lenisa $\cdots$

- Generic game semantics: $\neg$(**GM**) & (**UC**$'$)
  - system F (Abramsky-Jagadeesan '05) full compl. ML types

Impredicativity

- $\overbrace{\langle a, b\rangle \mapsto a}^{\pi_1} \quad : \quad \overbrace{\forall X \,.\, X \times X \to X}^{\mathsf{P}}$
  - at $\mathbb{N}$: $\langle 7, 3\rangle \mapsto 7 \quad : \quad \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  - at P : $\langle \pi_2, \pi_1\rangle \mapsto \pi_2 \quad : \quad \mathsf{P} \times \mathsf{P} \to \mathsf{P}$ ~~ML~~ , F

- { barbers : shave non-self-shavers }

- H is a move in H: "$H \in H$" (Zwicker)

Impredicativity: dynamic solution

- Static approaches not fully complete for F:
  - Domain models (continuity/approximation/limit)
  - PER models (families/$\cap$)
  - Abramsky-Jagadeesan '05 (no 2^{nd}-order moves)
- Dynamic solution: (**GM**) **G**ames as **M**oves

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \ \mathsf{P}$$

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X \,.\, X \times X \to X$$

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X \,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $\big(\forall X.X \times X \to X\big) \times \big(\forall Y.Y \times Y \to Y\big) \to \big(\forall Z.Z \times Z \to Z\big)$

at $\mathbb{N}$: $\big(\forall X.X \times X \to X\big) \times \big(\forall Y.Y \times Y \to Y\big) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X\,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $\big(\forall X.X \times X \to X\big) \times \big(\forall Y.Y \times Y \to Y\big) \to \big(\forall Z.Z \times Z \to Z\big)$

at $\mathbb{N}$: $\big(\forall X.X \times X \to X\big) \times \big(\forall Y.Y \times Y \to Y\big) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$n$$

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

$$\text{at } \mathsf{P} \;:\; \big(\forall X.X \times X \to X\big) \times \big(\forall Y.Y \times Y \to Y\big) \;\to\; \big(\forall Z.Z \times Z \to Z\big)$$

$$\text{at } \mathbb{N} \;:\; \big(\forall X.X \times X \to X\big) \times \big(\forall Y.Y \times Y \to Y\big) \;\to\; \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$n$$

O's hyper-move

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X \,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

$$\text{at } \mathsf{P} \;:\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \;\to\; (\forall Z.Z \times Z \to Z)$$

$$\text{at } \mathbb{N} \;:\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \;\to\; \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$n$$

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X \,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

$$\text{at } \mathsf{P} \;:\; (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$$

$$\text{at } \mathbb{N} \;:\; (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$n$$

$$\text{at } \mathbb{N} \;:\; \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $\big(\forall X.X\times X\to X\big) \times \big(\forall Y.Y\times Y\to Y\big) \to \big(\forall Z.Z\times Z\to Z\big)$

at $\mathbb{N}$: $\big(\forall X.X\times X\to X\big) \times \big(\forall Y.Y\times Y\to Y\big) \to \mathbb{N}\times\mathbb{N} \to \mathbb{N}$

$$n$$

at $\mathbb{N}$: $\mathbb{N}\times\mathbb{N} \to \mathbb{N} \times \big(\forall Y.Y\times Y\to Y\big) \to \mathbb{N}\times\mathbb{N} \to \mathbb{N}$

$$n$$

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X\,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

$$\text{at } \mathsf{P} \;:\; (\forall X. X \times X \to X) \times (\forall Y. Y \times Y \to Y) \;\to\; (\forall Z. Z \times Z \to Z)$$

$$\text{at } \mathbb{N} \;:\; (\forall X. X \times X \to X) \times (\forall Y. Y \times Y \to Y) \;\to\; \mathbb{N} \times \mathbb{N} \to \underset{n}{\mathbb{N}}$$

$$\text{at } \mathbb{N} \;:\; \mathbb{N} \times \mathbb{N} \to \underset{n}{\mathbb{N}} \times (\forall Y. Y \times Y \to Y) \;\to\; \mathbb{N} \times \mathbb{N} \to \mathbb{N} \;\Big\}\; \text{P's hyper-move}$$

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

$$\text{at } \mathsf{P} \; : \quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$$

$$\text{at } \mathbb{N} \; : \quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \underset{n}{\mathbb{N}}$$

$$\text{at } \mathbb{N} \; : \quad \mathbb{N} \times \mathbb{N} \to \underset{n}{\mathbb{N}} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

m

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\quad \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

m

m

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X \,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

m

m

4

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $\quad (\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\quad \mathbb{N} \times \mathbb{N} \to \mathbb{N} \quad \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

m

m

4

4

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \quad X \quad \times \quad X \quad \to \quad X$$

at P : $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

m

m

4

4

5

Impredicativity: dynamic solution

$$\langle a, b \rangle \mapsto a \quad : \quad \forall X \,.\, X \times X \to X \quad = \mathsf{P}$$

$$\forall X \,. \qquad X \qquad \times \qquad X \qquad \to \qquad X$$

at P : $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

m

m

4

4

5

5

Impredicativity: dynamic solution

$$\langle a, b\rangle \mapsto a \quad : \quad \forall X\,.\,X \times X \to X \quad = \mathsf{P}$$

$$\forall X\,. \quad X \quad \times \quad X \quad \to \quad X$$

at P : $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to (\forall Z.Z \times Z \to Z)$

at $\mathbb{N}$: $(\forall X.X \times X \to X) \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n

at $\mathbb{N}$: $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (\forall Y.Y \times Y \to Y) \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

n m

m

4 4

5

5

$\underbrace{\quad\quad\quad}_{\text{succ}}$

Full completeness for system F

- Uniform
- Innocent
- Total
- Compact

Full completeness for system F

- Uniform (UC)
- Innocent
- Total
- Compact

Full completeness for system F

- Uniform (UC)
- Innocent $\longleftarrow$ Hyland-Ong
- Total
- Compact

Full completeness for system F

- Uniform (UC)
- Innocent $\leftarrow$ Hyland-Ong
- Total $\leftarrow$ λ fragment
- Compact

Full completeness for system F

- Uniform (UC)
- Innocent $\longleftarrow$ Hyland-Ong
- Total $\longleftarrow$ λ fragment
- Compact $=$ finite ‘spine’

Full completeness for system F

- Uniform (UC)
- Innocent $\longleftarrow$ Hyland-Ong
- Total $\longleftarrow$ λ fragment
- Compact $=$ finite 'spine'

$$\forall X.\, X \times X \to X$$

at $\circ$: $\circ \times \circ \longrightarrow \circ$

Formal Hypergames

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\ X \to X$$

$$\Big\downarrow {\scriptstyle \forall Y.Y}$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\Big\downarrow \forall Y.Y$$

$$(\forall Y.Y) \to (\forall Y.Y)$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\Big\downarrow \forall Y.Y$$

$$(\forall Y.Y) \to (\forall Y.Y)$$

$$\rightsquigarrow \text{prenex}$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\big\downarrow \forall Y.Y$$

$$(\forall Y.Y) \to (\forall Y.Y)$$

$$\wr\!\downarrow \text{prenex}$$

$$\forall Y.\, (\forall Y.Y) \to Y$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\mathsf{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\Big\downarrow\, \forall Y.Y$$

$$\wr\, \mathsf{prenex}$$

$$\forall Y.\, (\forall Y.Y) \to Y$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)

- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$

- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\begin{array}{c} \forall X.\; X \to X \\ \Big\downarrow {\scriptstyle \forall Y.Y} \\ \forall Y.\; (\forall Y.Y) \to Y \end{array}$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)

- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$

- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\ X \to X$$

$$\Big\downarrow \ \forall Y.Y$$

$$\forall Y.\ (\forall Y.Y) \to Y$$

$$\Big\downarrow \ \mathbb{N} \to \mathbb{B}$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\Big\downarrow \; \forall Y.Y$$

$$\forall Y.\, (\forall Y.Y) \to Y$$

$$\Big\downarrow \; \mathbb{N}\to\mathbb{B}$$

$$(\forall Y.Y) \to (\mathbb{N}\to\mathbb{B})$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\ X \to X$$

$$\Big\downarrow \ \forall Y.Y$$

$$\Big\downarrow \ \mathbb{N} \to \mathbb{B}$$

$$(\forall Y.Y) \to (\mathbb{N} \to \mathbb{B})$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)

- $A \to \forall X.B \overset{\text{prenex}}{\rightsquigarrow} \forall X.(A \to B)$

- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\Big\downarrow \;\begin{matrix} \forall Y.Y \\ \mathbb{N}\to\mathbb{B} \end{matrix}$$

$$(\forall Y.Y) \to (\mathbb{N}\to\mathbb{B})$$

- **Assertion**: type occurrence
  - Lorenzen/Felscher dialogue games (1960, 1985)
- $A \to \forall X.B \overset{\text{prenex}}{\leadsto} \forall X.(A \to B)$
- **resolve** $=$ exhaustively instantiate leading $\forall$s

$$\forall X.\, X \to X$$

$$\Big\downarrow \begin{array}{l} \forall Y.Y \\ \mathbb{N} \to \mathbb{B} \end{array}$$

$$(\forall Y.Y) \to (\mathbb{N} \to \mathbb{B})$$

- n **branches**: $A_1 \to A_2 \to \ldots \to A_n \to X$

Hypergame $\mathcal{H}(A)$

Hypergame $\mathcal{H}(A)$

- O resolves A

Hypergame $\mathcal{H}(A)$

$$\forall X.\, X \to X \to X$$

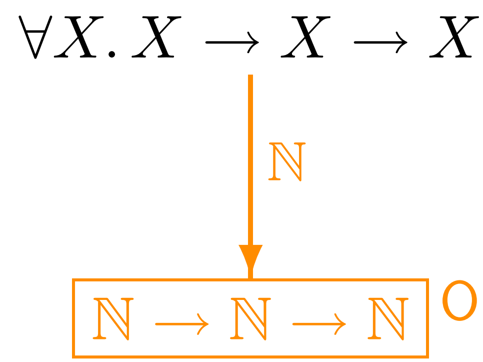
- O resolves A

Hypergame $\mathcal{H}(A)$

$$\forall X.\, X \to X \to X$$

- O resolves A

Hypergame $\mathcal{H}(A)$



- O resolves A

Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch

Hypergame $\mathcal{H}(A)$

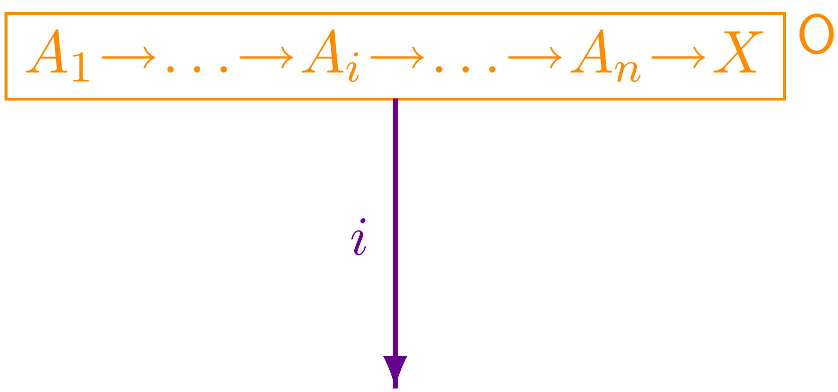
- O resolves A

- resolve opposing branch

$$\boxed{A_1 \to \ldots \to A_i \to \ldots \to A_n \to X}^{\,\mathsf{O}}$$

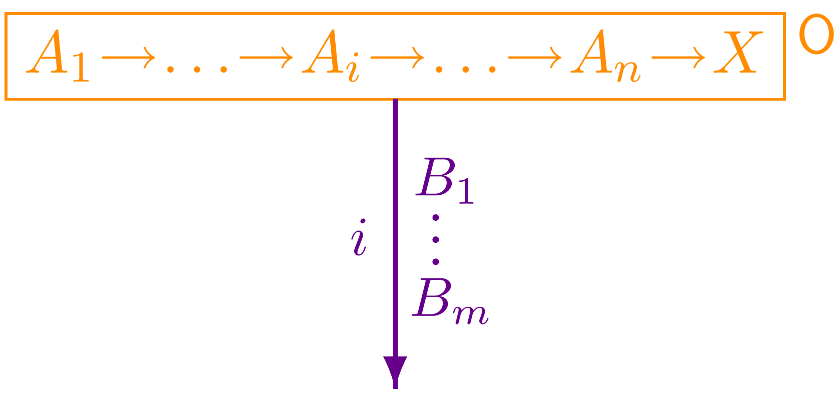
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



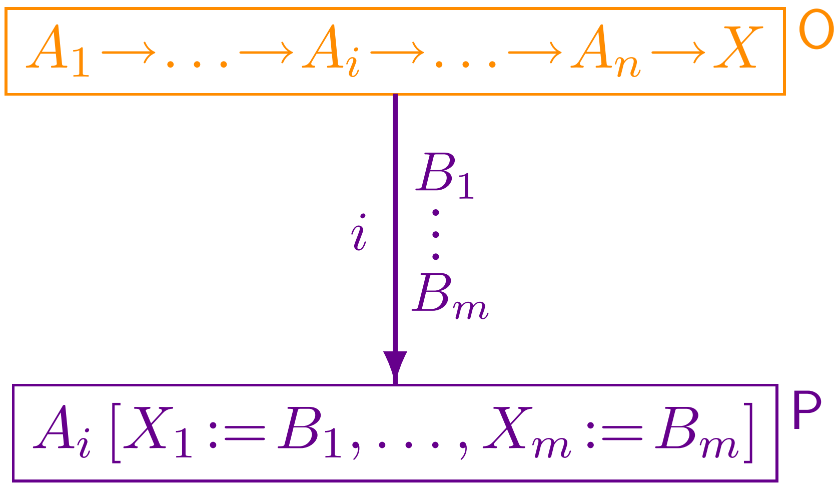
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



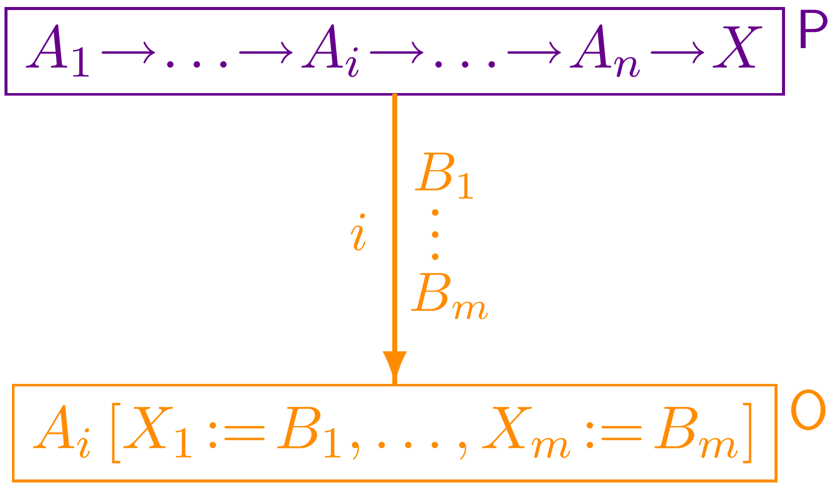
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



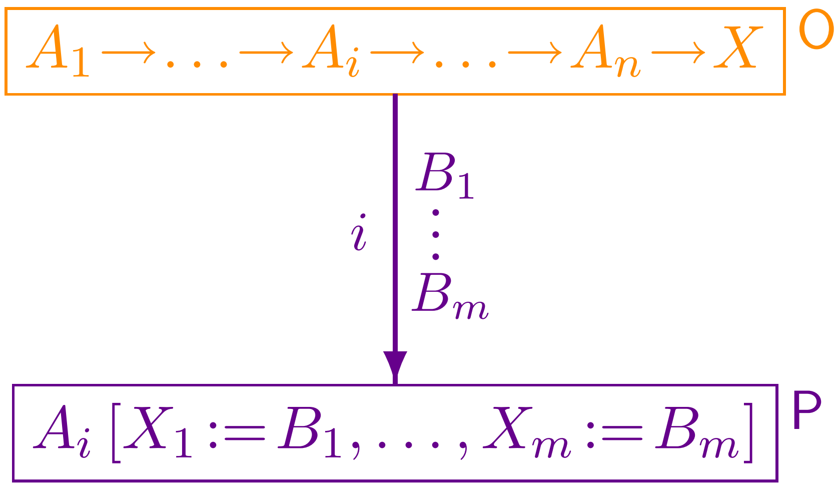
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



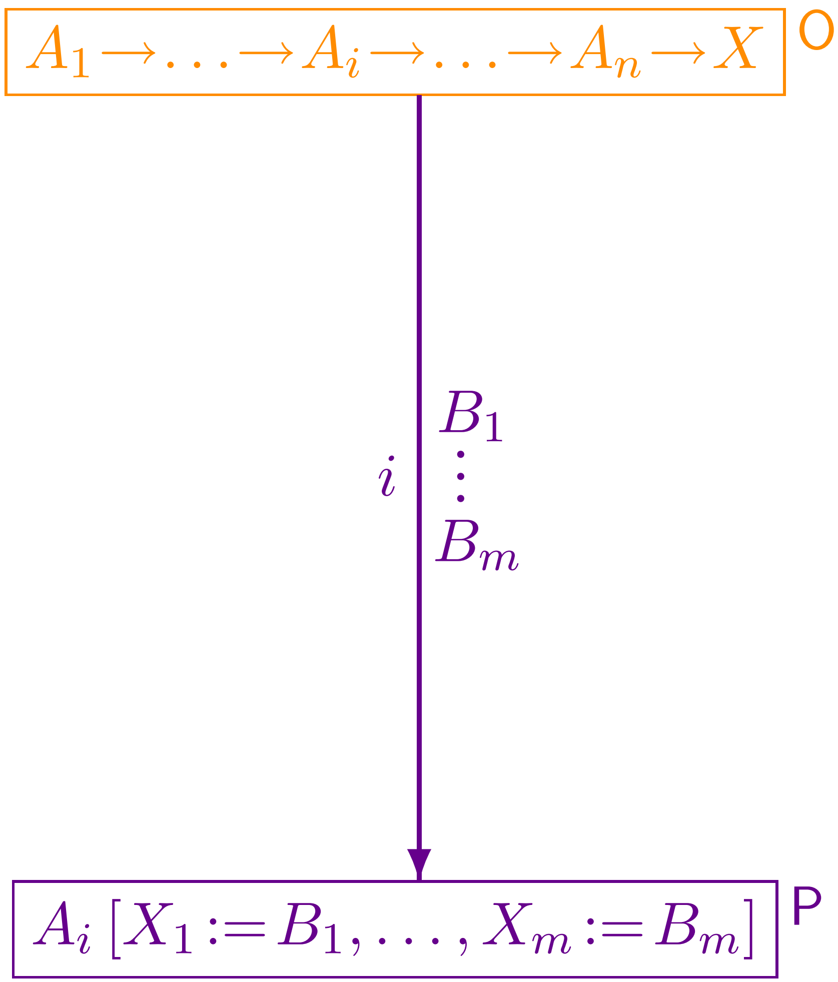
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



