

A classical sequent calculus free of structural rules

DOMINIC HUGHES
Stanford University

June 28, 2005

Gentzen's system **LK**, classical sequent calculus, has explicit structural rules for contraction and weakening. They can be absorbed (in a right-sided formulation) by replacing the axiom $P, \neg P$ by $\Gamma, P, \neg P$ for any context Γ , and replacing the original disjunction rule with Γ, A, B implies $\Gamma, A \vee B$.

This paper presents a classical sequent calculus which is also free of contraction and weakening, but more symmetrically: both contraction and weakening are absorbed into conjunction, leaving the axiom rule untouched. It uses a hybrid conjunction rule, combining the standard context-sharing and context-splitting rules: Γ, Δ, A and Γ, Σ, B implies $\Gamma, \Delta, \Sigma, A \wedge B$. We call this system *Hybrid Logic*. Hybrid conjunction is critical for the liberation from structural rules: relaxing it to the pair of standard conjunction rules breaks completeness.

We prove a *minimality theorem* for hybrid logic: any sequent calculus (within a standard class of right-sided calculi) is complete *iff* it contains hybrid logic. Thus one can view hybrid logic as a “complete core” of Gentzen's **LK**.

1 Introduction

The following Gentzen-Schütte-Tait [Gen39, Sch50, Tai68] system, denoted **GS1p** in [TS96], is a standard right-sided formulation of the propositional fragment of Gentzen's classical sequent calculus **LK**:

$\frac{}{P, \neg P}$	$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \ \&$	$\frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \ \oplus_i$
	$\frac{\Gamma}{\Gamma, A} \ \text{W}$	$\frac{\Gamma, A, A}{\Gamma, A} \ \text{C}$

Here P ranges over propositional variables, A, A_i, B range over formulas, Γ ranges over disjoint unions of formulas, and comma denotes disjoint union. (We have labelled the conjunction and disjunction rules with $\&$ and \oplus for reasons which will become apparent later.) By defining a sequent as a disjoint union of formulas, rather than an ordered list, we avoid an exchange/permutation rule (*cf.* [TS96, §1.1]). Negation is primitive on propositional variables P , and extends to compound formulas by de Morgan duality.

The structural rules, weakening W and contraction C , are absorbed in the following variant, a right-sided formulation of the propositional part of the calculus of [Ket44], called **GS3p** in [TS96].¹

System GS3p

$$\frac{}{\Gamma, P, \neg P} \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp$$

The new axiom $\Gamma, P, \neg P$ amounts to the original axiom $P, \neg P$ followed immediately by weakenings. This paper presents a classical propositional sequent calculus **Hp** which is also free of structural rules:

System Hp

$$\frac{}{P, \neg P} \quad \frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i$$

(By analogy with **GS3** and **GS3p** in [TS96], we reserve the symbol **H** for a full system with quantifiers, and use **Hp** to denote the propositional system. Also following [TS96], cut is treated separately.²) A distinguishing feature of **Hp** is the *hybrid conjunction rule*

$$\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge$$

which combines the standard context-sharing and context-splitting conjunction rules:

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes$$

Since it uses hybrid conjunction, we refer to **Hp** as (cut-free propositional) *Hybrid Logic*. The context-sharing rule $\&$ is the special case of the hybrid rule with Δ and Σ empty; the context-splitting rule \otimes is the special case with Γ empty. In contrast to

¹We have labelled the disjunction rule as \wp , to distinguish it from the disjunction rule \oplus of **GS1p**.

²To maximise emphasis on the hybrid conjunction rule, we omit quantifiers and cut in this paper.

GS3p, contraction and weakening are absorbed symmetrically: both are absorbed into the conjunction rule, leaving the axiom rule intact.

Our main theorem (page 10) is completeness:

Theorem (Completeness of cut-free propositional Hybrid Logic)

*Every true formula is provable in **Hp**.*

Hybrid conjunction is critical for the liberation from structural rules: relaxing it to the pair of standard conjunction rules (context-sharing $\&$ and context-splitting \otimes) breaks completeness. The following system fails to be complete:³

System \mathbf{Hp}^- (incomplete)

$$\begin{array}{c} \overline{P, \neg P} \\ \Gamma, A \quad \Gamma, B \\ \hline \Gamma, A \wedge B \quad \& \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \\ \Delta, A \quad \Sigma, B \\ \hline \Delta, \Sigma, A \wedge B \quad \otimes \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \end{array}$$

Minimality of Hybrid Logic. We make precise the sense in which hybrid logic represents a completeness threshold. Let a *standard system* be any propositional sequent calculus containing the axiom $\overline{P, \neg P}$ and any subset of the following *standard rules*:

$$\begin{array}{ccc} \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \quad \frac{\Gamma}{\Gamma, A} \mathbf{W} \\ \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \quad \quad \frac{\Gamma, A, A}{\Gamma, A} \mathbf{C} \end{array}$$

(Thus there are $2^6 = 64$ such systems.) We prove:

Minimality Theorem

*A standard system is complete iff it contains propositional hybrid logic **Hp**.*

³This precursor of **Hp** is (cut-free) multiplicative-additive linear logic [Gir87] with the formula constructors *tensor* \otimes and *with* $\&$ collapsed to \wedge , and the formula constructors *plus* \oplus and *par* \wp collapsed to \vee . This paper will not require any background knowledge of linear logic.

Here a system S contains a system T if each rule of T is a derived rule of S . For example, system **GS1p** (displayed on page 1) contains **Hp**, since the hybrid conjunction rule \wedge and the disjunction rule \wp of **Hp** can be derived in **GS1p**:

$$\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge \quad \longleftarrow \quad \frac{\frac{\Gamma, \Delta, A}{\Gamma, \Delta, \Sigma, A} W^* \quad \frac{\Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, B} W^*}{\Gamma, \Delta, \Sigma, A \wedge B} \&$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \longleftarrow \quad \frac{\frac{\Gamma, A, B}{\Gamma, A, A \vee B} \oplus_2 \quad \frac{\Gamma, A \vee B, A \vee B}{\Gamma, A \vee B} C}{\Gamma, A \vee B} \oplus_1$$

where W^* denotes a sequence of zero or more weakenings.

