A classical sequent calculus free of structural rules

DOMINIC HUGHES
Stanford University

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Gentzen's system **LK**, classical sequent calculus, has explicit structural rules for contraction and weakening. They can be absorbed (in a right-sided formulation) by replacing the axiom $P, \neg P$ by $\Gamma, P, \neg P$ for any context Γ, and replacing the original disjuction rule with Γ, A, B implies $\Gamma, A \vee B$.

This paper presents a classical sequent calculus which is also free of contraction and weakening, but more symmetrically: both contraction and weakening are absorbed into conjunction, leaving the axiom rule untouched. It uses a hybrid conjunction rule, combining the standard context-sharing and context-splitting rules: Γ, Δ, A and Γ, Σ, B implies $\Gamma, \Delta, \Sigma, A \wedge B$. We call this system *Hybrid Logic*. Hybrid conjunction is critical for the liberation from structural rules: relaxing it to the pair of standard conjunction rules breaks completeness.

We prove a *minimality theorem* for hybrid logic: any sequent calculus (within a standard class of right-sided calculi) is complete *iff* it contains hybrid logic. Thus one can view hybrid logic as a "complete core" of Gentzen's **LK**.

1 Introduction

The following Gentzen-Schütte-Tait [Gen39, Sch50, Tai68] system, denoted **GS1p** in [TS96], is a standard right-sided formulation of the propositional fragment of Gentzen's classical sequent calculus **LK**:

System GS1p

$$\frac{}{P, \neg P} \qquad\qquad \frac{\Gamma, A \qquad\qquad \Gamma, B}{\Gamma, A \wedge B}\ \& \qquad\qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i$$

$$\frac{\Gamma}{\Gamma, A}\ \mathsf{W} \qquad\qquad \frac{\Gamma, A, A}{\Gamma, A}\ \mathsf{C}$$

Here P ranges over propositional variables, A, A_i, B range over formulas, Γ ranges over disjoint unions of formulas, and comma denotes disjoint union. (We have labelled the conjunction and disjunction rules with $\&$ and $\oplus$ for reasons which will become apparent later.) By defining a sequent as a disjoint union of formulas, rather than an ordered list, we avoid an exchange/permutation rule (*cf.* [TS96, §1.1]). Negation is primitive on propositional variables P, and extends to compound formulas by de Morgan duality.

The structural rules, weakening W and contraction C, are absorbed in the following variant, a right-sided formulation of the propositional part of the calculus of [Ket44], called **GS3p** in [TS96].[1]

System GS3p

$$\frac{}{\Gamma, P, \neg P} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\,\& \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\,⅋$$

The new axiom $\Gamma, P, \neg P$ amounts to the original axiom $P, \neg P$ followed immediately by weakenings. This paper presents a classical propositional sequent calculus **Hp** which is also free of structural rules:

System Hp

$$\frac{}{P, \neg P} \qquad \frac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\,\wedge \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\,⅋ \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\,\oplus_i$$

(By analogy with **GS3** and **GS3p** in [TS96], we reserve the symbol **H** for a full system with quantifiers, and use **Hp** to denote the propositional system. Also following [TS96], cut is treated separately.[2]) A distinguishing feature of **Hp** is the *hybrid conjunction rule*

$$\frac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\,\wedge$$

which combines the standard context-sharing and context-splitting conjunction rules:

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\,\& \qquad\qquad \frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\,\otimes$$

Since it uses hybrid conjunction, we refer to **Hp** as (cut-free propositional) *Hybrid Logic*. The context-sharing rule $\&$ is the special case of the hybrid rule with Δ and Σ empty; the context-splitting rule $\otimes$ is the special case with Γ empty. In contrast to

[1] We have labelled the disjunction rule as ⅋, to distinguish it from the disjunction rule $\oplus$ of **GS1p**.

[2] To maximise emphasis on the hybrid conjunction rule, we omit quantifiers and cut in this paper.

GS3p, contraction and weakening are absorbed symmetrically: both are absorbed into the conjunction rule, leaving the axiom rule intact.

Our main theorem (page 10) is completeness:

Theorem (Completeness of cut-free propositional Hybrid Logic)

Every true formula is provable in **Hp**.

Hybrid conjunction is critical for the liberation from structural rules: relaxing it to the pair of standard conjunction rules (context-sharing $\&$ and context-splitting $\otimes$) breaks completeness. The following system fails to be complete:[3]

System Hp^{-} (incomplete)

$$\frac{}{P, \neg P} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \& \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ \invamp$$

$$\frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\ \otimes \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i$$

Minimality of Hybrid Logic. We make precise the sense in which hybrid logic represents a completeness threshold. Let a *standard system* be any propositional sequent calculus containing the axiom $\overline{P, \neg P}$ and any subset of the following *standard rules*:

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \& \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ \invamp \qquad \frac{\Gamma}{\Gamma, A}\ \mathsf{W}$$

$$\frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\ \otimes \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i \qquad \frac{\Gamma, A, A}{\Gamma, A}\ \mathsf{C}$$

(Thus there are $2^6 = 64$ such systems.) We prove:

Minimality Theorem

A standard system is complete iff it contains propositional hybrid logic **Hp**.

[3] This precursor of **Hp** is (cut-free) multiplicative-additive linear logic [Gir87] with the formula constructors *tensor* $\otimes$ and *with* $\&$ collapsed to $\wedge$, and the formula constructors *plus* $\oplus$ and *par* $\invamp$ collapsed to $\vee$. This paper will not require any background knowledge of linear logic.

Here a system S contains a system T if each rule of T is a derived rule of S. For example, system **GS1p** (displayed on page 1) contains **Hp**, since the hybrid conjunction rule $\wedge$ and the disjunction rule $\parr$ of **Hp** can be derived in **GS1p**:

$$\dfrac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\ \wedge \qquad \longleftarrow \qquad \dfrac{\dfrac{\Gamma, \Delta, A}{\Gamma, \Delta, \Sigma, A}\ \mathsf{W}^* \qquad \dfrac{\Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, B}\ \mathsf{W}^*}{\Gamma, \Delta, \Sigma, A \wedge B}\ \&$$

$$\dfrac{\Gamma, A, B}{\Gamma, A \vee B}\ \parr \qquad \longleftarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma, A, B}{\Gamma, A, A \vee B}\ \oplus_2}{\Gamma, A \vee B, A \vee B}\ \oplus_1}{\Gamma, A \vee B}\ \mathsf{C}$$

where W^* denotes a sequence of zero or more weakenings.