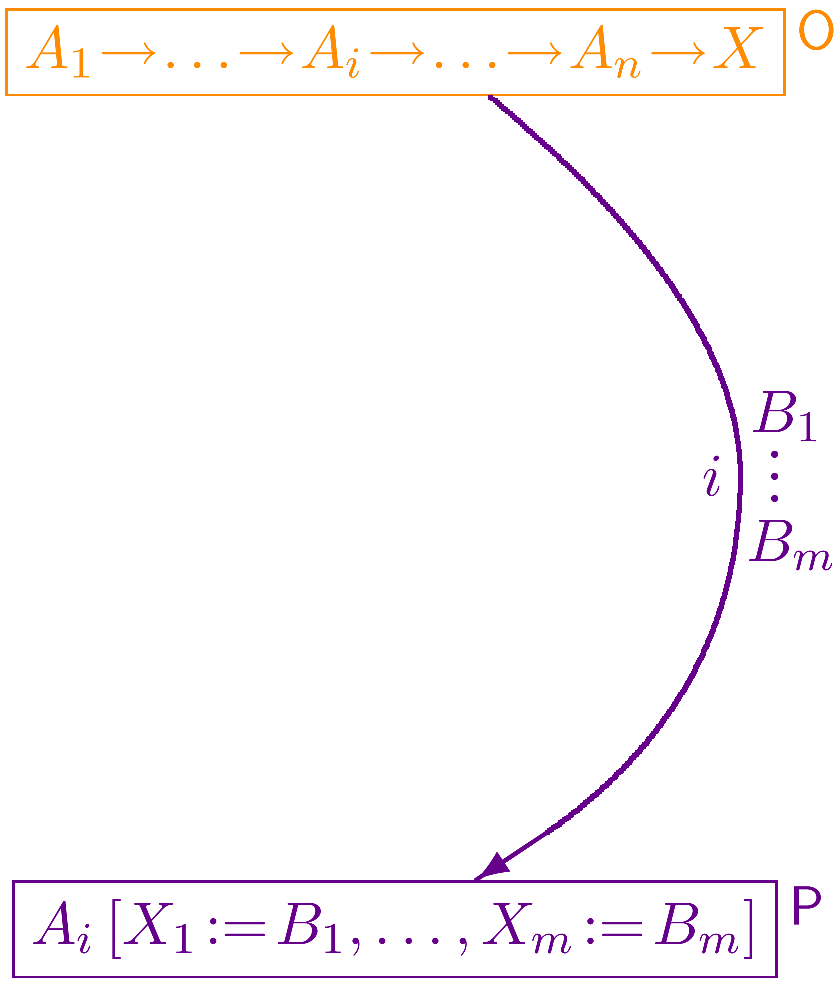
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



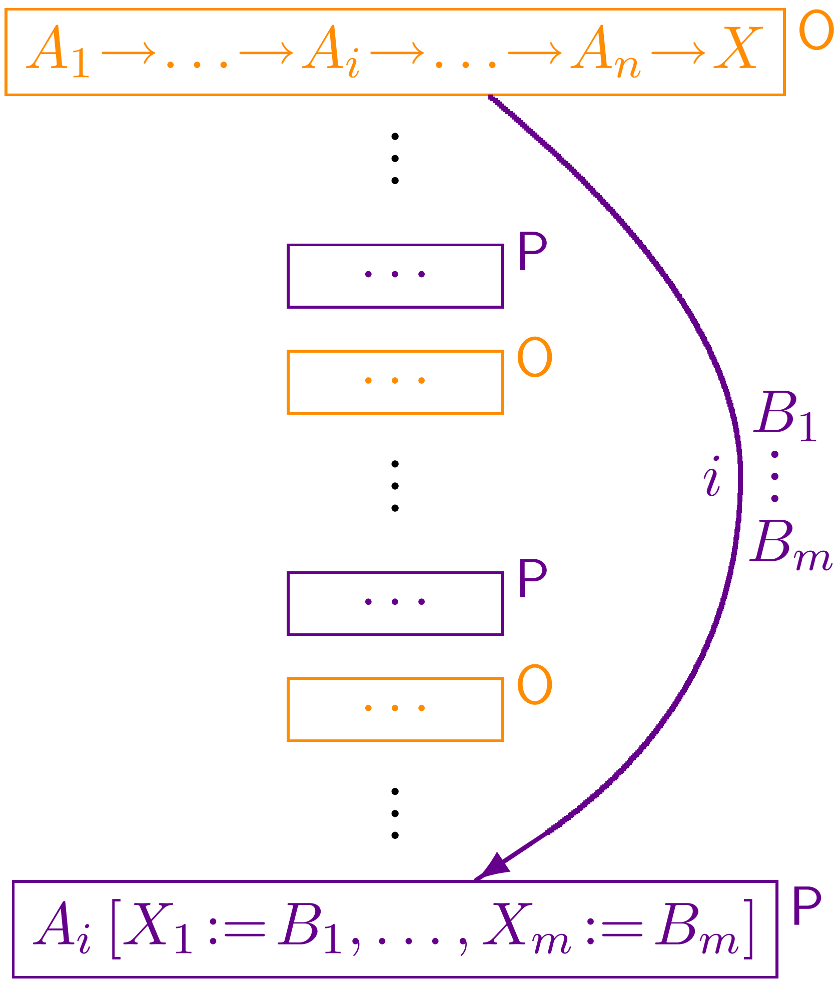
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch



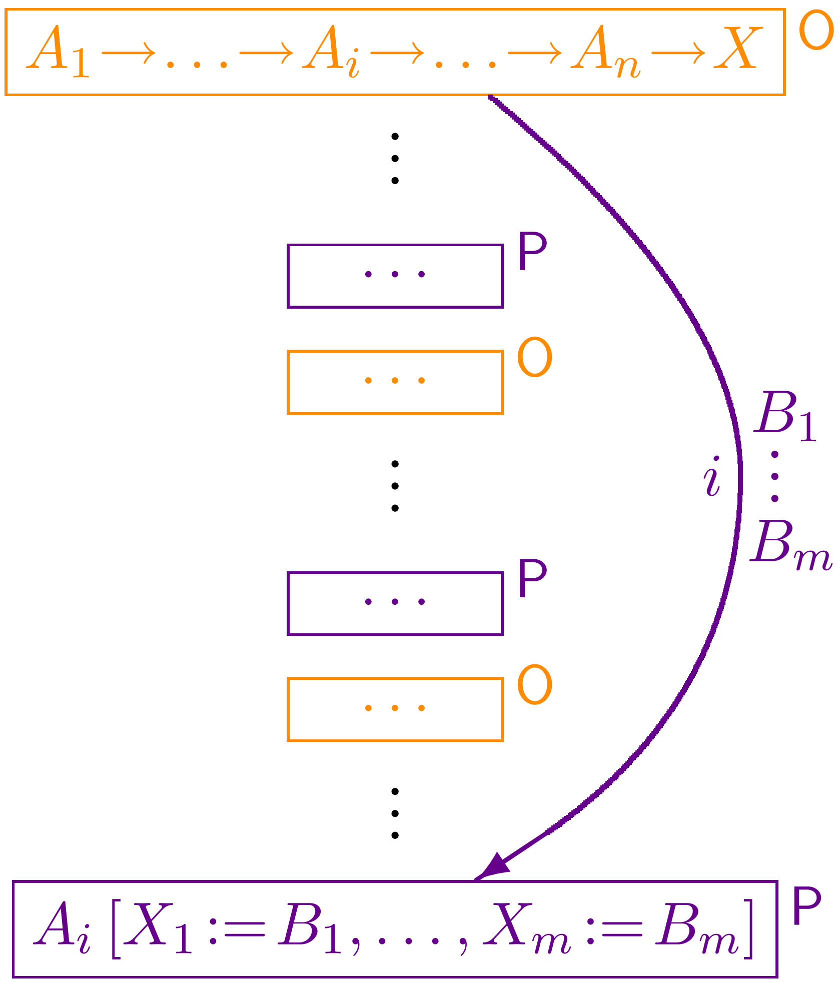
Hypergame $\mathcal{H}(A)$

- O resolves A
- resolve opposing branch

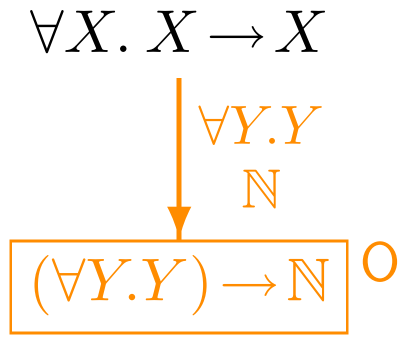


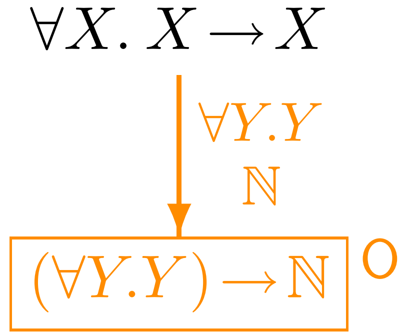
Hypergame $\mathcal{H}(A)$

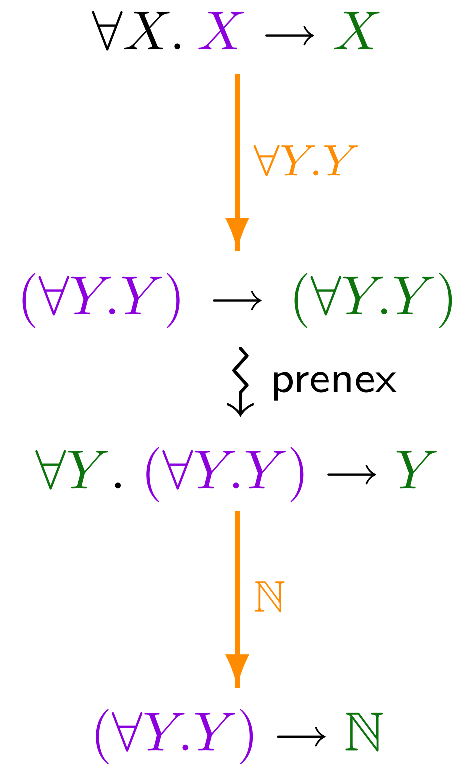
- O resolves A
- ↻ resolve opposing branch

