Deep inference proof theory equals categorical proof theory minus coherence

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Abstract

This paper links deep inference proof theory, as studied by Guglielmi *et al.*, to categorical proof theory in the sense of Lambek *et al.*. It observes how deep inference proof theory is categorical proof theory, minus the coherence diagrams/laws. Coherence yields a ready-made and well studied notion of equality on deep inference proofs.

The paper notes a precise correspondence between the symmetric deep inference system for multiplicative linear logic (the linear fragment of SKSg) and the presentation of $*$-autonomous categories as symmetric linearly distributive categories with negation. Contraction and weakening in SKSg corresponds precisely to the presence of (co)monoids.

1 Introduction

This paper observes that deep inference proof theory, as studied by Guglielmi *et al.* [Brü04], fits nicely into the tradition of categorical proof theory dating back to the late 1960's [Lam69]. It illustrates how deep inference proof theory is categorical proof theory, minus the coherence diagrams/laws.

Observing that deep inference proof theory is categorical proof theory brings two immediate benefits to the former. First, it places the subject within a mature and well-established area of mathematics, providing a broader audience and possibly precluding future instances of 'reinventing the wheel'. Second, existing techniques in categorical proof theory yield immediate results in deep inference proof theory. For example, coherence, a well-established and integral part of category theory [Mac71, Ch. VII], yields a ready-made notion of equality on deep inference proofs. Conversely, syntactic techniques used in the deep inference proof theory community, such as normalisation proofs and the medial rule of SKS [Brü04], should hopefully map back into categorical proof theory for new results. Ideas can flow in both directions.

A key feature of categorical proof theory is that inferences apply in any context. For example, associativity $A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ applies in the context $\big((-) \otimes X\big) \otimes Y$ to yield

$$\big((A \otimes (B \otimes C)) \otimes X\big) \otimes Y \to \big(((A \otimes B) \otimes C) \otimes X\big) \otimes Y.$$

In other words, inferences are 'deep'. This is due to functoriality (of $\otimes$, in the example above). The property called *symmetry* by deep inference proof theorists corresponds to what categorical proof theorists call a *duality*, a full and faithful contravariant endofunctor (the '$*$' of $*$-autonomous [Bar79]). For example, applying the duality functor to the 'left' linear distributivity natural transformation

$$\delta_{A,B,C} : A \otimes (B \parr C) \to (A \otimes B) \parr C$$

$$
\begin{array}{c}
p \otimes q \\
\downarrow \rho \\
(p \otimes q) \otimes 1 \\
\downarrow (p\otimes q)\otimes\overline{\gamma} \\
(p \otimes q) \otimes \big((\overline{p} \parr \overline{q}) \parr (p \otimes q)\big) \\
\downarrow \delta \\
\big((p \otimes q) \otimes (\overline{p} \parr \overline{q})\big) \parr (p \otimes q) \\
\downarrow \gamma\otimes(p\otimes q) \\
\bot \parr (p \otimes q) \\
\downarrow \overline{\lambda} \\
p \otimes q
\end{array}
\quad \leftrightarrow \quad
\begin{array}{cl}
p \otimes q & \\ \hline
(p \otimes q) \otimes \mathsf{t} & = \\ \hline
(p \otimes q) \otimes \big((\overline{p} \parr \overline{q}) \parr (p \otimes q)\big) & \mathsf{i}\downarrow \\ \hline
\big((p \otimes q) \otimes (\overline{p} \parr \overline{q})\big) \parr (p \otimes q) & \mathsf{s} \\ \hline
\mathsf{f} \parr (p \otimes q) & \mathsf{i}\uparrow \\ \hline
p \otimes q & =
\end{array}
\quad \leftrightarrow \quad
\begin{array}{cl}
(p, q) & \\ \hline
\big((p, q), \mathsf{t}\big) & = \\ \hline
\big((p, q), \big[[\overline{p}, \overline{q}], (p, q)\big]\big) & \mathsf{i}\downarrow \\ \hline
\big[\big((p, q), [\overline{p}, \overline{q}]\big), (p, q)\big] & \mathsf{s} \\ \hline
\big[\mathsf{f}, (p, q)\big] & \mathsf{i}\uparrow \\ \hline
(p, q) & =
\end{array}
$$

Figure 1: Illustrating the correspondence between categorical proofs in a $*$-autonomous category and proofs in the linear fragment of the deep inference system SKSg. The left column is a categorical proof, a composite of canonical maps. The right column is the corresponding SKSg proof. The centre column is to aid pattern matching. This example is explained in detail in the main text.