If we define systems S and T as equivalent if each contains the other (*i.e.*, every rule of S is derivable in T, and vice versa), then there are in fact only three complete standard systems, up to equivalence. The largest is **GS1p** (page 1). We refer to the other two, each contained in **GS1p**, as *Positive Logic* and *Negative Logic*, respectively:[4]

System Pp (Propositional Positive Logic)

$$\dfrac{}{P, \neg P} \qquad \dfrac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\ \otimes \qquad \dfrac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i \qquad \dfrac{\Gamma, A, A}{\Gamma, A}\ \mathsf{C}$$

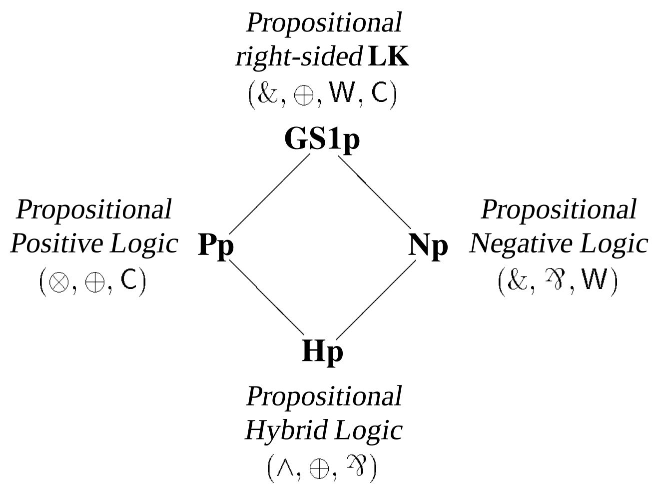
System Np (Propositional Negative Logic)

$$\dfrac{}{P, \neg P} \qquad \dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \& \qquad \dfrac{\Gamma, A, B}{\Gamma, A \vee B}\ \parr \qquad \dfrac{\Gamma}{\Gamma, A}\ \mathsf{W}$$

Together with propositional hybrid logic **Hp**, these systems sit in the following Hasse diagram of containments:[5]

[4] Our terminology here comes from polarity of connectives in linear logic [Gir87]: *tensor* $\otimes$ and *plus* $\oplus$ are positive, and *with* $\&$ and *par* $\parr$ are negative. (This validates Girard's choice of notation for the connectives.)

[5] These containments are strict since neither contraction C nor weakening W can be derived from the other standard rules.



Thus propositional hybrid logic **Hp** can be seen as a "complete core" of **GS1p**, hence of (propositional) Gentzen's **LK**.

1.1 Avenues for future work

1. *Cut.* Chapter 4 of [TS96] gives a detailed analysis of cut for Gentzen systems. One could pursue an analogous analysis of cut for hybrid logic. Aside from context-splitting and context-sharing cut rules

$$\frac{\Delta, A \qquad \Sigma, \neg A}{\Delta, \Sigma}\,\mathsf{cut}_{\otimes} \qquad\qquad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}\,\mathsf{cut}_{\&}$$

   one might also investigate a hybrid cut rule:

$$\frac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, \neg A}{\Gamma, \Delta, \Sigma}\,\mathsf{cut}$$

2. *Quantifiers.* Explore the various ways of adding quantifiers to **Hp**, for a full first-order system **H**.

3. *Mix (nullary multicut).* Gentzen's multicut rule

$$\frac{\Delta, A_1, \ldots, A_m \qquad \Sigma, \neg A_1, \ldots, \neg A_n}{\Delta, \Sigma}$$

   in the nullary case $m = n = 0$ has been of particular interest to linear logicians [Gir87], who call it the *mix* rule. One could investigate context-splitting, context-sharing and hybrid incarnations:

$$\frac{\Delta \qquad \Sigma}{\Delta, \Sigma}\,\mathsf{mix}_{\otimes} \qquad\qquad \frac{\Gamma \qquad \Gamma}{\Gamma}\,\mathsf{mix}_{\&} \qquad\qquad \frac{\Gamma, \Delta \qquad \Gamma, \Sigma}{\Gamma, \Delta, \Sigma}\,\mathsf{mix}$$

2 Notation and terminology

Formulas are built from literals (propositional variables $P, Q, R \ldots$ and their formal complements $\overline{P}, \overline{Q}, \overline{R}, \ldots$) by the binary connectives ***and*** $\wedge$ and ***or*** $\vee$. Define ***negation*** or ***not*** $\neg$ as an operation on formulas (rather than a syntactic formula construct): $\neg P = \overline{P}$ and $\neg\overline{P} = P$ for all propositional variables P, extending to arbitrary formulas by de Morgan duality $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ and $\neg(A \vee B) = (\neg A) \wedge (\neg B)$. Thus $\neg$ is not a connective; only $\wedge$ and $\vee$ are connectives.

We identify a formula with its parse tree, a tree labelled with literals at the leaves and connectives at the internal vertices. A ***sequent*** is a non-empty disjoint union of formulas. Thus a sequent is a particular kind of labelled forest.[6] We write comma for disjoint union, which is commutative and associative (as usual in graph theory). Throughout the document, $P, Q, \ldots$ range over propositional variables, $A, B, \ldots$ over formulas, and $\Gamma, \Delta, \ldots$ over (possibly empty) disjoint unions of formulas.

A formula is ***true*** (or *valid*) if, for all possible assignments of its propositional variables to 0 or 1, it evaluates to 1 (with the usual interpretation of $\wedge$ and $\vee$ on $\{0, 1\}$). A sequent $A_1, \ldots, A_n$ is true iff $A_1 \vee \ldots \vee A_n$ is true (with an arbitrary choice of bracketing). A sequent Γ is a ***subsequent*** of a sequent Δ if $\Delta = \Gamma, \Sigma$ for some (possibly empty) disjoint union Σ of formulas, and a ***proper subsequent*** of Δ if Σ is non-empty.