If we define systems S and T as equivalent if each contains the other (*i.e.*, every rule of S is derivable in T , and vice versa), then there are in fact only three complete standard systems, up to equivalence. The largest is **GS1p** (page 1). We refer to the other two, each contained in **GS1p**, as *Positive Logic* and *Negative Logic*, respectively:⁴

System Pp (Propositional Positive Logic)

$$\frac{}{P, \neg P} \quad \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \quad \frac{\Gamma, A, A}{\Gamma, A} C$$

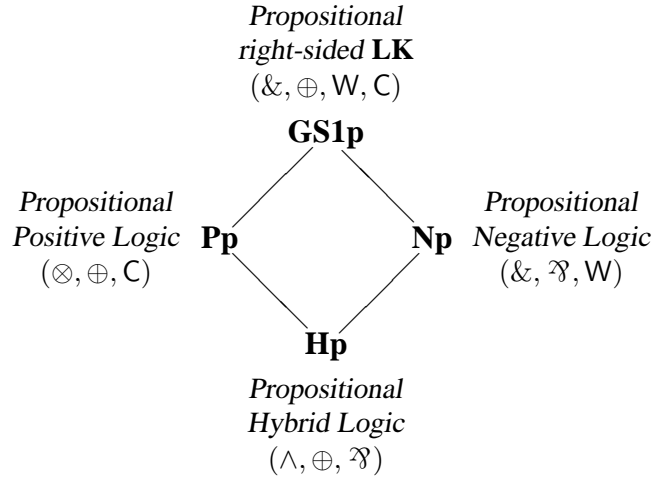
System Np (Propositional Negative Logic)

$$\frac{}{P, \neg P} \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \frac{\Gamma}{\Gamma, A} W$$

Together with propositional hybrid logic **Hp**, these systems sit in the following Hasse diagram of containments:⁵

⁴Our terminology here comes from polarity of connectives in linear logic [Gir87]: *tensor* \otimes and *plus* \oplus are positive, and *with* $\&$ and *par* \wp are negative. (This validates Girard's choice of notation for the connectives.)

⁵These containments are strict since neither contraction C nor weakening W can be derived from the other standard rules.



Thus propositional hybrid logic **Hp** can be seen as a “complete core” of **GS1p**, hence of (propositional) Gentzen’s **LK**.

1.1 Avenues for future work

1. *Cut*. Chapter 4 of [TS96] gives a detailed analysis of cut for Gentzen systems. One could pursue an analogous analysis of cut for hybrid logic. Aside from context-splitting and context-sharing cut rules

$$\frac{\Delta, A \quad \Sigma, \neg A}{\Delta, \Sigma} \text{cut}_{\otimes} \qquad \frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma} \text{cut}_{\&}$$

one might also investigate a hybrid cut rule:

$$\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, \neg A}{\Gamma, \Delta, \Sigma} \text{cut}$$

2. *Quantifiers*. Explore the various ways of adding quantifiers to **Hp**, for a full first-order system **H**.
3. *Mix (nullary multicut)*. Gentzen’s multicut rule

$$\frac{\Delta, A_1, \dots, A_m \quad \Sigma, \neg A_1, \dots, \neg A_n}{\Delta, \Sigma}$$

in the nullary case $m = n = 0$ has been of particular interest to linear logicians [Gir87], who call it the *mix* rule. One could investigate context-splitting, context-sharing and hybrid incarnations:

$$\frac{\Delta \quad \Sigma}{\Delta, \Sigma} \text{mix}_{\otimes} \qquad \frac{\Gamma \quad \Gamma}{\Gamma} \text{mix}_{\&} \qquad \frac{\Gamma, \Delta \quad \Gamma, \Sigma}{\Gamma, \Delta, \Sigma} \text{mix}$$

2 Notation and terminology

Formulas are built from literals (propositional variables P, Q, R, \dots and their formal complements $\overline{P}, \overline{Q}, \overline{R}, \dots$) by the binary connectives **and** \wedge and **or** \vee . Define **negation** or **not** \neg as an operation on formulas (rather than a syntactic formula construct): $\neg P = \overline{P}$ and $\neg \overline{P} = P$ for all propositional variables P , extending to arbitrary formulas by de Morgan duality $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ and $\neg(A \vee B) = (\neg A) \wedge (\neg B)$. Thus \neg is not a connective; only \wedge and \vee are connectives.

We identify a formula with its parse tree, a tree labelled with literals at the leaves and connectives at the internal vertices. A **sequent** is a non-empty disjoint union of formulas. Thus a sequent is a particular kind of labelled forest.⁶ We write comma for disjoint union, which is commutative and associative (as usual in graph theory). Throughout the document, P, Q, \dots range over propositional variables, A, B, \dots over formulas, and Γ, Δ, \dots over (possibly empty) disjoint unions of formulas.

A formula is **true** (or **valid**) if, for all possible assignments of its propositional variables to 0 or 1, it evaluates to 1 (with the usual interpretation of \wedge and \vee on $\{0, 1\}$). A sequent A_1, \dots, A_n is true iff $A_1 \vee \dots \vee A_n$ is true (with an arbitrary choice of bracketing). A sequent Γ is a **subsequent** of a sequent Δ if $\Delta = \Gamma, \Sigma$ for some (possibly empty) disjoint union Σ of formulas, and a **proper subsequent** of Δ if Σ is non-empty.

3 Hybrid logic

Define **Propositional Hybrid Logic** as the following system:

$$\boxed{\text{System Hp (Propositional Hybrid Logic)}} \quad \frac{}{P, \neg P} \quad \frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i$$

We refer to this conjunction rule as **hybrid conjunction**, as it is a hybrid of the usual context-sharing and context-splitting rules $\&$ and \otimes (see page 2 of the Introduction). The rule is used in [Urb00, UB01] to study cut elimination in classical logic via term assignment.

EXAMPLE 1 Here is a proof of Peirce's law $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ in **Hp**, where

⁶This foundational treatment of formulas and sequents as labelled trees and forests sidesteps the common problem of “formulas” versus “formula occurrences”: disjoint unions of graphs are well understood in graph theory [Bol02], so we avoid reinventing the wheel.