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yields 'right' linear distributivity

$$\overline{\delta}_{A,B,C} : (A \parr B) \otimes C \to A \parr (B \otimes C)$$

(up to isomorphism).

There is a precise correspondence between the symmetric deep inference system for multiplicative linear logic (the linear fragment of SKSg [Brü04]), and the presentation of $*$-autonomous categories [Bar79] as symmetric linearly distributive categories with negation [CS91, BCST96, CS97]. Figure 1 illustrates the correspondence. Contraction and weakening in SKSg corresponds to the presence of (co)monoids in the $*$-autonomous category [See89, Sel01, FP04]. See Figure 2.

2 Categorical proof theory

This section provides an overview of categorical proof theory, targetted at deep inference proof theorists. Central ideas of categorical proof theory [Lam69] include:

$$
\begin{array}{rcl}
\textit{proof} & = & \textit{morphism} \\
\textit{connective} & = & \textit{functor} \\
\textit{inference} & = & \textit{natural map}
\end{array}
$$

For example, in the categorical proof theory approach to linear logic [Gir87, See89, CS91, CS97], one has the proof of $p \otimes q$ from $p \otimes q$ in a $*$-autonomous category $\mathbb{C}$ [Bar79] shown in the left column of Figure 1. Tensor $\otimes$ and its dual, par $\parr$, are functors $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, with units 1 and $\bot$ respectively, and *duality* $A \mapsto \overline{A}$ is a full and faithful contravariant endofunctor on $\mathbb{C}$ (*i.e.*, $\overline{(-)} : \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$). In a $*$-autonomous category one has the isomorphisms $\overline{\overline{A}} \cong A$, $\overline{A \otimes B} \cong \overline{A} \parr \overline{B}$, $\overline{A \parr B} \cong \overline{A} \otimes \overline{B}$, and $\overline{1} \cong \bot$. We shall follow the standard (and convenient) approach to linear logic and assume these

$$
\begin{array}{c}
1 \\
\big\downarrow \overline{\gamma}_{x \parr x} \\
(\overline{x} \otimes \overline{x}) \parr (x \parr x) \\
\big\downarrow (\overline{x} \otimes \overline{x}) \parr c_x \\
(\overline{x} \otimes \overline{x}) \parr x \\
\big\downarrow (\overline{\rho_{\overline{x} f}} \otimes \overline{x}) \parr x \\
\big((\overline{x} \parr \bot) \otimes \overline{x}\big) \parr x \\
\big\downarrow \big((\overline{x} \parr w_y) \otimes \overline{x}\big) \parr x \\
\big((\overline{x} \parr y) \otimes \overline{x}\big) \parr x
\end{array}
\quad \leftrightarrow \quad
\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{t}}{(\overline{x} \otimes \overline{x}) \parr (x \parr x)}\ \mathsf{i}\downarrow}{(\overline{x} \otimes \overline{x}) \parr x}\ \mathsf{c}\downarrow}{\big((\overline{x} \parr \mathsf{f}) \otimes \overline{x}\big) \parr x}\ =}{\big((\overline{x} \parr y) \otimes \overline{x}\big) \parr x}\ \mathsf{w}\downarrow
\quad \leftrightarrow \quad
\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{t}}{\big[(\overline{x}, \overline{x}), [x, x]\big]}\ \mathsf{i}\downarrow}{\big[(\overline{x}, \overline{x}), x\big]}\ \mathsf{c}\downarrow}{\big[\big([\overline{x}, \mathsf{f}], \overline{x}\big), x\big]}\ =}{\big[\big([\overline{x}, y], \overline{x}\big), x\big]}\ \mathsf{w}\downarrow
$$

Figure 2: Illustrating how contraction and weakening in SKSg corresponds to the presence of (co)monoids in a $*$-autonomous category. The left column is a categorical proof of Peirce's law $\big((x \multimap y) \multimap x\big) \multimap x$, where $A \multimap B$ abbreviates $\overline{A} par B$ The right column is the corresponding SKSg proof. This example is explained in detail in the main text.

isomorphisms are equalities [Gir87, Blu91]: $\overline{\overline{A}} = A$, $\overline{A \otimes B} = \overline{A} \parr \overline{B}$, $\overline{A \parr B} = \overline{A} \otimes \overline{B}$, and $\overline{1} = \bot$. The inferences in the left column of Figure 1 are the natural maps