3 Hybrid logic

Define ***Propositional Hybrid Logic*** as the following system:

System Hp (Propositional Hybrid Logic)

$$\frac{}{P, \neg P} \qquad \frac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\wedge \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\invamp \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\oplus_i$$

We refer to this conjunction rule as ***hybrid conjunction***, as it is a hybrid of the usual context-sharing and context-splitting rules $\&$ and $\otimes$ (see page 2 of the Introduction). The rule is used in [Urb00, UB01] to study cut elimination in classical logic via term assignment.

EXAMPLE 1 Here is a proof of Peirce's law $\big((P \Rightarrow Q) \Rightarrow P\big) \Rightarrow P$ in **Hp**, where

[6]This foundational treatment of formulas and sequents as labelled trees and forests sidesteps the common problem of “formulas” versus “formula occurrences”: disjoint unions of graphs are well understood in graph theory [Bol02], so we avoid reinventing the wheel.

$A \Rightarrow B$ abbreviates $(\neg A) \vee B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\overline{P},\ P}}{\overline{P} \vee Q,\ P}\;\oplus_1 \qquad \dfrac{}{P,\ \overline{P}}}{(\overline{P} \vee Q) \wedge \overline{P},\ P}\;\wedge}{\big((\overline{P} \vee Q) \wedge \overline{P}\big) \vee P}\;\parr$$

This use of the hybrid conjunction rule

$$\dfrac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\;\wedge$$

is degenerate, since Σ and Δ are empty in the rule: it reduces to the special case of a context-sharing conjunction $\&$, with $\Gamma = P$.

EXAMPLE 2 The following example employs the full power of hybrid conjunction, in the final rule:

$$\dfrac{\dfrac{}{P,\ \overline{P}} \qquad \dfrac{\dfrac{}{Q,\ \overline{Q}} \qquad \dfrac{}{P,\ \overline{P}}}{Q,\ \overline{Q} \wedge P,\ \overline{P}}\;\wedge}{P \wedge Q,\ \overline{Q} \wedge P,\ \overline{P}}\;\wedge$$

The final rule is an instance of hybrid conjunction rule

$$\dfrac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\;\wedge$$

with $A = P$, $B = Q$, shared context $\Gamma = \overline{P}$, left-splitting context Δ empty and right-splitting context $\Sigma = \overline{Q} \wedge P$.

REMARK 1 Suppose Γ and Δ are disjoint unions of formulas (so each is either a sequent or empty). Also suppose that (by some additional annotation or otherwise) it is clear which formulas Γ and Δ have in common (if any). Define $\Gamma \cup \Delta$ as the (non-disjoint) union of Γ and Δ (so common formulas are not duplicated).[7] For example, if $\Gamma = \Sigma, \Gamma'$ and $\Delta = \Sigma, \Delta'$ (with Δ' and Γ' disjoint from one another, and from Σ)

[7] We leave possible foundational issues involved with "shared/common formulas" and the superimposing operation $\cup$ to the reader's mathematical common sense: any reader finding the definition of $\Gamma \cup \Delta$ ambiguous can simply skip this remark, as it is tangential to the main thread of the paper.

then $\Gamma \cup \Delta$ is Σ, Γ', Δ' (whereas the disjoint union Γ, Δ of Γ and Δ is $\Sigma, \Sigma, \Gamma', \Delta'$, duplicating Σ). We can then reformulate the hybrid conjunction rule as:

$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma \cup \Delta\,,\; A \wedge B}\ \wedge$$

(where $\cup$ binds more strongly than comma).

3.1 Soundness and completeness

THEOREM 1 (SOUNDNESS OF PROPOSITIONAL HYBRID LOGIC **Hp**) *Every formula provable in* **Hp** *is true.*

We obtain this theorem as a special case of:

PROPOSITION 1 *Every sequent provable in* **Hp** *is true.*

Proof. It is immediate that each rule is sound. Alternatively: every rule of **Hp** is either a rule or a derived rule of the sound system **GS1p** (see page 4 of the Introduction). □

To prove that **Hp** is complete, we require some auxiliary definitions and lemmas. A true sequent is ***minimal*** if no proper subsequent is true. For example, $P, \neg P$ is minimal, while $P, \neg P, Q$ is not. The sequent $P \wedge Q,\ \overline{Q} \wedge P,\ \overline{P}$ proved above in Example 2 is minimal.

LEMMA 1 *Every true sequent has a minimal subsequent.*

Proof. Immediate from the definition of minimal sequent. □

LEMMA 2 *Suppose a sequent* Γ *is a disjoint union of literals (*i.e.*,* Γ *contains no connectives*[8]*). Then* Γ *is minimal iff* $\Gamma = P, \neg P$ *for some propositional variable* P.

Proof. By definition of truth in terms of valuations, Γ is true iff it contains a complementary pair of literals, *i.e.*, iff $\Gamma = P, \neg P, \Delta$ with Δ a disjoint union of zero or more literals. Since $P, \neg P$ is true, Γ is minimal iff Δ is empty. □

Suppose Γ and Δ are disjoint unions of formulas (so each is either a sequent or empty). Write $\Gamma \subseteq \Delta$ if Δ is an extension of Γ with zero or more additional formulas, *i.e.*, $\Delta = \Gamma, A_1, \ldots, A_n$ for formulas A_i and $n \geq 0$. Let Γ be a disjoint union of formulas, and suppose $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$. Let $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ denote the union of Γ_1 and Γ_2 (with possible superposition/sharing), *i.e.*, the restriction of Γ to formulas (formula occurrences) in at least one of the Γ_i. (See Remark 1 above for related material.)

[8] Recall that $\neg$ is not a connective, but rather an operation on formulas (*i.e.*, on labelled trees). Only $\wedge$ and $\vee$ are connectives.