$A \Rightarrow B$ abbreviates $(\neg A) \vee B$:

$$\frac{\frac{\overline{\overline{P}, P}}{\overline{P \vee Q}, P} \oplus_1 \quad \overline{\overline{P}, \overline{P}}}{(\overline{P \vee Q}) \wedge \overline{P}, P} \wedge}{((\overline{P \vee Q}) \wedge \overline{P}) \vee P} \wp$$

This use of the hybrid conjunction rule

$$\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge$$

is degenerate, since Σ and Δ are empty in the rule: it reduces to the special case of a context-sharing conjunction $\&$, with $\Gamma = P$.

EXAMPLE 2 The following example employs the full power of hybrid conjunction, in the final rule:

$$\frac{\overline{\overline{P}, \overline{P}} \quad \frac{\overline{\overline{Q}, \overline{Q}} \quad \overline{\overline{P}, \overline{P}}}{Q, \overline{Q} \wedge P, \overline{P}} \wedge}{P \wedge Q, \overline{Q} \wedge P, \overline{P}} \wedge$$

The final rule is an instance of hybrid conjunction rule

$$\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge$$

with $A = P$, $B = Q$, shared context $\Gamma = \overline{P}$, left-splitting context Δ empty and right-splitting context $\Sigma = \overline{Q} \wedge P$.

REMARK 1 Suppose Γ and Δ are disjoint unions of formulas (so each is either a sequent or empty). Also suppose that (by some additional annotation or otherwise) it is clear which formulas Γ and Δ have in common (if any). Define $\Gamma \cup \Delta$ as the (non-disjoint) union of Γ and Δ (so common formulas are not duplicated).⁷ For example, if $\Gamma = \Sigma, \Gamma'$ and $\Delta = \Sigma, \Delta'$ (with Δ' and Γ' disjoint from one another, and from Σ)

⁷We leave possible foundational issues involved with “shared/common formulas” and the superimposing operation \cup to the reader’s mathematical common sense: any reader finding the definition of $\Gamma \cup \Delta$ ambiguous can simply skip this remark, as it is tangential to the main thread of the paper.

then $\Gamma \cup \Delta$ is Σ, Γ', Δ' (whereas the disjoint union Γ, Δ of Γ and Δ is $\Sigma, \Sigma, \Gamma', \Delta'$, duplicating Σ). We can then reformulate the hybrid conjunction rule as:

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma \cup \Delta, A \wedge B} \wedge$$

(where \cup binds more strongly than comma).

3.1 Soundness and completeness

THEOREM 1 (SOUNDNESS OF PROPOSITIONAL HYBRID LOGIC \mathbf{Hp}) *Every formula provable in \mathbf{Hp} is true.*

We obtain this theorem as a special case of:

PROPOSITION 1 *Every sequent provable in \mathbf{Hp} is true.*

Proof. It is immediate that each rule is sound. Alternatively: every rule of \mathbf{Hp} is either a rule or a derived rule of the sound system $\mathbf{GS1p}$ (see page 4 of the Introduction). \square

To prove that \mathbf{Hp} is complete, we require some auxiliary definitions and lemmas. A true sequent is *minimal* if no proper subsequent is true. For example, $P, \neg P$ is minimal, while $P, \neg P, Q$ is not. The sequent $P \wedge Q, \overline{Q} \wedge P, \overline{P}$ proved above in Example 2 is minimal.

LEMMA 1 *Every true sequent has a minimal subsequent.*

Proof. Immediate from the definition of minimal sequent. \square

LEMMA 2 *Suppose a sequent Γ is a disjoint union of literals (i.e., Γ contains no connectives⁸). Then Γ is minimal iff $\Gamma = P, \neg P$ for some propositional variable P .*

Proof. By definition of truth in terms of valuations, Γ is true iff it contains a complementary pair of literals, i.e., iff $\Gamma = P, \neg P, \Delta$ with Δ a disjoint union of zero or more literals. Since $P, \neg P$ is true, Γ is minimal iff Δ is empty. \square

Suppose Γ and Δ are disjoint unions of formulas (so each is either a sequent or empty). Write $\Gamma \subseteq \Delta$ if Δ is an extension of Γ with zero or more additional formulas, i.e., $\Delta = \Gamma, A_1, \dots, A_n$ for formulas A_i and $n \geq 0$. Let Γ be a disjoint union of formulas, and suppose $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$. Let $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ denote the union of Γ_1 and Γ_2 (with possible superposition/sharing), i.e., the restriction of Γ to formulas (formula occurrences) in at least one of the Γ_i . (See Remark 1 above for related material.)

⁸Recall that \neg is not a connective, but rather an operation on formulas (i.e., on labelled trees). Only \wedge and \vee are connectives.

LEMMA 3 *Suppose $\Gamma, A_1 \wedge A_2$ is a minimal sequent. Choose $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$ such that Γ_1, A_1 and Γ_2, A_2 are minimal (existing by Lemma 1, since Γ, A_1 and Γ, A_2 are true). Then $\Gamma_1 \cup \Gamma_2 = \Gamma$, i.e., every formula of Γ is in at least one of the Γ_i .*

Proof. Otherwise $\Gamma_1 \cup \Gamma_2, A_1 \wedge A_2$ is a true proper subsequent of $\Gamma, A_1 \wedge A_2$, contradicting its minimality. (The sequent $\Gamma_1 \cup \Gamma_2, A_1 \wedge A_2$ is true since Γ_1, A_1 and Γ_2, A_2 are true.) \square

LEMMA 4 *Suppose $\Gamma, A \vee B$ is a minimal sequent, and that Γ, A is true. Then Γ, A is minimal.*

Proof. If not, some proper subsequent Δ of Γ, A is true. If Δ does not contain A , then it is also a proper subsequent of $\Gamma, A \vee B$, contradicting minimality. Otherwise let Δ' be the result of replacing A in Δ by $A \vee B$. Since Δ is true, so also is Δ' . Thus Δ' is a true proper subsequent of $\Gamma, A \vee B$, contradicting minimality. \square

LEMMA 5 *Suppose $\Gamma, A \vee B$ is a minimal sequent, and that neither Γ, A nor Γ, B is true. Then Γ, A, B is minimal.*

Proof. Suppose Γ, A, B had a true proper subsequent Δ . Since neither Γ, A nor Γ, B is true, Δ must contain both A and B . Let Δ' result from replacing A, B by $A \vee B$ in Δ . Then Δ' is a true proper subsequent of $\Gamma, A \vee B$, contradicting minimality. \square

The Completeness Theorem will be a special case of the following proposition.