ρ	:	A	$\rightarrow$	$A \otimes 1$	(tensor unit-right)
λ	:	$\bot \parr A$	$\rightarrow$	A	(par unit-left)
γ	:	$A \otimes \overline{A}$	$\rightarrow$	$\bot$	(cut)
$\overline{\gamma}$	:	1	$\rightarrow$	$\overline{A} \parr A$	(axiom)
δ	:	$A \otimes (B \parr C)$	$\rightarrow$	$(A \otimes B) \parr C$	(linear distributiv.)

The first two are isomorphisms. See [CS91, BCST96, CS97] for more details and examples.[1] Linear distributivity is also known as weak distributivity. In Figure 1 the identity morphism $A \rightarrow A$ is denoted A.

Inference in any context. A central feature of categorical proof theory is that inferences apply in any context. For example, linear distributivity

$$\delta : A \otimes (B \parr C) \rightarrow (A \otimes B) \parr C$$

applies here:

$$
\begin{array}{c}
\big(\big(A \otimes (B \parr C)\big) \parr X\big) \otimes Y \\
\big\downarrow (\delta \parr X) \otimes Y \\
\big(\big((A \otimes B) \parr C\big) \parr X\big) \otimes Y
\end{array}
$$

[1][BCST96] uses the notation τ (*tertium non datur*) for axiom (here $\overline{\gamma}$), $\oplus$ for $\parr$, and $A^{\perp}$ for $\overline{A}$. Also their $u^L_{\parr}$ is $\overline{\lambda}$, and $(u^R_{\otimes})^{-1}$ is ρ. Our choice of notation and terminology highlights the correspondence with the deep inference system SKSg. (In particular, do not confuse *cut* $\gamma : A \otimes \overline{A} \rightarrow \bot$ here with categorical composition.) Also remember that we have made the usual assumptions $\overline{A \otimes B} = \overline{A} \parr \overline{B}$, $\overline{\overline{A}} = A$ *etc.*, as in [Gir87, Blu91].

In the morphism $(\delta \parr X) \otimes Y$, the X denotes the identity morphism $X \to X$ and the Y denotes the identity $Y \to Y$. That inferences apply in context amounts to functoriality (in the example above, of $\parr, \otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, yielding the context (compound functor) $((-) \parr X) \otimes Y : \mathbb{C} \to \mathbb{C}$). In the notation of SKSg [Brü04], the inference above is written thus:

$$\frac{\big(\big[\big(A,[B,C]\big),X\big],Y\big)}{\big(\big[\big[(A,B),C\big],X\big],Y\big)}\,\mathsf{s}$$

The in-context nature of inference in categorical logic is not merely an aesthetic property. It has also been found to be useful technically. In-context linear distributivity $\delta : A \otimes (B \parr C) \to (A \otimes B) \parr C$ is pivotal in the argument of full completeness for unit-free MLL (multiplicative linear logic) with MIX in [AJ94, Lem. 2], and for unit-free MLL in [DHPP99]. The latter paper axiomatised unit-free MLL with *system S*, a subformula rewriting system (*i.e.*, deep inference system) based on the natural maps $\delta : A \otimes (B \parr C) \to (A \otimes B) \parr C$ *etc.*; see Appendix A. The linear fragment of SKSg is an extension of system S with units.

Duality. Duality is an important feature of the categorical proof theory of linear logic. For example, proof in Figure 1 can be 'flipped' from top to bottom, yielding a dual proof of $p \parr q$ from $p \parr q$:

$$\begin{array}{c} p \parr q \\ \big\downarrow{\scriptstyle \lambda} \\ 1 \otimes (p \parr q) \\ \big\downarrow{\scriptstyle \overline{\gamma} \otimes (p \parr q)} \\ \big((p \parr q) \parr (\overline{p} \otimes \overline{q})\big) \otimes (p \parr q) \\ \big\downarrow{\scriptstyle \overline{\delta}} \\ (p \parr q) \parr \big((\overline{p} \otimes \overline{q}) \otimes (p \parr q)\big) \\ \big\downarrow{\scriptstyle (p \parr q) \parr \gamma} \\ (p \parr q) \parr \bot \\ \big\downarrow{\scriptstyle \overline{\rho}} \\ p \parr q \end{array} \tag{1}$$

We have simply applied the duality functor $\overline{(-)} : \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$ to the original diagram; since contravaraince reverses the arrows, we have then drawn the resulting proof upside down. (We have also exchanged the atoms $p \leftrightarrow \overline{p}$ and $q \leftrightarrow \overline{q}$.)