LEMMA 3 *Suppose* $\Gamma,\ A_1 \wedge A_2$ *is a minimal sequent. Choose* $\Gamma_1 \subseteq \Gamma$ *and* $\Gamma_2 \subseteq \Gamma$ *such that* Γ_1, A_1 *and* Γ_2, A_2 *are minimal (existing by Lemma 1, since* Γ, A_1 *and* Γ, A_2 *are true). Then* $\Gamma_1 \cup \Gamma_2 = \Gamma$ *,* i.e.*, every formula of* Γ *is in at least one of the* Γ_i *.*

Proof. Otherwise $\Gamma_1 \cup \Gamma_2,\ A_1 \wedge A_2$ is a true proper subsequent of $\Gamma,\ A_1 \wedge A_2$, contradicting its minimality. (The sequent $\Gamma_1 \cup \Gamma_2,\ A_1 \wedge A_2$ is true since Γ_1, A_1 and Γ_2, A_2 are true.) $\square$

LEMMA 4 *Suppose* $\Gamma,\ A \vee B$ *is a minimal sequent, and that* $\Gamma,\ A$ *is true. Then* $\Gamma,\ A$ *is minimal.*

Proof. If not, some proper subsequent Δ of Γ, A is true. If Δ does not contain A, then it is also a proper subsequent of $\Gamma,\ A \vee B$, contradicting minimality. Otherwise let Δ' be the result of replacing A in Δ by $A \vee B$. Since Δ is true, so also is Δ'. Thus Δ' is a true proper subsequent of $\Gamma,\ A \vee B$, contradicting minimality. $\square$

LEMMA 5 *Suppose* $\Gamma,\ A \vee B$ *is a minimal sequent, and that neither* Γ, A *nor* Γ, B *is true. Then* Γ, A, B *is minimal.*

Proof. Suppose Γ, A, B had a true proper subsequent Δ. Since neither Γ, A nor Γ, B is true, Δ must contain both A and B. Let Δ' result from replacing A, B by $A \vee B$ in Δ. Then Δ' is a true proper subsequent of $\Gamma,\ A \vee B$, contradicting minimality. $\square$

The Completeness Theorem will be a special case of the following proposition.

PROPOSITION 2 *Every minimal sequent is provable in* **Hp***.*

Proof. Suppose Γ is a minimal sequent. We proceed by induction on the number of connectives in Γ.

- *Induction base (no connective).* Since Γ is minimal, Lemma 2 implies $\Gamma = P, \neg P$, the conclusion of the axiom rule $\overline{P, \neg P}$.
- *Induction step (at least one connective).*

  1. *Case:* $\Gamma = \Delta, A_1 \wedge A_2$. By Lemma 3, $\Gamma = \Sigma, \Delta_1, \Delta_2, A_1 \wedge A_2$ for Σ, Δ_1, A_1 and Σ, Δ_2, A_2 minimal. Write down the conjunction rule

$$\frac{\Sigma, \Delta_1, A_1 \qquad \Sigma, \Delta_2, A_2}{\Sigma, \Delta_1, \Delta_2, A_1 \vee A_2}\ \wedge$$

     and appeal to induction with the two hypothesis sequents.

2. *Case:* $\Gamma = \Delta, A_1 \vee A_2$.
   (a) *Case:* Δ, A_i *is true for some* $i \in \{1, 2\}$. Write down the disjunction rule
   $$\frac{\Delta, A_i}{\Delta, A_1 \vee A_2} \oplus_i$$
   then appeal to induction with Δ, A_i, which is minimal by Lemma 4.
   (b) *Case:* Δ, A_i *is not true for each* $i \in \{1, 2\}$. Thus Δ, A_1, A_2 is minimal, by Lemma 5. Write down the disjunction rule
   $$\frac{\Delta, A_1, A_2}{\Delta, A_1 \vee A_2} \parr$$
   then appeal to induction with Δ, A_1, A_2.

(Γ may match both case 1 and case 2 in the inductive step, permitting some choice in the construction of the proof. There is also choice within case 2(a) if Δ, A_1 and Δ, A_2 are both true.) □

Since a true formula (singleton sequent) is a special case of a minimal sequent, the following is an immediate corollary of Proposition 2:

THEOREM 2 (COMPLETENESS OF CUT-FREE PROPOSITIONAL HYBRID LOGIC)
Every true formula is provable in **Hp**.

Completeness is for true formulas (singleton sequents), or more generally minimal sequents (Proposition 2). It does not hold for arbitrary sequents. For example, the sequent $P, \neg P, Q$ is true but not provable in **Hp**. See Section 6 for details.

4 Minimality of Hybrid Logic

Hybrid conjunction is critical for the liberation from structural rules: relaxing it to the pair of standard conjunction rules (context-sharing $\&$ and context-splitting $\otimes$) breaks completeness. Let **Hp**$^-$ be the following subsystem of **Hp**:[9]

[9] This precursor of **Hp** is (cut-free) multiplicative-additive linear logic [Gir87] with the formula constructors *tensor* $\otimes$ and *with* $\&$ collapsed to $\wedge$, and the formula constructors *plus* $\oplus$ and *par* $\parr$ collapsed to $\vee$.

System Hp$^-$

$$\frac{}{P, \neg P} \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B}\,\& \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\,⅋$$

$$\frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B}\,\otimes \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\,\oplus_i$$

PROPOSITION 3 *System* **Hp$^-$** *is incomplete.*

Proof. We show that the true formula $A = \big((P \wedge Q) \vee (\overline{Q} \wedge P)\big) \vee \overline{P}$ is not provable in **Hp$^-$**. The placement of the two outermost $\vee$ connectives forces the last two rules of a potential proof to be disjunction rules. Since $P \wedge Q, \overline{Q} \wedge P, \overline{P}$ is minimal (no proper subsequent is true), the two disjunction rules must be ⅋ rather than $\oplus$:

$$\cfrac{\cfrac{P \wedge Q, \overline{Q} \wedge P, \overline{P}}{(P \wedge Q) \vee (\overline{Q} \wedge P), \overline{P}}\,⅋}{\big((P \wedge Q) \vee (\overline{Q} \wedge P)\big) \vee \overline{P}}\,⅋$$

It remains to show that $P \wedge Q, \overline{Q} \wedge P, \overline{P}$ is not provable in **Hp$^-$**.[10] There are only two connectives, both $\wedge$, so the last rule must be a conjunction.