PROPOSITION 2 *Every minimal sequent is provable in **Hp**.*

Proof. Suppose Γ is a minimal sequent. We proceed by induction on the number of connectives in Γ .

- *Induction base (no connective).* Since Γ is minimal, Lemma 2 implies $\Gamma = P, \neg P$, the conclusion of the axiom rule $\overline{P, \neg P}$.
- *Induction step (at least one connective).*
 1. *Case: $\Gamma = \Delta, A_1 \wedge A_2$.* By Lemma 3, $\Gamma = \Sigma, \Delta_1, \Delta_2, A_1 \wedge A_2$ for Σ, Δ_1, A_1 and Σ, Δ_2, A_2 minimal. Write down the conjunction rule

$$\frac{\Sigma, \Delta_1, A_1 \quad \Sigma, \Delta_2, A_2}{\Sigma, \Delta_1, \Delta_2, A_1 \wedge A_2} \wedge$$

and appeal to induction with the two hypothesis sequents.

2. *Case:* $\Gamma = \Delta, A_1 \vee A_2$.

(a) *Case:* Δ, A_i is true for some $i \in \{1, 2\}$. Write down the disjunction rule

$$\frac{\Delta, A_i}{\Delta, A_1 \vee A_2} \oplus_i$$

then appeal to induction with Δ, A_i , which is minimal by Lemma 4.

(b) *Case:* Δ, A_i is not true for each $i \in \{1, 2\}$. Thus Δ, A_1, A_2 is minimal, by Lemma 5. Write down the disjunction rule

$$\frac{\Delta, A_1, A_2}{\Delta, A_1 \vee A_2} \wp$$

then appeal to induction with Δ, A_1, A_2 .

(Γ may match both case 1 and case 2 in the inductive step, permitting some choice in the construction of the proof. There is also choice within case 2(a) if Δ, A_1 and Δ, A_2 are both true.) □

Since a true formula (singleton sequent) is a special case of a minimal sequent, the following is an immediate corollary of Proposition 2:

THEOREM 2 (COMPLETENESS OF CUT-FREE PROPOSITIONAL HYBRID LOGIC)
Every true formula is provable in \mathbf{Hp} .

Completeness is for true formulas (singleton sequents), or more generally minimal sequents (Proposition 2). It does not hold for arbitrary sequents. For example, the sequent $P, \neg P, Q$ is true but not provable in \mathbf{Hp} . See Section 6 for details.

4 Minimality of Hybrid Logic

Hybrid conjunction is critical for the liberation from structural rules: relaxing it to the pair of standard conjunction rules (context-sharing $\&$ and context-splitting \otimes) breaks completeness. Let \mathbf{Hp}^- be the following subsystem of \mathbf{Hp} :⁹

⁹This precursor of \mathbf{Hp} is (cut-free) multiplicative-additive linear logic [Gir87] with the formula constructors *tensor* \otimes and *with* $\&$ collapsed to \wedge , and the formula constructors *plus* \oplus and *par* \wp collapsed to \vee .

System \mathbf{Hp}^-

$$\begin{array}{c}
 \overline{P, \neg P} \\
 \\
 \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \\
 \\
 \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i
 \end{array}$$

PROPOSITION 3 *System \mathbf{Hp}^- is incomplete.*

Proof. We show that the true formula $A = ((P \wedge Q) \vee (\overline{Q} \wedge P)) \vee \overline{P}$ is not provable in \mathbf{Hp}^- . The placement of the two outermost \vee connectives forces the last two rules of a potential proof to be disjunction rules. Since $P \wedge Q, \overline{Q} \wedge P, \overline{P}$ is minimal (no proper subsequent is true), the two disjunction rules must be \wp rather than \oplus :

$$\frac{\frac{P \wedge Q, \overline{Q} \wedge P, \overline{P}}{(P \wedge Q) \vee (\overline{Q} \wedge P), \overline{P}} \wp}{((P \wedge Q) \vee (\overline{Q} \wedge P)) \vee \overline{P}} \wp$$

It remains to show that $P \wedge Q, \overline{Q} \wedge P, \overline{P}$ is not provable in \mathbf{Hp}^- .¹⁰ There are only two connectives, both \wedge , so the last rule must be a conjunction.

1. *Case: the last rule is a context-sharing $\&$ -rule.*

(a) *Case: The last rule introduces $P \wedge Q$. We have:*

$$\frac{P, \overline{Q} \wedge P, \overline{P} \quad Q, \overline{Q} \wedge P, \overline{P}}{P \wedge Q, \overline{Q} \wedge P, \overline{P}} \&$$

The left hypothesis $P, \overline{Q} \wedge P, \overline{P}$ cannot be proved in \mathbf{Hp}^- , since there is no Q to match the \overline{Q} (and no weakening).

(b) *Case: The last rule introduces $\overline{Q} \wedge P$. The same as the previous case, by symmetry, and exchanging $Q \leftrightarrow \overline{Q}$.*

2. *Case: the last rule is a context-splitting \otimes -rule.*

¹⁰Note, in contrast, that an \mathbf{Hp} proof of this sequent was given in Example 2 on page 7.

(a) *Case: The last rule introduces $P \wedge Q$. We have:*

$$\frac{P, \Gamma \quad Q, \Delta}{P \wedge Q, \overline{Q} \wedge P, \overline{P}} \otimes$$

where we must allocate each of $\overline{Q} \wedge P$ and \overline{P} either to Γ or to Δ . If $\overline{Q} \wedge P$ is in Γ , then P, Γ is not provable in \mathbf{Hp}^- , since it contains no Q to match the \overline{Q} . So $\overline{Q} \wedge P$ is in Δ . But then the \overline{P} is required in both Γ and Δ .