Equalities on proofs (coherence). Categorical proof theory involves the formal assertion of equalities between proofs. These equalities are called ***coherence laws*** or ***coherence diagrams***. For example, by coherence, the categorical proof in Figure 1 is formally equal to the identity proof $p \otimes q \to p \otimes q$:

$$
\begin{array}{c}
p \otimes q \\
\big\downarrow{\scriptstyle \rho} \\
(p \otimes q) \otimes 1 \\
\big\downarrow{\scriptstyle (p\otimes q)\otimes\overline{\gamma}} \\
(p \otimes q) \otimes \big((\overline{p} \parr \overline{q}) \parr (p \otimes q)\big) \\
\big\downarrow{\scriptstyle \delta} \\
\big((p \otimes q) \otimes (\overline{p} \parr \overline{q})\big) \parr (p \otimes q) \\
\big\downarrow{\scriptstyle \gamma\otimes(p\otimes q)} \\
\bot \parr (p \otimes q) \\
\big\downarrow{\scriptstyle \overline{\lambda}} \\
p \otimes q
\end{array}
\quad = \quad
\begin{array}{c}
p \otimes q \\
\Big\downarrow{\scriptstyle p\otimes q} \\
p \otimes q
\end{array}
\tag{2}
$$

See page 4 of [BCST96] for the relevant coherence diagram/law (and remember that we assumed negation is strict, with $\overline{\overline{A}} = A$ and $\overline{A \otimes B} = \overline{A} \parr \overline{B}$ *etc.*, as in [Gir87, Blu91]). In the notation of SKSg, this corresponds to asserting the following formal equality (in all contexts):

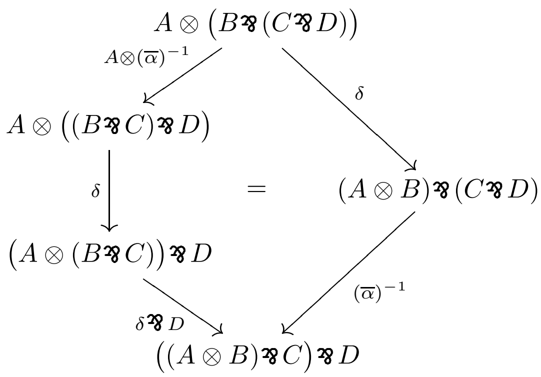
$$
\begin{array}{cl}
(p, q) & \\
\hline
\big((p, q), \mathsf{t}\big) & = \\
\hline
\big((p, q), \big[[\overline{p}, \overline{q}], (p, q)\big]\big) & \mathsf{i}\downarrow \\
\hline
\big[\big((p, q), [\overline{p}, \overline{q}]\big), (p, q)\big] & \mathsf{s} \\
\hline
\big[\mathsf{f}, (p, q)\big] & \mathsf{i}\uparrow \\
\hline
(p, q) & =
\end{array}
\qquad = \qquad (p, q)
$$

where the right-hand side could alternatively be written as $\frac{(p,q)}{(p,q)}=$, the 'identity' derivation of (p, q) from itself.

Examples of coherence laws include the well-known associativity pentagon diagram

$$
\begin{array}{ccc}
 & A \otimes \big(B \otimes (C \otimes D)\big) & \\
\swarrow{\scriptstyle A\otimes\alpha} & & \searrow{\scriptstyle \alpha} \\
A \otimes \big((B \otimes C) \otimes D\big) & & \\
\big\downarrow{\scriptstyle \alpha} & = & (A \otimes B) \otimes (C \otimes D) \\
\big(A \otimes (B \otimes C)\big) \otimes D & & \\
\searrow{\scriptstyle \alpha\otimes D} & & \swarrow{\scriptstyle \alpha} \\
 & \big((A \otimes B) \otimes C\big) \otimes D &
\end{array}
\tag{3}
$$

and the associativity/linear-distributivity pentagon



(4)

i.e., the equations

$$\mathsf{id}_A \otimes \alpha_{B,C,D} \; ; \; \alpha_{A,B\otimes C,D} \; ; \; \alpha_{A,B,C} \otimes \mathsf{id}_D \\ = \\ \alpha_{A,B,C\otimes D} \; ; \; \alpha_{A\otimes B,C,D}$$