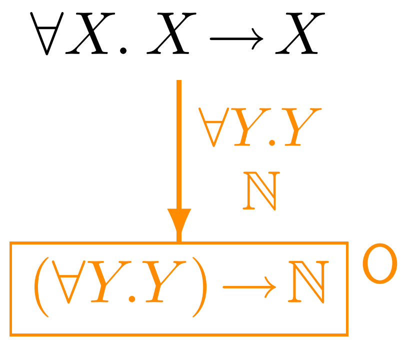


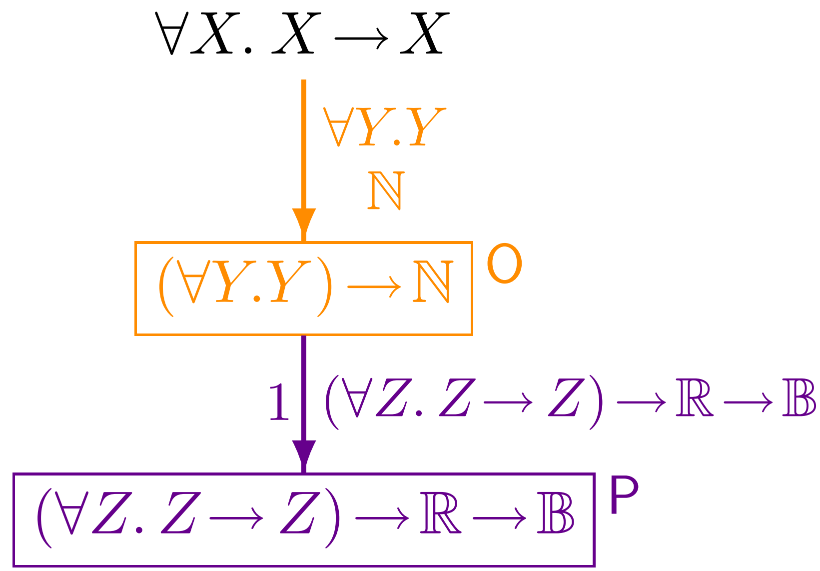
$$\forall X.\, X \to X$$

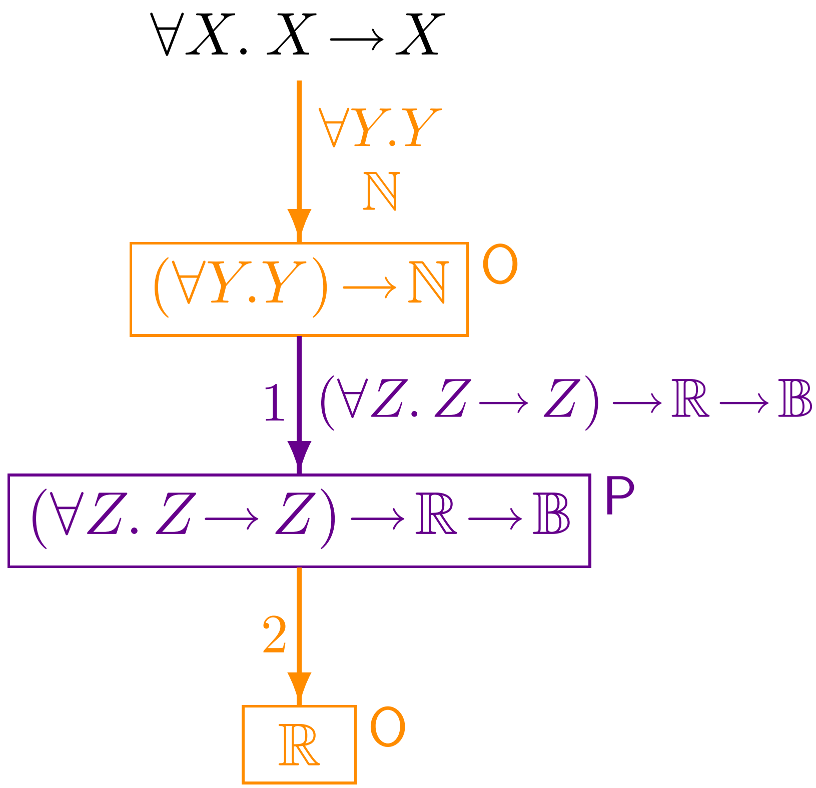

∀X. X → X
∀Y.Y
ℕ
(∀Y.Y) → ℕ
O

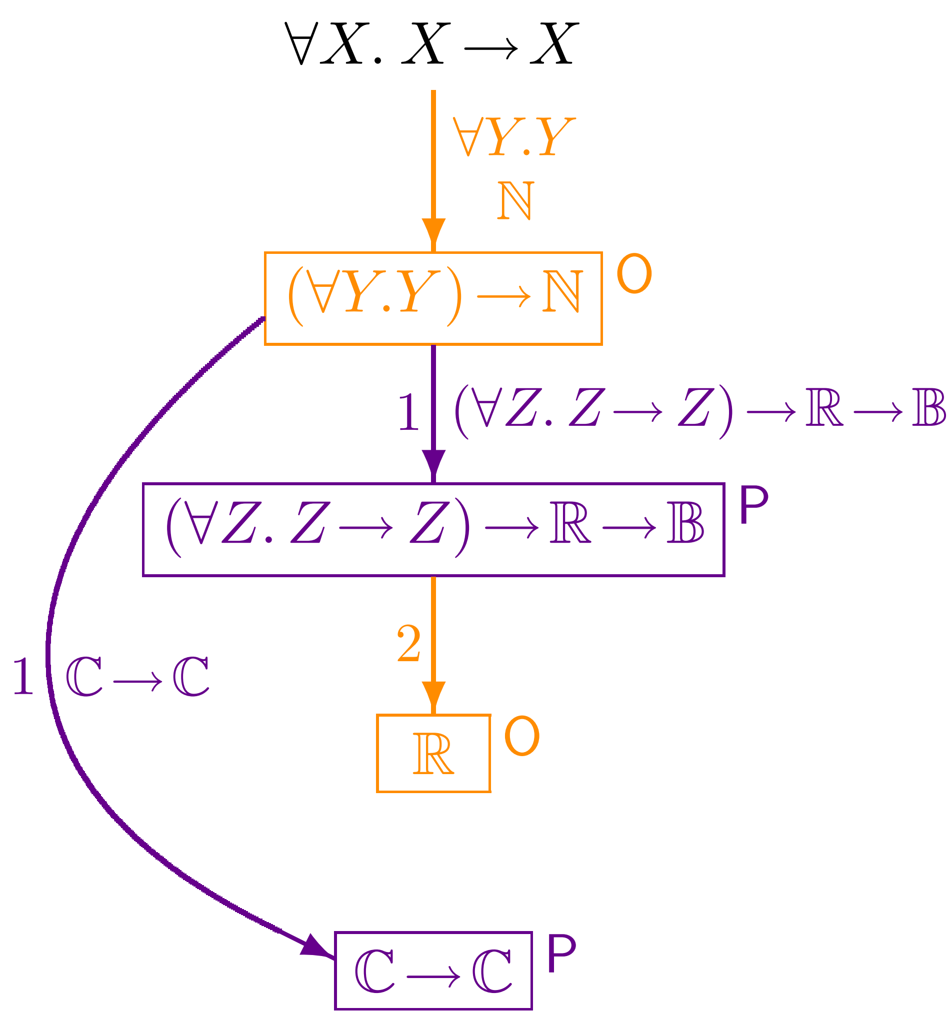

∀X. X → X
∀Y.Y
ℕ
(∀Y.Y) → ℕ
O


∀X. X → X
∀Y.Y
(∀Y.Y) → (∀Y.Y)
prenex
∀Y. (∀Y.Y) → Y
ℕ
(∀Y.Y) → ℕ

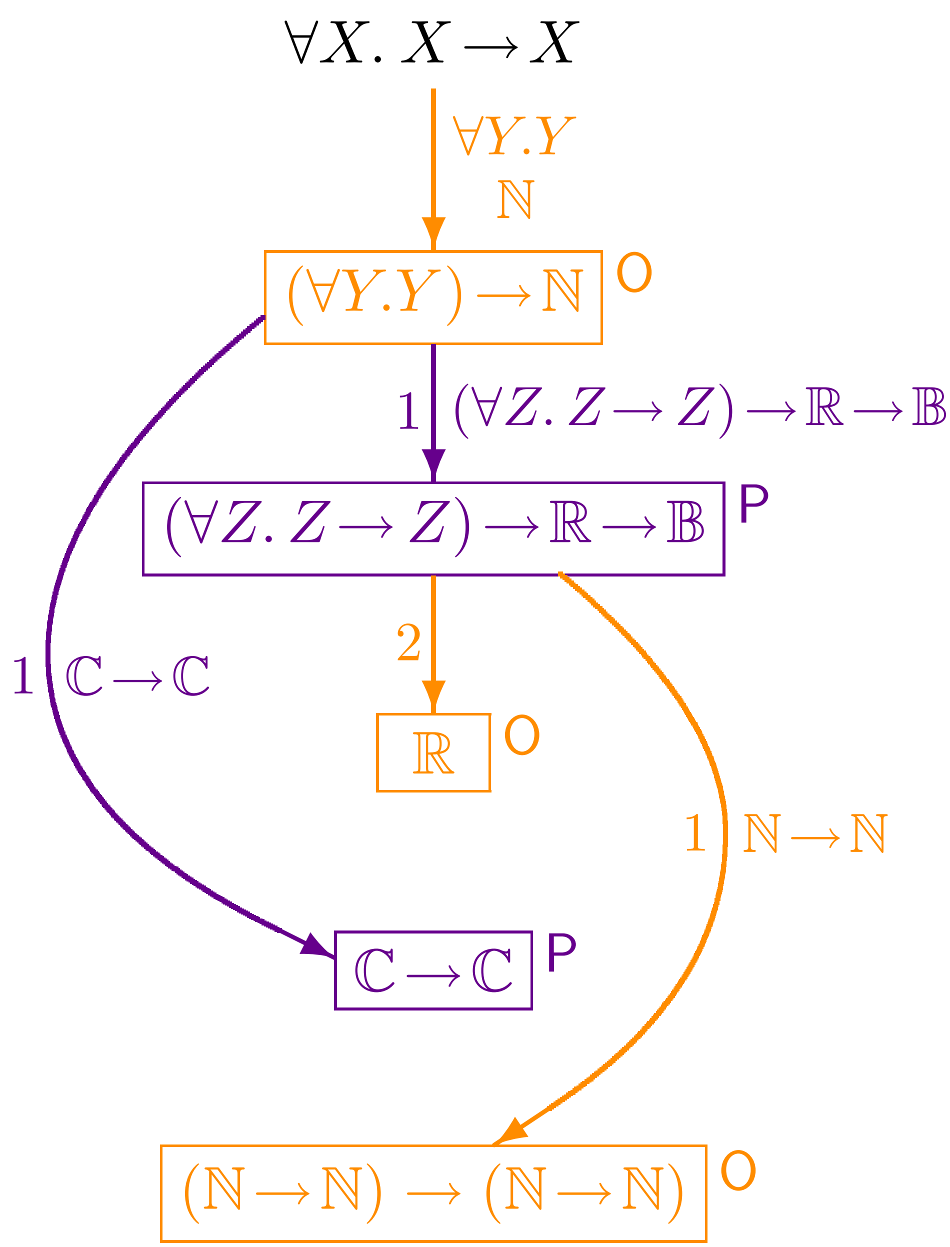

∀X. X → X
∀Y.Y
ℕ
(∀Y.Y) → ℕ
O


$\forall X.\, X \to X$
$\forall Y. Y$
$\mathbb{N}$
$(\forall Y. Y) \to \mathbb{N}$
O
1
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
P


∀X. X → X
∀Y.Y
ℕ
(∀Y.Y) → ℕ
O
1
(∀Z. Z → Z) → ℝ → 𝔹
(∀Z. Z → Z) → ℝ → 𝔹
P
2
ℝ
O



∀X. X → X
∀Y.Y
ℕ
(∀Y.Y) → ℕ
O
1
(∀Z. Z → Z) → ℝ → 𝔹
(∀Z. Z → Z) → ℝ → 𝔹
P
2
ℝ
O
1
ℂ → ℂ
ℂ → ℂ
P

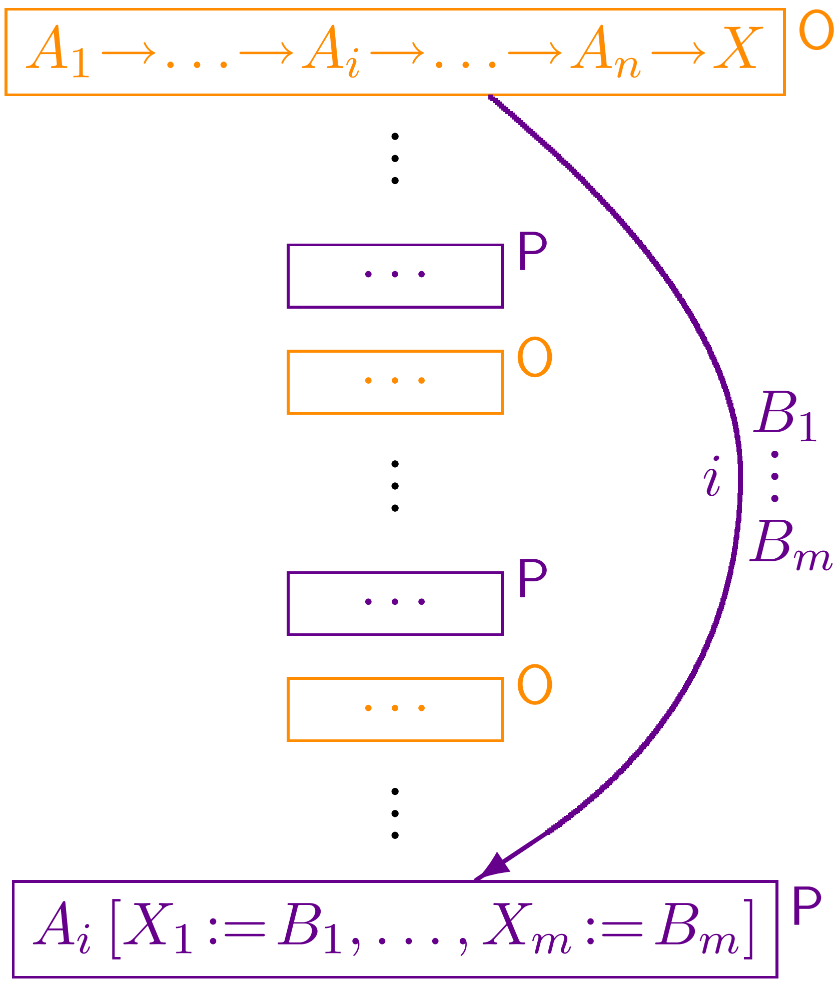


$\forall X.\, X \to X$
$\forall Y.Y$
$\mathbb{N}$
$(\forall Y.Y) \to \mathbb{N}$
O
1
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
P
1
$\mathbb{C} \to \mathbb{C}$
2
$\mathbb{R}$
O
1
$\mathbb{N} \to \mathbb{N}$
$\mathbb{C} \to \mathbb{C}$
P
$(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$
O

Hypergame $\mathcal{H}(A)$

- O resolves A

- ↻ resolve opposing branch



Composition

- 2^{nd}-order extension of Hyland-Ong (λ fragment):
  Interaction = Hyland-Ong interaction

- Originally (LICS'97, PhD) abstract machine style:
  (uniform) view $\leftrightarrow$ (uniform) view
  [Danos-Herbelin-Regnier, LICS'96; Nickau PhD '96]

- This tutorial:
  - back to Hyland-Ong style (all info shared)
  - **Transition (system) games**