1. *Case: the last rule is a context-sharing $\&$-rule.*

   (a) *Case: The last rule introduces $P \wedge Q$.* We have:

   $$\frac{P, \overline{Q} \wedge P, \overline{P} \qquad Q, \overline{Q} \wedge P, \overline{P}}{P \wedge Q, \overline{Q} \wedge P, \overline{P}}\,\&$$

   The left hypothesis $P, \overline{Q} \wedge P, \overline{P}$ cannot be proved in **Hp$^-$**, since there is no Q to match the $\overline{Q}$ (and no weakening).

   (b) *Case: The last rule introduces $\overline{Q} \wedge P$.* The same as the previous case, by symmetry, and exchanging $Q \leftrightarrow \overline{Q}$.

2. *Case: the last rule is a context-splitting $\otimes$-rule.*

[10] Note, in contrast, that an **Hp** proof of this sequent was given in Example 2 on page 7.

(a) *Case: The last rule introduces $P \wedge Q$.* We have:

$$\frac{P,\,\Gamma \qquad Q,\,\Delta}{P \wedge Q,\, \overline{Q} \wedge P,\, \overline{P}}\ \otimes$$

where we must allocate each of $\overline{Q} \wedge P$ and $\overline{P}$ either to Γ or to Δ. If $\overline{Q} \wedge P$ is in Γ, then P, Γ is not provable in **Hp**$^-$, since it contains no Q to match the $\overline{Q}$. So $\overline{Q} \wedge P$ is in Δ. But then the $\overline{P}$ is required in both Γ and Δ.

(b) *Case: The last rule introduces $\overline{Q} \wedge P$.* The same as the previous case, by symmetry, and exchanging $Q \leftrightarrow \overline{Q}$.

$\square$

4.1 The Minimality Theorem

This section makes precise the sense in which hybrid logic represents a completeness threshold. Define a ***standard system*** as any propositional sequent calculus containing the axiom $\overline{P, \neg P}$ and any of the ***standard rules***:

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \& \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ \invamp \qquad \frac{\Gamma}{\Gamma, A}\ \mathsf{W}$$

$$\frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\ \otimes \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i \qquad \frac{\Gamma, A, A}{\Gamma, A}\ \mathsf{C}$$

Thus there are $2^6 = 64$ such systems (many of which will not be complete).

Say that a system S ***contains*** a system T if each rule of T is a derived rule of S. For example, as we noted in the Introduction, system **GS1p** (displayed on page 1) contains **Hp** (page 6), since the hybrid conjunction rule $\wedge$ and the disjunction rule $\invamp$ of **Hp** can be derived in **GS1p**:

$$\frac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\ \wedge \qquad \longleftarrow \qquad \frac{\dfrac{\Gamma, \Delta, A}{\Gamma, \Delta, \Sigma, A}\ \mathsf{W}^* \qquad \dfrac{\Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, B}\ \mathsf{W}^*}{\Gamma, \Delta, \Sigma, A \wedge B}\ \&$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \invamp \qquad \longleftarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma, A, B}{\Gamma, A, A \vee B}\ \oplus_2}{\Gamma, A \vee B, A \vee B}\ \oplus_1}{\Gamma, A \vee B}\ \mathsf{C}$$

where W^* denotes a sequence of zero or more weakenings.

THEOREM 3 (MINIMALITY THEOREM) *A standard system is complete iff it contains propositional hybrid logic* **Hp**.

We prove this theorem in the next section.

4.2 Proof of the Minimality Theorem

Define two systems as ***equivalent*** if each contains the other. (Recall that S *contains* T if every rule of T is a derived rule of S.) For example, it is well known that **GS1p** (displayed on page 1) is equivalent to the system[11]

$$\frac{}{P, \neg P} \qquad \frac{\Delta, A \qquad \Sigma, B}{\Delta,\ \Sigma,\ A \wedge B}\ \otimes \qquad \frac{\Gamma, A_1, A_2}{\Gamma,\ A_1 \vee A_2}\ \parr \qquad \frac{\Gamma}{\Gamma, A}\ \mathsf{W} \qquad \frac{\Gamma, A, A}{\Gamma, A}\ \mathsf{C}$$

by deriving the following rules:

$$\frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\ \otimes \qquad \longleftarrow \qquad \frac{\dfrac{\Delta, A}{\Delta, \Sigma, A}\ \mathsf{W}^* \qquad \dfrac{\Sigma, B}{\Delta, \Sigma, B}\ \mathsf{W}^*}{\Delta, \Sigma, A \wedge B}\ \&$$

$$\frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i \qquad \longleftarrow \qquad \frac{\dfrac{\Gamma, A_i}{\Gamma, A_1, A_2}\ \mathsf{W}}{\Gamma, A_1 \vee A_2}\ \parr$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \parr \qquad \longleftarrow \qquad \frac{\dfrac{\dfrac{\Gamma, A, B}{\Gamma, A, A \vee B}\ \oplus_2}{\Gamma, A \vee B, A \vee B}\ \oplus_1}{\Gamma, A \vee B}\ \mathsf{C}$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \& \qquad \longleftarrow \qquad \frac{\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, \Gamma, A \wedge B}\ \otimes}{\Gamma, A \wedge B}\ \mathsf{C}^*$$

We can abbreviate these four rule derivations as:

[11] This system is multiplicative linear logic [Gir87] plus contraction and weakening (with the connectives denoted $\wedge$ and $\vee$ instead of $\otimes$ and $\parr$).

Standard derived rules

$$\begin{array}{ccccccc} \otimes & \longleftarrow & \&\,\mathsf{W} & \qquad & \& & \longleftarrow & \otimes\,\mathsf{C} \\ \oplus & \longleftarrow & ⅋\,\mathsf{W} & \qquad & ⅋ & \longleftarrow & \oplus\,\mathsf{C} \end{array}$$

(Context-splitting conjunction $\otimes$ is derivable from context-sharing conjunction $\&$ and weakening W, and so on.)