(b) *Case: The last rule introduces $\overline{Q} \wedge P$. The same as the previous case, by symmetry, and exchanging $Q \leftrightarrow \overline{Q}$.*

□

4.1 The Minimality Theorem

This section makes precise the sense in which hybrid logic represents a completeness threshold. Define a *standard system* as any propositional sequent calculus containing the axiom $\overline{P}, \neg\overline{P}$ and any of the *standard rules*:

$$\begin{array}{ccc} \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \quad \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \quad \quad \frac{\Gamma}{\Gamma, A} \text{W} \\ \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \quad \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \quad \quad \quad \frac{\Gamma, A, A}{\Gamma, A} \text{C} \end{array}$$

Thus there are $2^6 = 64$ such systems (many of which will not be complete).

Say that a system S *contains* a system T if each rule of T is a derived rule of S . For example, as we noted in the Introduction, system $\mathbf{GS1p}$ (displayed on page 1) contains \mathbf{Hp} (page 6), since the hybrid conjunction rule \wedge and the disjunction rule \wp of \mathbf{Hp} can be derived in $\mathbf{GS1p}$:

$$\begin{array}{ccc} \frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge & \longleftarrow & \frac{\frac{\Gamma, \Delta, A}{\Gamma, \Delta, \Sigma, A} \text{W}^* \quad \frac{\Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, B} \text{W}^*}{\Gamma, \Delta, \Sigma, A \wedge B} \& \\ \\ \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp & \longleftarrow & \frac{\frac{\Gamma, A, B}{\Gamma, A, A \vee B} \oplus_2}{\frac{\Gamma, A \vee B, A \vee B}{\Gamma, A \vee B} \text{C}} \oplus_1 \end{array}$$

where W^* denotes a sequence of zero or more weakenings.

THEOREM 3 (MINIMALITY THEOREM) *A standard system is complete iff it contains propositional hybrid logic **Hp**.*

We prove this theorem in the next section.

4.2 Proof of the Minimality Theorem

Define two systems as *equivalent* if each contains the other. (Recall that S contains T if every rule of T is a derived rule of S .) For example, it is well known that **GS1p** (displayed on page 1) is equivalent to the system¹¹

$$\frac{}{P, \neg P} \quad \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \frac{\Gamma, A_1, A_2}{\Gamma, A_1 \vee A_2} \wp \quad \frac{\Gamma}{\Gamma, A} \text{W} \quad \frac{\Gamma, A, A}{\Gamma, A} \text{C}$$

by deriving the following rules:

$$\frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \longleftarrow \quad \frac{\frac{\Delta, A}{\Delta, \Sigma, A} \text{W}^* \quad \frac{\Sigma, B}{\Delta, \Sigma, B} \text{W}^*}{\Delta, \Sigma, A \wedge B} \&$$

$$\frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \quad \longleftarrow \quad \frac{\frac{\Gamma, A_i}{\Gamma, A_1, A_2} \text{W}}{\Gamma, A_1 \vee A_2} \wp$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \longleftarrow \quad \frac{\frac{\Gamma, A, B}{\Gamma, A, A \vee B} \oplus_2}{\Gamma, A \vee B, A \vee B} \oplus_1 \quad \text{C}$$

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \longleftarrow \quad \frac{\frac{\Gamma, A \quad \Gamma, B}{\Gamma, \Gamma, A \wedge B} \otimes}{\Gamma, A \wedge B} \text{C}^*$$

We can abbreviate these four rule derivations as:

¹¹This system is multiplicative linear logic [Gir87] plus contraction and weakening (with the connectives denoted \wedge and \vee instead of \otimes and \wp).

Standard derived rules

$$\begin{array}{cc} \otimes \leftarrow \& W & \& \leftarrow \otimes C \\ \oplus \leftarrow \wp W & \wp \leftarrow \oplus C \end{array}$$

(Context-splitting conjunction \otimes is derivable from context-sharing conjunction $\&$ and weakening W , and so on.)

4.2.1 Only three complete standard systems, up to equivalence

As a stepping stone towards the Minimality Theorem, in this section we prove that, up to equivalence, there are only three complete standard systems.

We abbreviate a system by listing its non-axiom rules. For example, **GS1p** = $(\&, \oplus, W, C)$ and **Hp** = (\wedge, \oplus, \wp) . Besides **GS1p**, we shall pay particular attention to the systems (\otimes, \oplus, C) and $(\&, \wp, W)$, which we refer to as *positive propositional logic*, denoted **Pp**, and *negative propositional logic*, denoted **Np**, respectively.¹² In full, for ease of reference, the systems are:

System Pp (Propositional Positive Logic)

$$\frac{}{P, \neg P} \quad \frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \quad \frac{\Gamma, A, A}{\Gamma, A} C$$

System Np (Propositional Negative Logic)

$$\frac{}{P, \neg P} \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \& \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp \quad \frac{\Gamma}{\Gamma, A} W$$

PROPOSITION 4 *Up to equivalence:*

- (1) the Gentzen-Schütte-Tait system **GS1p** = $(\&, \oplus, C, W)$ is the only complete standard system with both contraction C and weakening W ;
- (2) propositional positive logic **Pp** = (\otimes, \oplus, C) is the only complete standard system without weakening W ;
- (3) propositional negative logic **Np** = $(\&, \wp, W)$ is the only complete standard system without contraction C .

¹²Our terminology here comes from polarity of connectives in linear logic [Gir87]: *tensor* \otimes and *plus* \oplus are positive, and *with* $\&$ and *par* \wp are negative.

We prove this proposition via a number of lemmas.

LEMMA 6 *Propositional hybrid logic $\mathbf{Hp} = (\wedge, \oplus, \wp)$ is contained in propositional positive logic $\mathbf{Pp} = (\otimes, \oplus, C)$, propositional negative logic $\mathbf{Np} = (\&, \wp, W)$ and $\mathbf{GS1p} = (\&, \oplus, C, W)$.*

Proof. \mathbf{Pp} contains \mathbf{Hp} since the hybrid conjunction rule \wedge is derivable from C and \otimes ,

$$\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge \quad \longleftarrow \quad \frac{\frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Gamma, \Delta, \Sigma, A \wedge B} \otimes}{\Gamma, \Delta, \Sigma, A \wedge B} C^*$$

(where C^* denotes zero or more consecutive contractions) and \wp is derivable from \oplus and C (see page 4).