and

$$\mathsf{id}_A \otimes (\overline{\alpha})^{-1}_{B,C,D} \; ; \; \delta_{A,B \parr C,D} \; ; \; \delta_{A,B,C} \parr \mathsf{id}_D \\ = \\ \delta_{A,B,C \parr D} \; ; \; (\overline{\alpha})^{-1}_{A\otimes B,C,D}$$

respectively. In SKSg notation, these coherence diagrams/laws assert the equalities

$$\cfrac{\cfrac{\cfrac{\big(A,\big(B,(C,D)\big)\big)}{\big(A,\big((B,C),D\big)\big)}=}{\big(\big(A,(B,C)\big),D\big)}=}{\big(\big((A,B),C\big),D\big)}= \qquad = \qquad \cfrac{\cfrac{\big(A,\big(B,(C,D)\big)\big)}{\big((A,B),(C,D)\big)}=}{\big(\big((A,B),C\big),D\big)}=$$

and

$$\cfrac{\cfrac{\cfrac{\big(A,\big[B,[C,D]\big]\big)}{\big(A,\big[[B,C],D\big]\big)}=}{\big[\big(A,[B,C]\big),D\big]}\mathsf{s}}{\big[\big[(A,B),C\big],D\big]}\mathsf{s} \qquad = \qquad \cfrac{\cfrac{\big(A,\big[B,[C,D]\big]\big)}{\big[(A,B),[C,D]\big)\big]}\mathsf{s}}{\big[\big[(A,B),C\big],D\big]}=$$

(respectively) in all contexts.

Coherence laws imply, for example, that all uses of cut and axiom reduce to the atomic case. The uses of cut and axiom

$$\begin{array}{c} (p \otimes q) \otimes (\overline{p} \parr \overline{q}) \\ \big\downarrow \gamma_{p\otimes q} \\ \bot \end{array} \qquad\qquad \begin{array}{c} 1 \\ \big\downarrow \overline{\gamma}_{p\otimes q} \\ (\overline{p} \parr \overline{q}) \parr (p \otimes q) \end{array} \tag{5}$$

in the categorical proof in Figure 1 are formally equal, by coherence laws, to the following, in which each cut γ and axiom $\overline{\gamma}$ is atomic (*i.e.*, they are applied to the atoms p and q only):

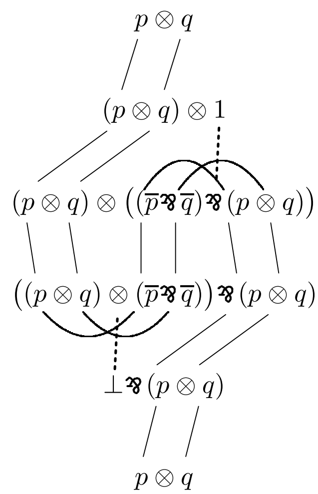
$$
\begin{array}{c}
(p \otimes q) \otimes (\overline{p} \parr \overline{q}) \\
\big\downarrow {\scriptstyle \alpha^{-1}} \\
p \otimes \big(q \otimes (\overline{p} \parr \overline{q})\big) \\
\big\downarrow {\scriptstyle p \otimes (q \otimes \sigma)} \\
p \otimes \big(q \otimes (\overline{q} \parr \overline{p})\big) \\
\big\downarrow {\scriptstyle p \otimes \delta} \\
p \otimes \big((q \otimes \overline{q}) \parr \overline{p}\big) \\
\big\downarrow {\scriptstyle p \otimes \sigma} \\
p \otimes \big(\overline{p} \parr (q \otimes \overline{q})\big) \\
\big\downarrow {\scriptstyle \delta} \\
(p \otimes \overline{p}) \parr (q \otimes \overline{q}) \\
\big\downarrow {\scriptstyle \gamma_p \parr \gamma_q} \\
\bot \parr \bot \\
\big\downarrow {\scriptstyle \overline{\lambda}} \\
\bot
\end{array}
\qquad\qquad
\begin{array}{c}
1 \\
\big\downarrow {\scriptstyle \lambda} \\
1 \otimes 1 \\
\big\downarrow {\scriptstyle \overline{\gamma}_p \otimes \overline{\gamma}_q} \\
(\overline{p} \parr p) \otimes (\overline{q} \parr q) \\
\big\downarrow {\scriptstyle \overline{\delta}} \\
\overline{p} \parr \big(p \otimes (\overline{q} \parr q)\big) \\
\big\downarrow {\scriptstyle \overline{p} \parr \overline{\sigma}} \\
\overline{p} \parr \big((\overline{q} \otimes q) \otimes p\big) \\
\big\downarrow {\scriptstyle \overline{p} \parr \overline{\delta}} \\
\overline{p} \parr \big(\overline{q} \parr (q \otimes p)\big) \\
\big\downarrow {\scriptstyle \overline{p} \parr (\overline{q} \parr \overline{\sigma})} \\
\overline{p} \parr \big(\overline{q} \parr (p \otimes q)\big) \\
\big\downarrow {\scriptstyle (\overline{\alpha})^{-1}} \\
(\overline{p} \parr \overline{q}) \parr (p \otimes q)
\end{array}
\tag{6}
$$