4.2.1 Only three complete standard systems, up to equivalence

As a stepping stone towards the Minimality Theorem, in this section we prove that, up to equivalence, there are only three complete standard systems.

We abbreviate a system by listing its non-axiom rules. For example, **GS1p** $= (\&, \oplus, \mathsf{W}, \mathsf{C})$ and **Hp** $= (\wedge, \oplus, ⅋)$. Besides **GS1p**, we shall pay particular attention to the systems $(\otimes, \oplus, \mathsf{C})$ and $(\&, ⅋, \mathsf{W})$, which we refer to as ***positive propositional logic***, denoted **Pp**, and ***negative propositional logic***, denoted **Np**, respectively.[12] In full, for ease of reference, the systems are:

System Pp (Propositional Positive Logic)

$$\frac{}{P, \neg P} \qquad \frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\,\otimes \qquad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\,\oplus_i \qquad \frac{\Gamma, A, A}{\Gamma, A}\,\mathsf{C}$$

System Np (Propositional Negative Logic)

$$\frac{}{P, \neg P} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\,\& \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\,⅋ \qquad \frac{\Gamma}{\Gamma, A}\,\mathsf{W}$$

PROPOSITION 4 *Up to equivalence:*

(1) *the Gentzen-Schütte-Tait system* **GS1p** $= (\&, \oplus, \mathsf{C}, \mathsf{W})$ *is the only complete standard system with both contraction* C *and weakening* W*;*

(2) *propositional positive logic* **Pp** $= (\otimes, \oplus, \mathsf{C})$ *is the only complete standard system without weakening* W*;*

(3) *propositional negative logic* **Np** $= (\&, ⅋, \mathsf{W})$ *is the only complete standard system without contraction* C.

[12] Our terminology here comes from polarity of connectives in linear logic [Gir87]: *tensor* $\otimes$ and *plus* $\oplus$ are positive, and *with* $\&$ and *par* $⅋$ are negative.

We prove this proposition via a number of lemmas.

LEMMA 6 *Propositional hybrid logic* **Hp** $= (\wedge, \oplus, \parr)$ *is contained in propositional positive logic* **Pp** $= (\otimes, \oplus, \mathsf{C})$*, propositional negative logic* **Np** $= (\&, \parr, \mathsf{W})$ *and* **GS1p** $= (\&, \oplus, \mathsf{C}, \mathsf{W})$.

Proof. **Pp** contains **Hp** since the hybrid conjunction rule $\wedge$ is derivable from C and $\otimes$,

$$\dfrac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\,\wedge \qquad \longleftarrow \qquad \dfrac{\dfrac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Gamma, \Delta, \Sigma, A \wedge B}\,\otimes}{\Gamma, \Delta, \Sigma, A \wedge B}\,\mathsf{C}^*$$

(where C^* denotes zero or more consecutive contractions) and $\parr$ is derivable from $\oplus$ and C (see page 4).

Np contains propositional hybrid logic since the hybrid conjunction rule $\wedge$ is derivable from W and $\&$ (see page 12), and $\oplus$ is derivable from W and $\parr$ (see page 14).

GS1p $= (\&, \oplus, \mathsf{C}, \mathsf{W})$ is equivalent to $(\otimes, \&, \oplus, \parr, \mathsf{C}, \mathsf{W})$, since $\otimes$ and $\parr$ are derivable, therefore contains **Pp** (and **Np**), hence **Hp**. $\square$

LEMMA 7 *Propositional positive logic* **Pp** $= (\mathsf{C}, \otimes, \oplus)$ *and propositional negative logic* **Np** $= (\&, \parr, \mathsf{W})$ *are complete.*

Proof. Both contain propositional hybrid logic **Hp** (Lemma 6), which is complete (Theorem 2). $\square$

LEMMA 8 *Up to equivalence, system* **GS1p** $= (\&, \oplus, \mathsf{C}, \mathsf{W})$ *is the only complete standard system with both contraction* C *and weakening* W.

Proof. **GS1p** is complete (see *e.g.* [TS96], or by the fact that **GS1p** contains **Hp** which is complete). Any complete system must have a conjunction rule ($\otimes$ or $\&$) and a disjunction rule ($\oplus$ or $\parr$). In the presence of C and W, the two conjunctions are derivable from one other, as are the two disjunctions (see page 14). $\square$

LEMMA 9 *A complete standard system without weakening* W *must contain propositional positive logic* **Pp** $= (\otimes, \oplus, \mathsf{C})$.

Proof. System **Hp**$^-$ $= (\otimes, \oplus, \&, \parr)$, with both conjunction rules and both disjunction rules, is incomplete (Proposition 3, page 11), therefore we must have contraction C.

Without the $\oplus$ rule, the true formula $(P \vee \overline{P}) \vee Q$ is not provable: the last rule must be $\parr$, leaving us to prove $P \vee \overline{P}, Q$, which is impossible without weakening W (*i.e.*, with at most $\parr$, $\&$, $\otimes$ and C available), since, after a necessary axiom $\overline{P, \overline{P}}$ at the top of the proof, there is no way to introduce the formula Q.

Without the context-splitting $\otimes$ rule, the true formula $P \vee (Q \vee (\overline{P} \wedge \overline{Q}))$ is not provable. The last two rules must be $\parr$, for if we use a $\oplus$ we will not be able to match complementary literals in the axioms at the top of the proof. Thus we are left to prove $P, Q, \overline{P} \wedge \overline{Q}$, using $\&$ and C. The proof must contain an axiom rule $\overline{P, \overline{P}}$. The next rule can only be a $\&$ (since $P, \overline{P}$ cannot be the hypothesis sequent of a contraction C rule). Since the only $\wedge$-formula in the final concluding sequent $P, Q, \overline{P} \wedge \overline{Q}$ is $\overline{P} \wedge \overline{Q}$, and the $\&$ rule is context sharing, the $\&$-rule must be

$$\dfrac{\dfrac{}{P,\overline{P}} \qquad \begin{array}{c}\vdots\\ P,\overline{Q}\end{array}}{P,\overline{P}\wedge\overline{Q}}\;\&$$

but $P, \overline{Q}$ is not provable. □

LEMMA 10 *Up to equivalence, propositional positive logic* $\mathbf{Pp} = (\otimes, \oplus, \mathsf{C})$ *is the only complete standard system without weakening* W.