\mathbf{Np} contains propositional hybrid logic since the hybrid conjunction rule \wedge is derivable from W and $\&$ (see page 12), and \oplus is derivable from W and \wp (see page 14).

$\mathbf{GS1p} = (\&, \oplus, C, W)$ is equivalent to $(\otimes, \&, \oplus, \wp, C, W)$, since \otimes and \wp are derivable, therefore contains \mathbf{Pp} (and \mathbf{Np}), hence \mathbf{Hp} . \square

LEMMA 7 *Propositional positive logic $\mathbf{Pp} = (C, \otimes, \oplus)$ and propositional negative logic $\mathbf{Np} = (\&, \wp, W)$ are complete.*

Proof. Both contain propositional hybrid logic \mathbf{Hp} (Lemma 6), which is complete (Theorem 2). \square

LEMMA 8 *Up to equivalence, system $\mathbf{GS1p} = (\&, \oplus, C, W)$ is the only complete standard system with both contraction C and weakening W .*

Proof. $\mathbf{GS1p}$ is complete (see e.g. [TS96], or by the fact that $\mathbf{GS1p}$ contains \mathbf{Hp} which is complete). Any complete system must have a conjunction rule (\otimes or $\&$) and a disjunction rule (\oplus or \wp). In the presence of C and W , the two conjunctions are derivable from one other, as are the two disjunctions (see page 14). \square

LEMMA 9 *A complete standard system without weakening W must contain propositional positive logic $\mathbf{Pp} = (\otimes, \oplus, C)$.*

Proof. System $\mathbf{Hp}^- = (\otimes, \oplus, \&, \wp)$, with both conjunction rules and both disjunction rules, is incomplete (Proposition 3, page 11), therefore we must have contraction C .

Without the \oplus rule, the true formula $(P \vee \overline{P}) \vee Q$ is not provable: the last rule must be \wp , leaving us to prove $P \vee \overline{P}, Q$, which is impossible without weakening W (i.e., with at most $\wp, \&, \otimes$ and C available), since, after a necessary axiom $\overline{P}, \overline{P}$ at the top of the proof, there is no way to introduce the formula Q .

Without the context-splitting \otimes rule, the true formula $P \vee (Q \vee (\overline{P} \wedge \overline{Q}))$ is not provable. The last two rules must be \wp , for if we use a \oplus we will not be able to match complementary literals in the axioms at the top of the proof. Thus we are left to prove $P, Q, \overline{P} \wedge \overline{Q}$, using $\&$ and C . The proof must contain an axiom rule $\overline{P}, \overline{P}$. The next rule can only be a $\&$ (since P, \overline{P} cannot be the hypothesis sequent of a contraction C rule). Since the only \wedge -formula in the final concluding sequent $P, Q, \overline{P} \wedge \overline{Q}$ is $\overline{P} \wedge \overline{Q}$, and the $\&$ rule is context sharing, the $\&$ -rule must be

$$\frac{\overline{P}, \overline{P} \quad \begin{array}{c} \vdots \\ P, \overline{Q} \end{array}}{P, \overline{P} \wedge \overline{Q}} \&$$

but P, \overline{Q} is not provable. □

LEMMA 10 *Up to equivalence, propositional positive logic $\mathbf{Pp} = (\otimes, \oplus, C)$ is the only complete standard system without weakening W .*

Proof. By Lemma 7, \mathbf{Pp} is complete. By Lemma 9, every W -free complete standard system contains \mathbf{Pp} . All other W -free standard systems containing \mathbf{Pp} are equivalent to \mathbf{Pp} , since the standard rule derivations $\& \leftarrow \otimes C$ and $\wp \leftarrow \oplus C$ yield $\&$ and \wp (see page 14). □

LEMMA 11 *A complete standard system without contraction C must contain propositional negative logic $\mathbf{Np} = (\&, \wp, W)$.*

Proof. System $\mathbf{Hp}^- = (\otimes, \oplus, \&, \wp)$, with both conjunction rules and both disjunction rules, is incomplete (Proposition 3, page 11), therefore we must have weakening W .

Without the \wp rule, the true formula $P \vee \overline{P}$ would not be provable.

Without the $\&$ rule the true formula $P \vee (\overline{P} \wedge \overline{P})$ would not be provable. The last rule must be a \wp (rather than a \oplus , otherwise we are stranded either without a P or without a \overline{P}), so we are left to prove $P, \overline{P} \wedge \overline{P}$. The last rule cannot be \wp or \oplus , as the only connective is \wedge . It cannot be W , or we are stranded either without a P or without a \overline{P} . It cannot be \otimes , as one of the two hypotheses will be the single formula \overline{P} . □

LEMMA 12 *Up to equivalence, propositional negative logic $\mathbf{Np} = (\&, \wp, W)$ is the only complete standard system without contraction C .*

Proof. By Lemma 7, \mathbf{Np} is complete. By Lemma 11, every C -free complete standard system contains \mathbf{Np} . All other C -free standard systems containing \mathbf{Np} are equivalent to \mathbf{Np} , since the standard rule derivations $\otimes \leftarrow \& W$ and $\oplus \leftarrow \wp W$ yield \otimes and \oplus (see page 14). □

*Proof of Proposition 4, page 14 (up to equivalence, the standard systems **GS1p**/**Pp**/**Np** are unique with C and W / without W / without C). Parts (1), (2) and (3) of Proposition 4 are Lemmas 8, 10 and 12, respectively. \square*

LEMMA 13 *A standard system without contraction C or weakening W is incomplete.*

Proof. Any such system is contained in $\mathbf{Hp}^- = (\otimes, \oplus, \&, \wp)$, which is incomplete (Proposition 3, page 11). \square

THEOREM 4 *Up to equivalence, there are only three complete standard systems:*

1. *The Gentzen-Schütte-Tait system **GS1p** = ($\&, \oplus, C, W$).*
2. *Propositional positive logic **Pp** = (\otimes, \oplus, C).*
3. *Propositional negative logic **Np** = ($\&, \wp, W$).*