Here σ and $\overline{\sigma}$ are the symmetry natural isomorphisms $\sigma : A \parr B \to B \parr A$ and $\overline{\sigma} : A \otimes B \to B \otimes A$. Note that, by duality of axiom and cut, the right diagram results from applying the duality functor $\overline{(-)} : \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$ to the left one.

In SKSg notation, the cut-reducing equality above corresponds to the following derived equality on proofs (in all contexts):

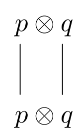
$$
\dfrac{\big((p,q),[\overline{p},\overline{q}]\big)}{\mathsf{f}}\ \mathsf{i}\uparrow
\qquad = \qquad
\begin{array}{cl}
\big((p,q),[\overline{p},\overline{q}]\big) & \\ \hline
\big(p,(q,[\overline{p},\overline{q}])\big) & = \\ \hline
\big(p,(q,[\overline{q},\overline{p}])\big) & = \\ \hline
\big(p,\big[(q,\overline{q}),\overline{p}\big]\big) & \mathsf{s} \\ \hline
\big(p,\big[\overline{p},(q,\overline{q})\big]\big) & = \\ \hline
\big[(p,\overline{p}),(q,\overline{q})\big] & \mathsf{s} \\ \hline
[\mathsf{f},\mathsf{f}] & \mathsf{ai}\uparrow\ \text{(twice)} \\ \hline
\mathsf{f} & =
\end{array}
$$

Abstract normal forms: proof nets. Proof nets for MLL [Gir87] provide a convenient graphical representation of categorical MLL proofs. For example, the proof in Figure 1 becomes:



(7)

which, with the obvious path composition, becomes the identity morphism (proof net) $p \otimes q \to p \otimes q$,



(8)

Aside from the 'unit attachments' [GSS91, Gir92, BCST96], shown above as dotted lines, the underlying graphs are Kelly-MacLane graphs, with the usual composition (path composition) [KM71, Blu91].[2] To obtain the proof net for the dual proof (1), one simply 'flips' the diagram of proof nets upside down (dualising $\parr \leftrightarrow \otimes$ and $\bot \leftrightarrow 1$). Proof nets are a convenient way to tell when two proofs are equal with respect to coherence [BCST96]; for example, the path composition above shows that the proof in Figure 1 is equal to the identity $p \otimes q \to p \otimes q$.[3]

---

[2] *Formal definition of proof net.* A ***link*** is a complementary pair of literal occurrences. A ***linking*** on an MLL formula A (viewed as a labelled tree) is a partitioning of the literal occurrences of A into links, together with a function from the $\bot$-occurrences of A to the vertices of A. A ***switching*** of a linking λ on A is any subgraph of the graph $\lambda \cup A$ obtained by deleting one argument edge of each $\parr$. A ***proof net*** is a linking whose every switching is a tree. A proof net $f : A \to B$ is a proof net f on $\overline{A} \parr B$ (drawn in the obvious two-sided manner in Figure (7)). This definition is an amalgam of [KM71, Gir87, DR89, GSS91, Gir92, BCST96, HG03].