Proof. By Lemma 7, $\mathbf{Pp}$ is complete. By Lemma 9, every W-free complete standard system contains $\mathbf{Pp}$. All other W-free standard systems containing $\mathbf{Pp}$ are equivalent to $\mathbf{Pp}$, since the standard rule derivations $\& \leftarrow \otimes\mathsf{C}$ and $\parr \leftarrow \oplus\mathsf{C}$ yield $\&$ and $\parr$ (see page 14). □

LEMMA 11 *A complete standard system without contraction* C *must contain propositional negative logic* $\mathbf{Np} = (\&, \parr, \mathsf{W})$.

Proof. System $\mathbf{Hp}^- = (\otimes, \oplus, \&, \parr)$, with both conjunction rules and both disjunction rules, is incomplete (Proposition 3, page 11), therefore we must have weakening W.

Without the $\parr$ rule, the true formula $P \vee \overline{P}$ would not be provable.

Without the $\&$ rule the true formula $P \vee (\overline{P} \wedge \overline{P})$ would not be provable. The last rule must be a $\parr$ (rather than a $\oplus$, otherwise we are stranded either without a P or without a $\overline{P}$), so we are left to prove $P, \overline{P} \wedge \overline{P}$. The last rule cannot be $\parr$ or $\oplus$, as the only connective is $\wedge$. It cannot be W, or we are stranded either without a P or without a $\overline{P}$. It cannot be $\otimes$, as one of the two hypotheses will be the single formula $\overline{P}$. □

LEMMA 12 *Up to equivalence, propositional negative logic* $\mathbf{Np} = (\&, \parr, \mathsf{W})$ *is the only complete standard system without contraction* C.

Proof. By Lemma 7, $\mathbf{Np}$ is complete. By Lemma 11, every C-free complete standard system contains $\mathbf{Np}$. All other C-free standard systems containing $\mathbf{Np}$ are equivalent to $\mathbf{Np}$, since the standard rule derivations $\otimes \leftarrow \&\mathsf{W}$ and $\oplus \leftarrow \parr\mathsf{W}$ yield $\otimes$ and $\oplus$ (see page 14). □

Proof of Proposition 4, page 14 (up to equivalence, the standard systems **GS1p**/**Pp**/**Np** *are unique with* C *and* W */ without* W */ without* C*).* Parts (1), (2) and (3) of Proposition 4 are Lemmas 8, 10 and 12, respectively. $\square$

LEMMA 13 *A standard system without contraction* C *or weakening* W *is incomplete.*

Proof. Any such system is contained in $\mathbf{Hp}^- = (\otimes, \oplus, \&, \parr)$, which is incomplete (Proposition 3, page 11). $\square$

THEOREM 4 *Up to equivalence, there are only three complete standard systems:*

1. *The Gentzen-Schütte-Tait system* **GS1p** $= (\&, \oplus, \mathsf{C}, \mathsf{W})$.
2. *Propositional positive logic* **Pp** $= (\otimes, \oplus, \mathsf{C})$.
3. *Propositional negative logic* **Np** $= (\&, \parr, \mathsf{W})$.

Proof. Proposition 4 and Lemma 13. $\square$

4.2.2 Proof of the Minimality Theorem

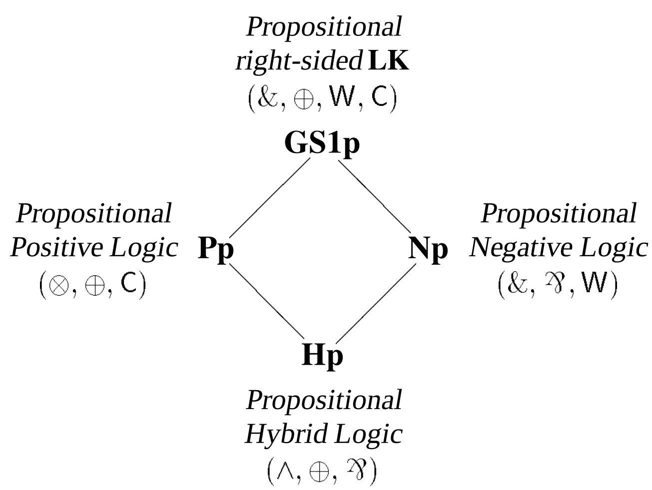
Now we can prove Theorem 3, the Minimality Theorem.

MINIMALITY THEOREM *A standard system is complete iff it contains propositional hybrid logic* **Hp**.

Proof. Up to equivalence, there are only three complete standard systems (Theorem 4), and each contains **Hp** (Lemma 6). $\square$

4.3 Propositional hybrid logic Hp as a "complete core" of Gentzen's LK

The three inequivalent complete standard systems **GS1p**, **Pp** and **Np**, together with propositional hybrid logic **Hp**, sit in the following Hasse diagram of containments:



Thus we can view propositional hybrid logic **Hp** as a “complete core” of **GS1p**, hence of (propositional) Gentzen’s **LK**.

5 Extended Minimality Theorem

Define an ***extended system*** as one containing the axiom rule $\overline{P, \overline{P}}$ and any of the following rules:

Extended system rules

$$\frac{\Delta, A \qquad \Sigma, B}{\Delta, \Sigma, A \wedge B}\ \otimes \qquad \frac{\Gamma, \Delta, A \qquad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B}\ \wedge \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \&$$

$$\frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2}\ \oplus_i \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ \mathbin{⅋}$$

$$\frac{\Gamma, A, A}{\Gamma, A}\ \mathsf{C} \qquad \frac{\Gamma}{\Gamma, A}\ \mathsf{W}$$

Thus there are $2^7 = 128$ extended systems, of which the $2^6 = 64$ without the hybrid conjunction rule $\wedge$ are the standard systems defined in Section 4.1.[13] The Minimality Theorem (Theorem 3, page 13) extends as follows.