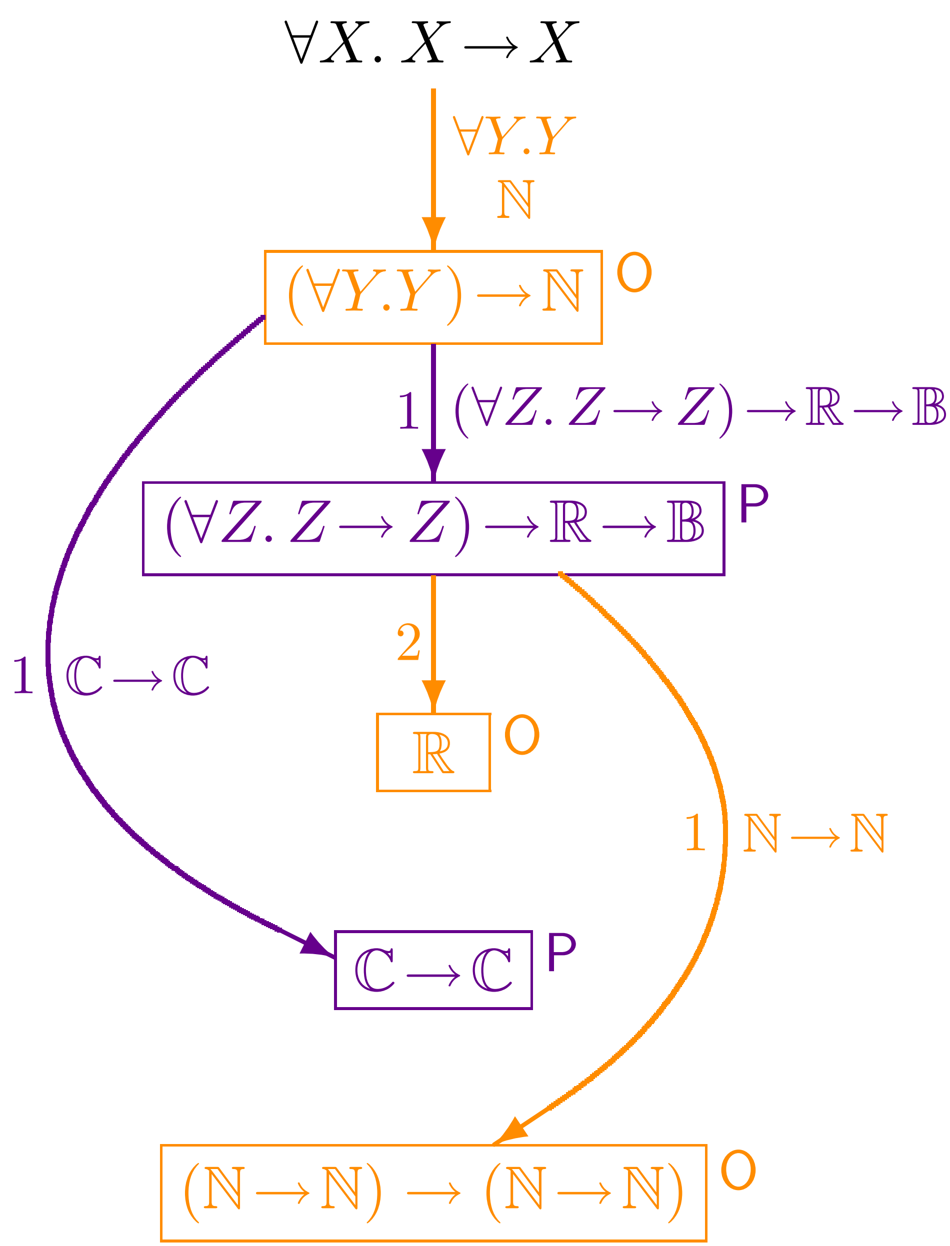
Transition game $\mathsf{LTS}(A)$

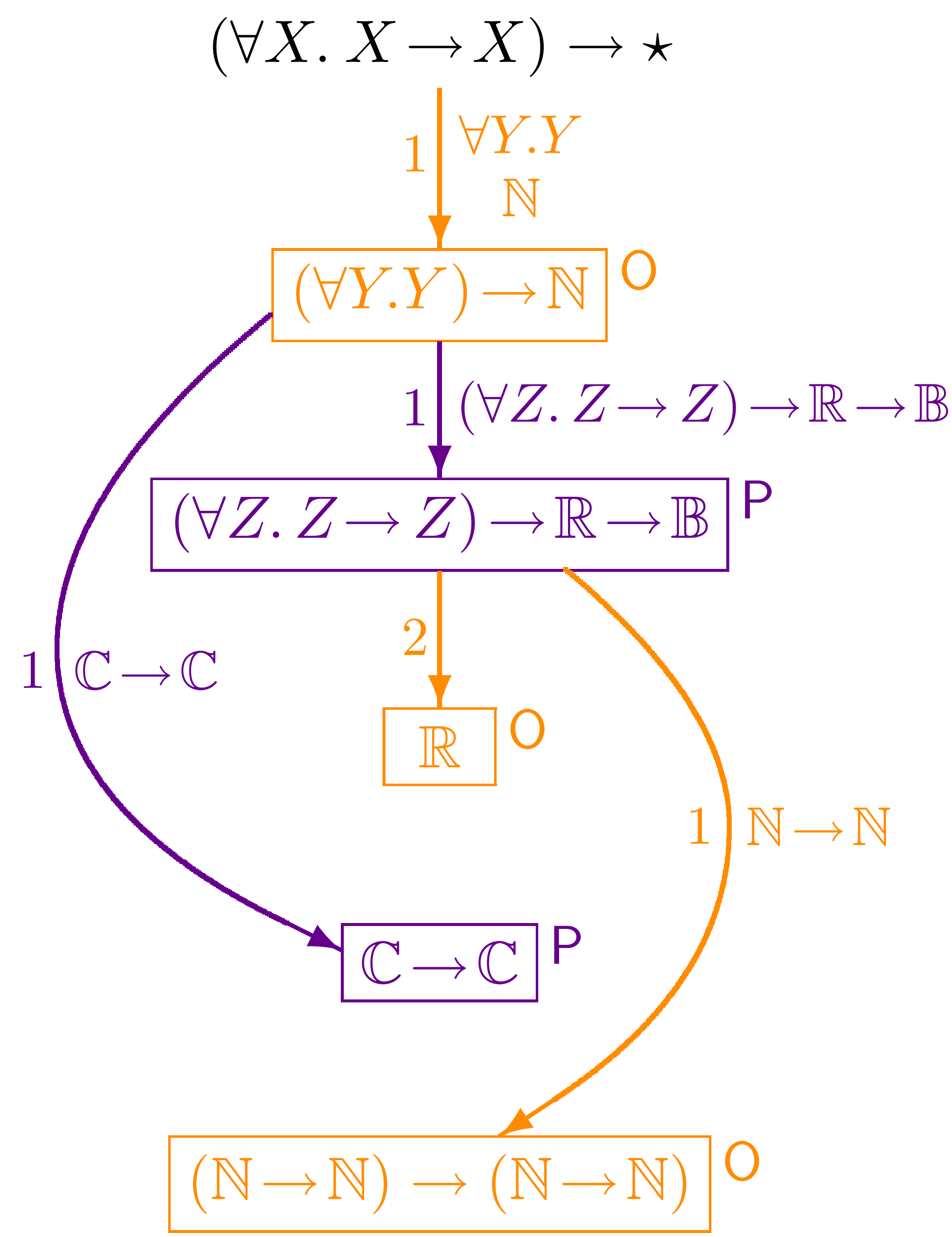
- States: $\textsc{Types}(F \cup \{\star\})$

- Transitions: $A_1 \to \ldots \to A_n \to X$

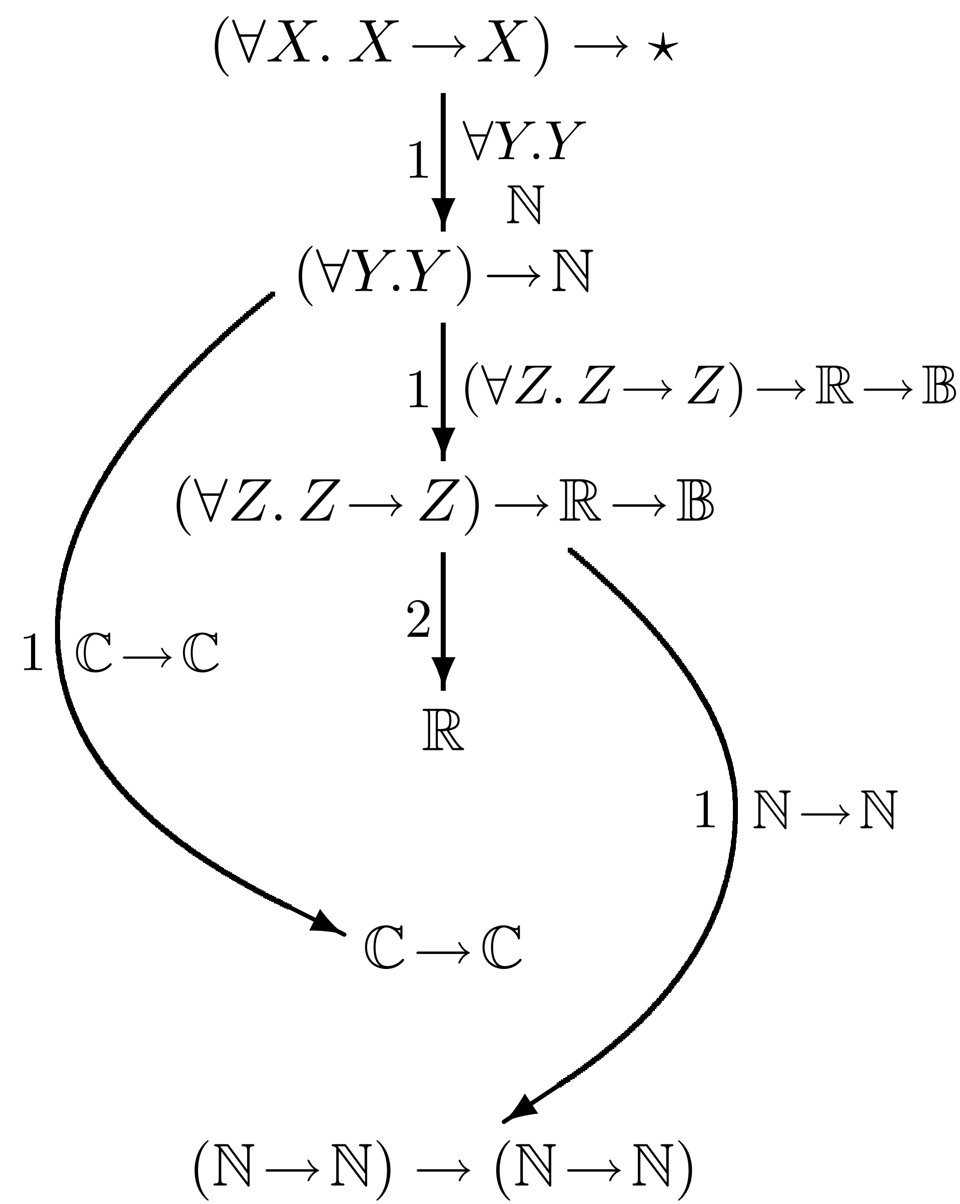
$$A_1 \to \ldots \to A_n \to X \quad \xrightarrow[\;i\;]{B_1 \cdots B_m} \quad A_i\,[X_1 := B_1, \ldots, X_m := B_m]$$