Proof. Proposition 4 and Lemma 13. \square

4.2.2 Proof of the Minimality Theorem

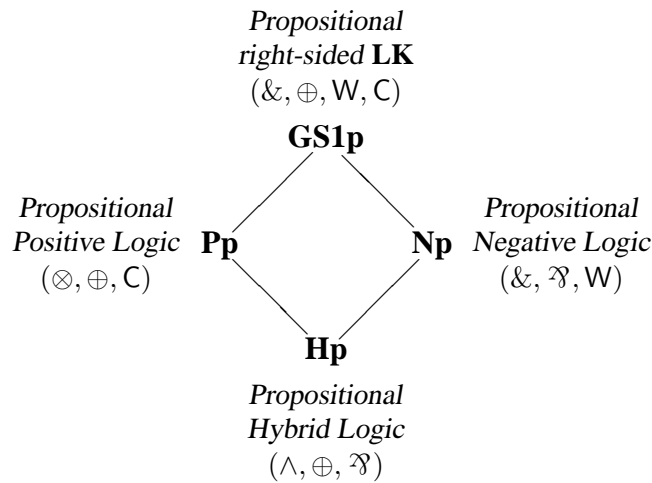
Now we can prove Theorem 3, the Minimality Theorem.

MINIMALITY THEOREM *A standard system is complete iff it contains propositional hybrid logic **Hp**.*

Proof. Up to equivalence, there are only three complete standard systems (Theorem 4), and each contains **Hp** (Lemma 6). \square

4.3 Propositional hybrid logic **Hp** as a “complete core” of Gentzen’s LK

The three inequivalent complete standard systems **GS1p**, **Pp** and **Np**, together with propositional hybrid logic **Hp**, sit in the following Hasse diagram of containments:



Thus we can view propositional hybrid logic **Hp** as a “complete core” of **GS1p**, hence of (propositional) Gentzen’s **LK**.

5 Extended Minimality Theorem

Define an *extended system* as one containing the axiom rule $\overline{P, \overline{P}}$ and any of the following rules:

Extended system rules

$$\frac{\Delta, A \quad \Sigma, B}{\Delta, \Sigma, A \wedge B} \otimes \quad \frac{\Gamma, \Delta, A \quad \Gamma, \Sigma, B}{\Gamma, \Delta, \Sigma, A \wedge B} \wedge \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \&$$

$$\frac{\Gamma, A_i}{\Gamma, A_1 \vee A_2} \oplus_i \quad \frac{\Gamma, A, B}{\Gamma, A \vee B} \wp$$

$$\frac{\Gamma, A, A}{\Gamma, A} C \quad \frac{\Gamma}{\Gamma, A} W$$

Thus there are $2^7 = 128$ extended systems, of which the $2^6 = 64$ without the hybrid conjunction rule \wedge are the standard systems defined in Section 4.1.¹³ The Minimality Theorem (Theorem 3, page 13) extends as follows.

THEOREM 5 (EXTENDED MINIMALITY THEOREM) *An extended system is complete iff it contains propositional hybrid logic **Hp**.*

To prove this theorem, we require two lemmas.

LEMMA 14 *Suppose S is a complete extended system with the hybrid conjunction rule \wedge , and with at least one of contraction C or weakening W . Then S is equivalent to a standard system.*

Proof. If S has weakening W , let S' be the result of replacing the hybrid conjunction rule \wedge in S by context-sharing conjunction $\&$; otherwise S has contraction, and let S' result from replacing \wedge by context-splitting \otimes . Then S' is equivalent to S , since \wedge is derivable from \otimes and C (see page 15) and from $\&$ and W (see page 12). \square

¹³Note, however, that many of these standard systems have hybrid conjunction \wedge as a derived rule.

LEMMA 15 *Suppose S is a complete extended system without contraction C or weakening W . Then S is equivalent to propositional hybrid logic \mathbf{Hp} .*

Proof. Since $\mathbf{Hp}^- = (\otimes, \&, \oplus, \wp)$ is incomplete (Proposition 3, page 11), S must have the hybrid conjunction rule \wedge either directly or as a derived rule. Since S is complete, it must have a disjunction rule, therefore it could only fail to be equivalent to $\mathbf{Hp} = (\wedge, \oplus, \wp)$ if (a) it has \oplus and \wp is not derivable, *i.e.*, S is equivalent to (\wedge, \oplus) , or (b) it has \wp and \oplus is not derivable, *i.e.*, S is equivalent to (\wedge, \wp) . In case (a), the true formula $P \vee \overline{P}$ would not be provable, and in case (b) the true formula $(P \vee \overline{P}) \vee Q$ would not be provable, either way contradicting the completeness of S . \square

Proof of the Extended Minimality Theorem (Theorem 5). Suppose S is a complete extended system. If S has contraction C or weakening W then it is equivalent to a standard system by Lemma 14, hence contains \mathbf{Hp} by the original Minimality Theorem. Otherwise S is equivalent to \mathbf{Hp} by Lemma 15, hence in particular contains \mathbf{Hp} .

Conversely, suppose S is an extended system containing \mathbf{Hp} . Then S is complete since \mathbf{Hp} is complete. \square

We also have an extension of Theorem 4 (page 17), which stated that, up to equivalence, there are only three complete standard systems, $\mathbf{GS1p}$, \mathbf{Pp} and \mathbf{Np} .

THEOREM 6 *Up to equivalence, there are only four complete extended systems:*

1. *The Gentzen-Schütte-Tait system $\mathbf{GS1p} = (\&, \oplus, C, W)$.*
2. *Propositional positive logic $\mathbf{Pp} = (\otimes, \oplus, C)$.*
3. *Propositional negative logic $\mathbf{Np} = (\&, \wp, W)$.*
4. *Propositional hybrid logic $\mathbf{Hp} = (\wedge, \oplus, \wp)$.*

Proof. Theorem 4 together with Lemmas 14 and 15. \square

6 Relative degrees of completeness

We defined a system as *complete* if every true formula (singleton sequent) is provable. To avoid ambiguity with forthcoming definitions, let us refer to this default notion of completeness as *formula-complete*. Define a system as *minimal-complete* if every minimal¹⁴ sequent is provable, and *sequent-complete* if every true sequent is provable. (Thus *sequent-complete* implies *minimal-complete* implies *formula-complete*.)