THEOREM 5 (EXTENDED MINIMALITY THEOREM) *An extended system is complete iff it contains propositional hybrid logic* **Hp**.

To prove this theorem, we require two lemmas.

LEMMA 14 *Suppose S is a complete extended system with the hybrid conjunction rule $\wedge$, and with at least one of contraction C or weakening W. Then S is equivalent to a standard system.*

Proof. If S has weakening W, let S' be the result of replacing the hybrid conjunction rule $\wedge$ in S by context-sharing conjunction $\&$; otherwise S has contraction, and let S' result from replacing $\wedge$ by context-splitting $\otimes$. Then S' is equivalent to S, since $\wedge$ is derivable from $\otimes$ and C (see page 15) and from $\&$ and W (see page 12). $\square$

[13]Note, however, that many of these standard systems have hybrid conjunction $\wedge$ as a derived rule.

LEMMA 15 *Suppose S is a complete extended system without contraction C or weakening W. Then S is equivalent to propositional hybrid logic* **Hp**.

Proof. Since $\mathbf{Hp}^- = (\otimes, \&, \oplus, \parr)$ is incomplete (Proposition 3, page 11), S must have the hybrid conjunction rule $\wedge$ either directly or as a derived rule. Since S is complete, it must have a disjunction rule, therefore it could only fail to be equivalent to $\mathbf{Hp} = (\wedge, \oplus, \parr)$ if (a) it has $\oplus$ and $\parr$ is not derivable, *i.e.*, S is equivalent to $(\wedge, \oplus)$, or (b) it has $\parr$ and $\oplus$ is not derivable, *i.e.*, S is equivalent to $(\wedge, \parr)$. In case (a), the true formula $P \vee \overline{P}$ would not be provable, and in case (b) the true formula $(P \vee \overline{P}) \vee Q$ would not be provable, either way contradicting the completeness of S. □

Proof of the Extended Minimality Theorem (Theorem 5). Suppose S is a complete extended system. If S has contraction C or weakening W then it is equivalent to a standard system by Lemma 14, hence contains **Hp** by the original Minimality Theorem. Otherwise S is equivalent to **Hp** by Lemma 15, hence in particular contains **Hp**.

Conversely, suppose S is an extended system containing **Hp**. Then S is complete since **Hp** is complete. □

We also have an extension of Theorem 4 (page 17), which stated that, up to equivalence, there are only three complete standard systems, **GS1p**, **Pp** and **Np**.

THEOREM 6 *Up to equivalence, there are only four complete extended systems:*

1. *The Gentzen-Schütte-Tait system* $\mathbf{GS1p} = (\&, \oplus, \mathsf{C}, \mathsf{W})$.
2. *Propositional positive logic* $\mathbf{Pp} = (\otimes, \oplus, \mathsf{C})$.
3. *Propositional negative logic* $\mathbf{Np} = (\&, \parr, \mathsf{W})$.
4. *Propositional hybrid logic* $\mathbf{Hp} = (\wedge, \oplus, \parr)$.

Proof. Theorem 4 together with Lemmas 14 and 15. □

6 Relative degrees of completeness

We defined a system as *complete* if every true formula (singleton sequent) is provable. To avoid ambiguity with forthcoming definitions, let us refer to this default notion of completeness as ***formula-complete***. Define a system as ***minimal-complete*** if every minimal[14] sequent is provable, and ***sequent-complete*** if every true sequent is provable. (Thus *sequent-complete* implies *minimal-complete* implies *formula-complete*.)

[14]Recall that a true sequent is minimal if no proper subsequent is true.

PROPOSITION 5 *Propositional positive logic* **Pp** $= (\otimes, \oplus, \mathsf{C})$ *and propositional hybrid logic* **Hp** $= (\wedge, \oplus, \parr)$ *are formula-complete and minimal-complete, but not sequent-complete.*

Proof. We have already proved that **Hp** (hence also **Pp**, by containment) is minimal-complete (Proposition 2).

We show that the true (non-minimal) sequent $P, \overline{P}, Q$ is unprovable in **Pp** (hence also in **Hp**). A proof must contain an axiom rule $\overline{P, \overline{P}}$. This cannot be followed by a $\otimes$ or $\oplus$ rule, otherwise we introduce a connective $\wedge$ or $\vee$ which cannot subsequently be removed by any other rule before the concluding sequent $P, \overline{P}, Q$. Neither can it be followed by contraction C, since there is nothing to contract. □

PROPOSITION 6 *Propositional negative logic* **Np** $= (\&, \parr, \mathsf{W})$ *is formula-complete, minimal-complete and sequent-complete.*

Proof. **Np** is minimal-complete since it contains minimal-complete **Hp**. Suppose Γ is a true but not minimal sequent. Choose a minimal subsequent Δ of Γ (see Lemma 1, page 8). By minimal-completeness, Δ has a proof. Follow this with weakenings to obtain Γ from Δ. □

Below we have annotated our Hasse diagram with completeness strengths.

Relationships between complete inequivalent systems

Propositional
right-sided **LK**
$(\&, \oplus, \mathsf{W}, \mathsf{C})$
formula-complete
minimal-complete
sequent-complete

$$\begin{tikzcd}[column sep=small] & \textbf{GS1p} \arrow[dl, dash] \arrow[dr, dash] & \\ \textbf{Pp} \arrow[dr, dash] & & \textbf{Np} \arrow[dl, dash] \\ & \textbf{Hp} & \end{tikzcd}$$

Propositional
Positive Logic
$(\otimes, \oplus, \mathsf{C})$
formula-complete
minimal-complete

Propositional
Negative Logic
$(\&, \parr, \mathsf{W})$
formula-complete
minimal-complete
sequent-complete

Propositional
Hybrid Logic
$(\wedge, \oplus, \parr)$
formula-complete
minimal-complete

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