- Initial state: $A \to \star$


$\forall X.\, X \to X$
$\forall Y.Y$
$\mathbb{N}$
$(\forall Y.Y) \to \mathbb{N}$
O
1
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
P
1
$\mathbb{C} \to \mathbb{C}$
2
$\mathbb{R}$
O
1
$\mathbb{N} \to \mathbb{N}$
$\mathbb{C} \to \mathbb{C}$
P
$(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$
O

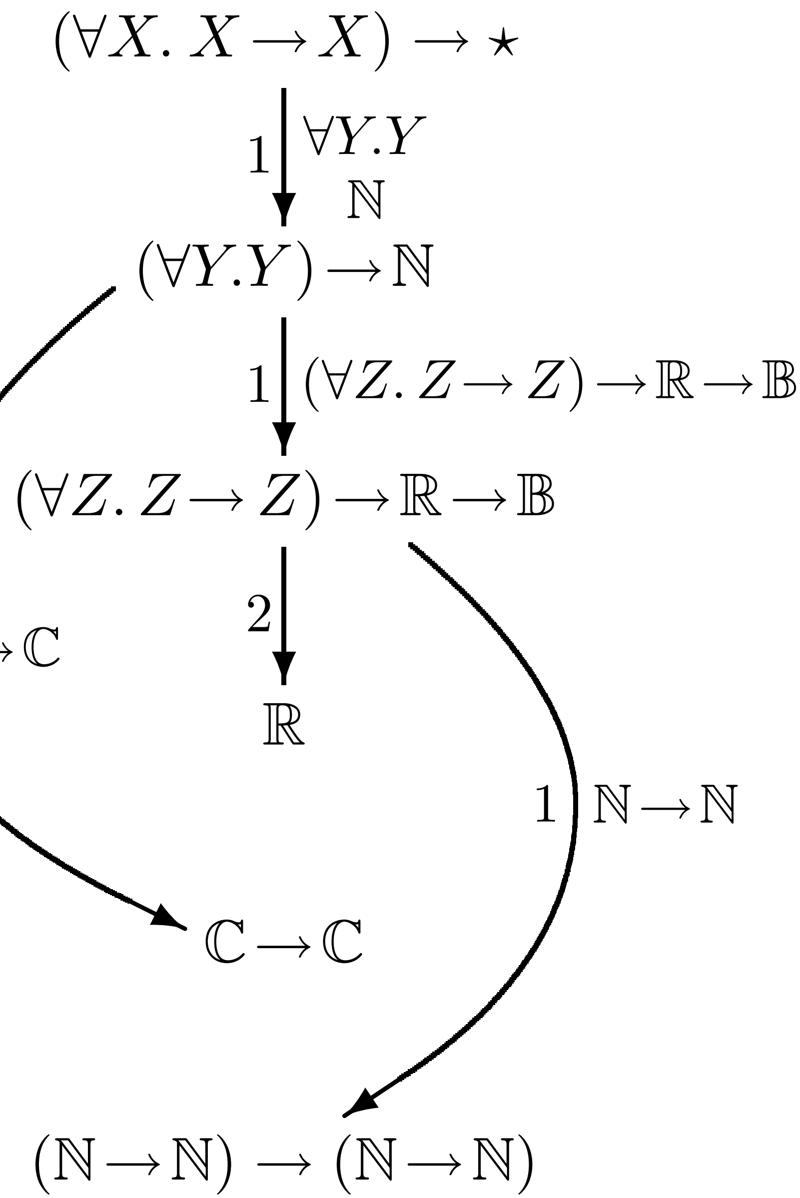


$(\forall X.\, X \to X) \to \star$
1
$\forall Y.Y$
$\mathbb{N}$
$(\forall Y.Y) \to \mathbb{N}$
O
1
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
P
1
$\mathbb{C} \to \mathbb{C}$
2
$\mathbb{R}$
O
1
$\mathbb{N} \to \mathbb{N}$
$\mathbb{C} \to \mathbb{C}$
P
$(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$
O

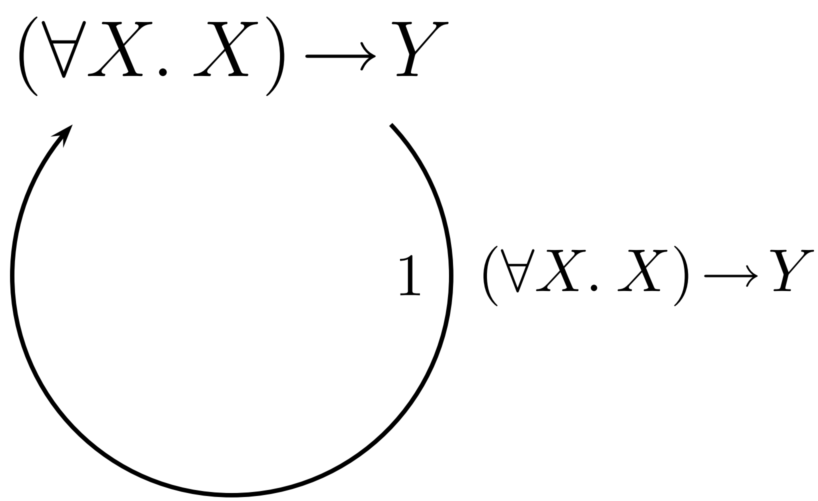

$(\forall X.\, X \to X) \to \star$
1
$\forall Y.Y$
$\mathbb{N}$
$(\forall Y.Y) \to \mathbb{N}$
1
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
2
$\mathbb{R}$
1 $\mathbb{C} \to \mathbb{C}$
1 $\mathbb{N} \to \mathbb{N}$
$\mathbb{C} \to \mathbb{C}$
$(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$

part of

LTS$(\forall X.\ X \to X)$



Loops in $\mathsf{LTS}(A)$



Transition game $\mathcal{G}(A)$

- Hyland-Ong arena $\mathsf{Ar}(LTS) \;:=\; \mathsf{Traces}_{ne}(LTS)$

- Hyland-Ong game $\mathcal{G}(LTS)$

- $\mathcal{G}(A) \;:=\; \mathcal{G}(\mathsf{LTS}(A)) \;\cong\; \mathcal{H}(A)$

- Arena isos: $\mathsf{Ar}(A \to B) \quad\cong\quad \mathsf{Ar}(A) \Rightarrow \mathsf{Ar}(B)$

$$\mathsf{Ar}(\forall X.A) \quad\cong\quad \prod_{\text{Types } B} \mathsf{Ar}(A[B/X])$$

Transition game $\mathcal{G}(A)$

- Hyland-Ong arena $\mathsf{Ar}(LTS) \ := \ \mathsf{Traces}_{ne}(LTS)$
- Hyland-Ong game $\mathcal{G}(LTS)$ (−VISIBILITY)
- $\mathcal{G}(A) \ := \ \mathcal{G}(\mathsf{LTS}(A)) \ \cong \ \mathcal{H}(A)$
- Arena isos: $\mathsf{Ar}(A \rightarrow B) \quad \cong \quad \mathsf{Ar}(A) \Rightarrow \mathsf{Ar}(B)$

$$\mathsf{Ar}(\forall X.A) \quad \cong \quad \prod_{\mathsf{Types}\ B} \mathsf{Ar}(A[B/X])$$

Hypergame composition

$$\mathsf{Ar}(B \to C) \quad \cong \quad \mathsf{Ar}(B) \Rightarrow \mathsf{Ar}(C)$$

Composition $=$ Hyland-Ong composition

Uniformity & Full Completeness

Uniformity precondition

- **COPYCAT** Condition [LICS'97, PhD]:

$$X \text{ is rightmost var in } \mathsf{O}\text{'s assertion}$$
$$\Downarrow$$
$$X \text{ is rightmost var in } \mathsf{P}\text{'s assertion}$$

Strengthens (D10) in Felscher'85 $\leftarrow$ Lorenz(en)