¹⁴Recall that a true sequent is minimal if no proper subsequent is true.

PROPOSITION 5 *Propositional positive logic $\mathbf{Pp} = (\otimes, \oplus, C)$ and propositional hybrid logic $\mathbf{Hp} = (\wedge, \oplus, \wp)$ are formula-complete and minimal-complete, but not sequent-complete.*

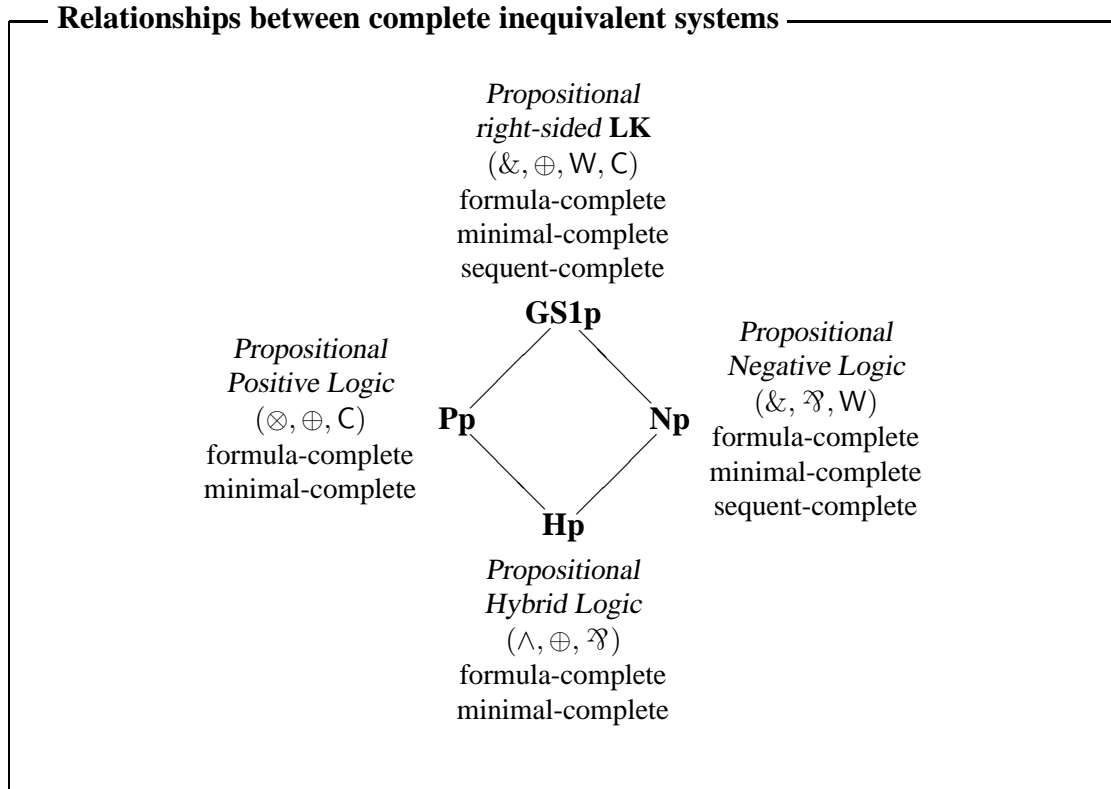
Proof. We have already proved that \mathbf{Hp} (hence also \mathbf{Pp} , by containment) is minimal-complete (Proposition 2).

We show that the true (non-minimal) sequent P, \overline{P}, Q is unprovable in \mathbf{Pp} (hence also in \mathbf{Hp}). A proof must contain an axiom rule $\overline{P}, \overline{P}$. This cannot be followed by a \otimes or \oplus rule, otherwise we introduce a connective \wedge or \vee which cannot subsequently be removed by any other rule before the concluding sequent P, \overline{P}, Q . Neither can it be followed by contraction C , since there is nothing to contract. \square

PROPOSITION 6 *Propositional negative logic $\mathbf{Np} = (\&, \wp, W)$ is formula-complete, minimal-complete and sequent-complete.*

Proof. \mathbf{Np} is minimal-complete since it contains minimal-complete \mathbf{Hp} . Suppose Γ is a true but not minimal sequent. Choose a minimal subsequent Δ of Γ (see Lemma 1, page 8). By minimal-completeness, Δ has a proof. Follow this with weakenings to obtain Γ from Δ . \square

Below we have annotated our Hasse diagram with completeness strengths.



References

- [Bol02] B. Bollobás. *Modern Graph Theory*. Springer-Verlag, 2002.
- [Gen39] G. Gentzen. Untersuchungen über das logische Schließen I, II. *Mathematische Zeitschrift*, 39:176–210 405–431, 1939. Translation in [Gen69, 68–131].
- [Gen69] G. Gentzen. *The Collected Papers of Gerhard Gentzen*. North-Holland, 1969. Edited and introduced by M.E. Szabo.
- [Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
- [Ket44] J. Ketonen. Untersuchungen zum Prädikatenkalkül. *Annales Academiae Scientiarum Fennicae, ser. A, I. Mathematica-physica*, 23, 1944.
- [Sch50] K. Schütte. Schlußweisen-Kalküle der Prädikatenlogik. *Mathematische Annalen*, 123:166–186 405–431, 1950.
- [Tai68] W. W. Tait. Normal derivability in classical logic. In *The Syntax and Semantics of Infinitary Languages*, volume 72 of *Lecture Notes in Computer Science*, pages 204–236. Springer-Verlag, 1968.
- [TS96] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, Cambridge, U.K., 1996.
- [Urb00] C. Urban. *Classical Logic and Computation*. Ph.D. thesis, University of Cambridge, 2000.
- [UB01] C. Urban and G. Bierman. Strong Normalisation of Cut-Elimination in Classical Logic. *Fundamenta Informaticae*, 45:123–155, 2001.