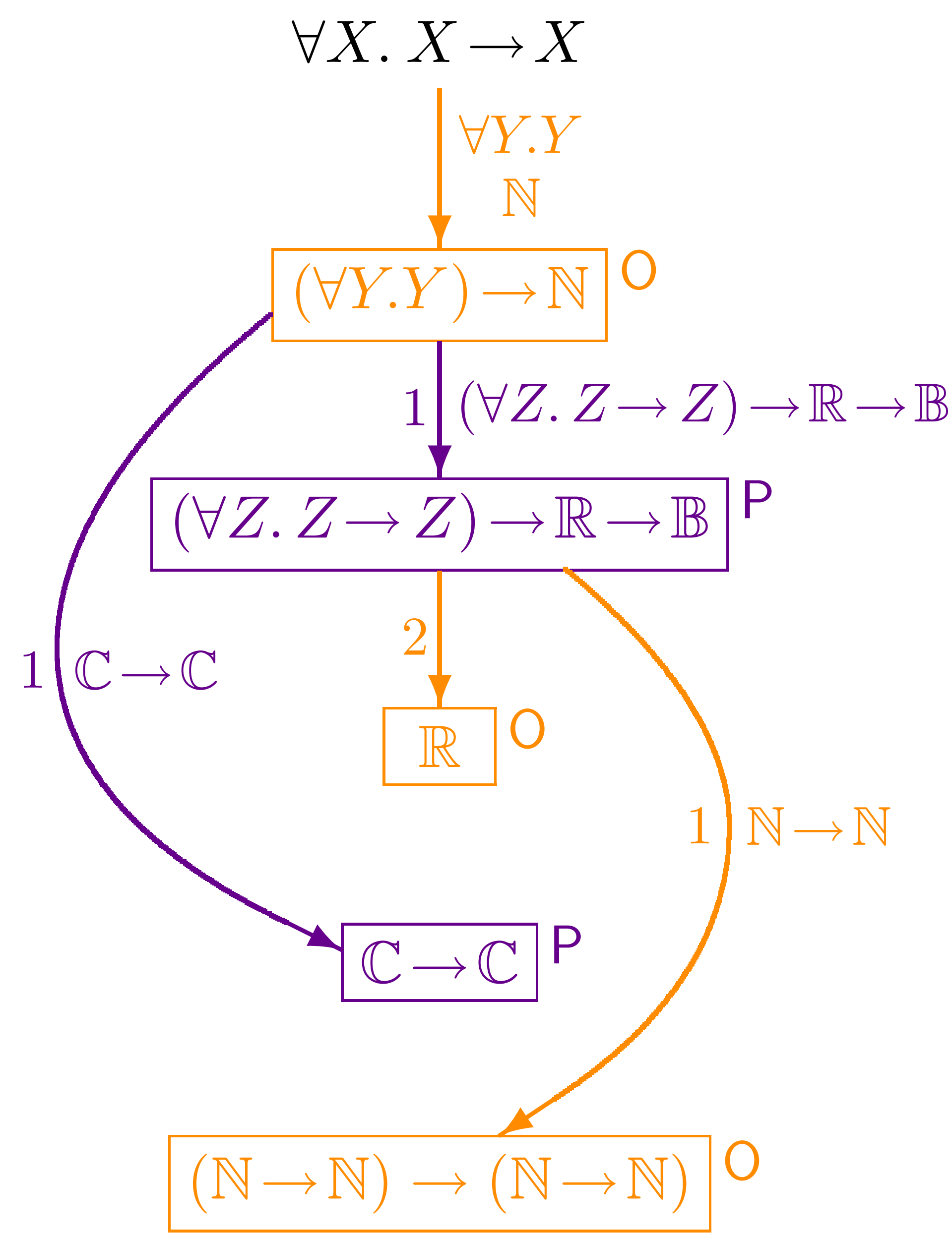
Uniformity precondition

- COPYCAT Condition [LICS'97, PhD]:

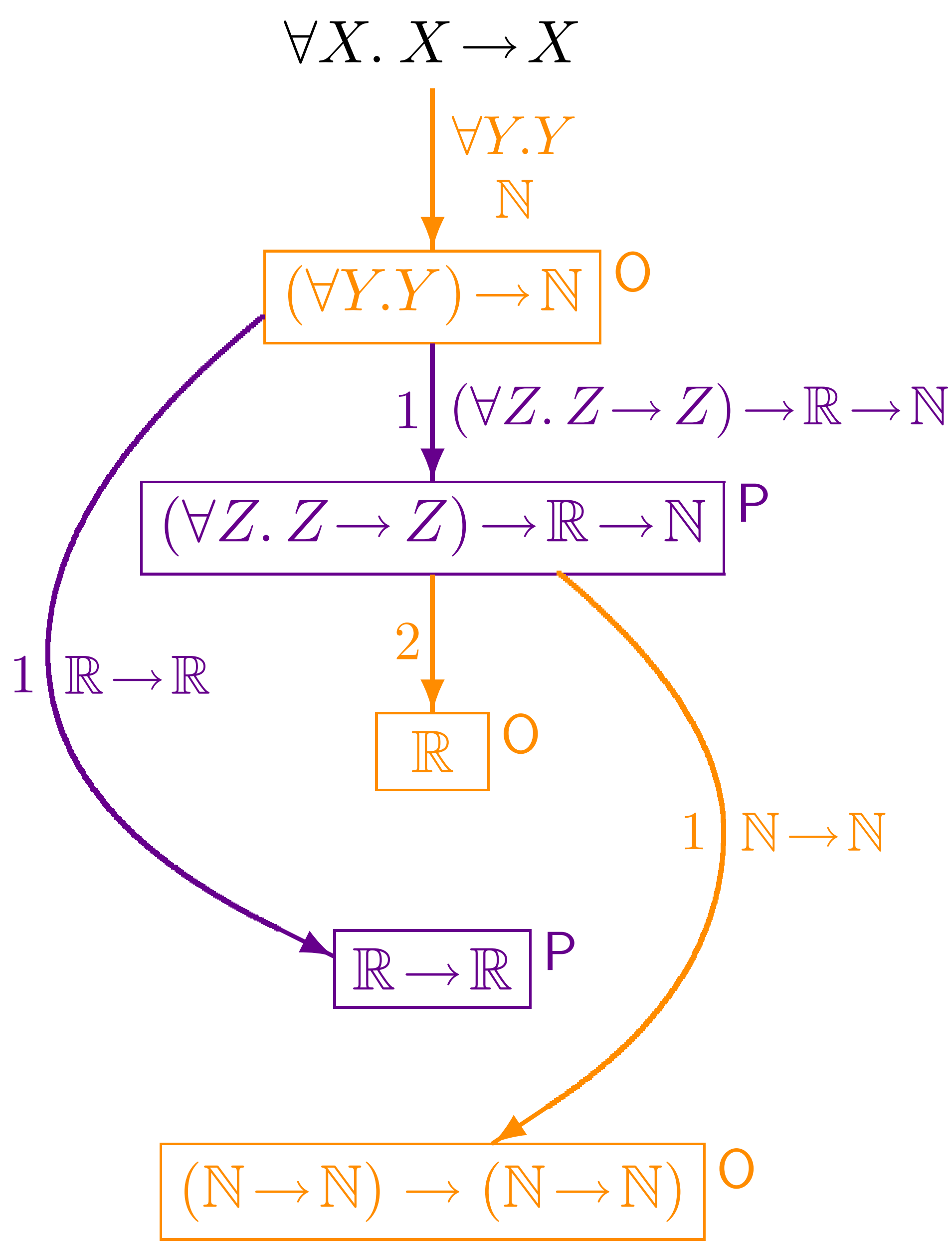
$$X \text{ is rightmost var in } \mathsf{O}\text{'s assertion}$$
$$\Downarrow$$
$$X \text{ is rightmost var in } \mathsf{P}\text{'s assertion}$$

Strengthens (D10) in Felscher'85 $\longleftarrow$ Lorenz(en)

Note: henceforth read $\mathbb{N}, \mathbb{B}, \mathbb{R}, \ldots$ as type variables



$\forall X.\, X \to X$
$\forall Y.Y$
$\mathbb{N}$
$(\forall Y.Y) \to \mathbb{N}$
O
1
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
$(\forall Z.\, Z \to Z) \to \mathbb{R} \to \mathbb{B}$
P
1
$\mathbb{C} \to \mathbb{C}$
2
$\mathbb{R}$
O
1
$\mathbb{N} \to \mathbb{N}$
$\mathbb{C} \to \mathbb{C}$
P
$(\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$
O



∀X. X → X
∀Y.Y
ℕ
(∀Y.Y) → ℕ
O
1
(∀Z. Z → Z) → ℝ → ℕ
(∀Z. Z → Z) → ℝ → ℕ
P
2
ℝ
O
1 ℝ → ℝ
1 ℕ → ℕ
ℝ → ℝ
P
(ℕ → ℕ) → (ℕ → ℕ)
O

Copycat $\rightarrow$ (UC) Uniformity by Copycat expansion

- Suppose
  - play p satisfies Copycat
  - X free, first imported by O
- **Copycat expansion** $p(X \mapsto A)$: copycat $A^{+} \leftrightarrow A^{-}$
  $\sim$ Felscher's *skeleton* expansion
- Strategy **uniform** if:
  - satisfies Copycat
  - closed under copycat expansions

Copycat expansion $\sim \eta$ expansion

$$\Lambda X.\lambda f^{X}.f \quad : \quad \forall X.X \rightarrow X \qquad (\eta\text{-exp. } \beta\text{-normal})$$

$$\Big\downarrow \; \mathbb{N} \rightarrow \mathbb{N}$$

$$\lambda f^{\mathbb{N}\rightarrow\mathbb{N}}.f \quad : \quad (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N}) \qquad (\text{not } \eta\text{-exp.})$$

Copycat expansion $\sim \eta$ expansion

$$\Lambda X.\lambda f^{X}.f \quad : \quad \forall X.X \to X \qquad (\eta\text{-exp. } \beta\text{-normal})$$

$$\Big\downarrow \; \mathbb{N} \to \mathbb{N}$$

$$\lambda f^{\mathbb{N}\to\mathbb{N}}.f \quad : \quad (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}) \qquad (\text{not } \eta\text{-exp.})$$

$$\Big\downarrow \; \eta$$

$$\lambda f^{\mathbb{N}\to\mathbb{N}}.\lambda x^{\mathbb{N}}.f x \quad : \quad (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}) \qquad (\eta\text{-exp.})$$

Full completeness

An innocent uniform strategy is **compact** if it is generated from a finite set of views by copycat expansion

THEOREM:

System F terms $\xrightarrow{\text{onto}}$ total compact strategies