[3] One defect of the proof net approach which remains unresolved is that, in the presence of units, MLL proof nets are *not quite* normal forms. To obtain coherence (*i.e.*, to characterise correctly the proof equalities in a $*$-autonomous category) one must permit a $\bot$-attachment on the output side (or 1-attachment on the input side) to 'slide' around its so-called empire, *i.e.*, one can move the attachment point around so long as one does not break the graphical correctness criterion [BCST96]. A variation in [SL04] permits a more complex attachment of units (replacing the links and dotted lines above with secondary formula trees), but still suffers from the same problem: normal forms are defined only modulo a rewrite which, as in [BCST96], is permissible when the correctness criterion is not violated. Normal forms for proof nets with units remains an interesting open problem.

Reduction to atomic identity links. Analogous to the reduction for cut γ and axiom $\overline{\gamma}$, identities also reduce to atomic identities. For example,

$$\begin{array}{c} p \otimes q \\ \Big\downarrow \mathsf{id}_{p\otimes q} \\ p \otimes q \end{array} \quad = \quad \begin{array}{c} p \\ \mathsf{id}_p \Big\downarrow \\ p \end{array} \quad \otimes \quad \begin{array}{c} q \\ \Big\downarrow \mathsf{id}_q \\ q \end{array} \tag{9}$$

This is simply functoriality of tensor ($\mathsf{id}_{p\otimes q} = \mathsf{id}_p \otimes \mathsf{id}_q : p \otimes q \to p \otimes q$). Graphically, in terms of proof nets:

$$\begin{array}{c} p \otimes q \\ \Big| \\ p \otimes q \end{array} \quad = \quad \begin{array}{c} p \otimes q \\ \Big|\;\;\Big| \\ p \otimes q \end{array} \tag{10}$$

(The link on the left is the obvious generalisation of an atomic axiom link, between identical subformulas.)

Contraction and weakening. Intuitively, the six obvious candidates for contraction and weakening for a categorical proof theory of classical logic are:

$$\begin{array}{rclcll} c & : & A \parr A & \to & A & (\parr\text{-contraction}) \\ w & : & \bot & \to & A & (\bot\text{-weakening}) \\ w' & : & A & \to & A \parr B & (\parr\text{-weakening}) \\ \overline{c} & : & A & \to & A \otimes A & (\otimes\text{-contraction}) \\ \overline{w} & : & A & \to & 1 & (1\text{-weakening}) \\ \overline{w'} & : & A \otimes B & \to & A & (\otimes\text{-weakening}) \end{array}$$

The weakening maps w and w' are equivalent (via $\bot \parr X \cong X$); dually, $\overline{w}$ and $\overline{w'}$ are equivalent (via $X \cong X \otimes 1$). Historically, the standard categorical approach to contraction and weakening (*e.g.* [See89, Sel01, FP04]) has been to choose

$$\begin{array}{rclcll} c & : & A \parr A & \to & A & (\parr\text{-contraction}) \\ w & : & \bot & \to & A & (\bot\text{-weakening}) \\ \overline{c} & : & A & \to & A \otimes A & (\otimes\text{-contraction}) \\ \overline{w} & : & A & \to & 1 & (1\text{-weakening}) \end{array}$$

essentially because (c, w) provides a monoid structure on A with respect to $\parr : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $(\overline{c}, \overline{w})$ provides a comonoid structure on A with respect to $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.[4,5]

As an example, here is a categorical proof of Peirce's law $\big((x \multimap y) \multimap x\big) \multimap x$, where $A \multimap B$

[4] Monoids were central in the early development of category theory. For example, rings are monoids with respect to tensor $\otimes$ in the category of abelian groups. For an introduction to monoids see Chapter VII of [Mac71].

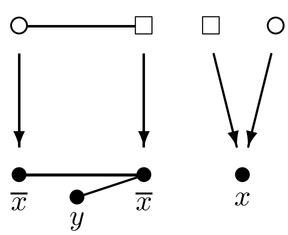
[5] In the case of categorical linear logic [See89], with the exponential functor (cotriple/comonad) $!$ and its dual $?$, one axiomatises comonoids only on objects $?A$ and monoids only on objects $!A$. Thus contraction and weakening are restricted to $!A \to !A \otimes !A$, $!A \to 1$, $?A \parr ?A \to ?A$ and $\bot \to ?A$.

abbreviates $\overline{A} \parr B$:

$$\begin{array}{c} 1 \\ \Big\downarrow{\scriptstyle \overline{\gamma}_{x \parr x}} \\ (\overline{x} \otimes \overline{x}) \parr (x \parr x) \\ \Big\downarrow{\scriptstyle (\overline{x}\otimes\overline{x}) \parr c_x} \\ (\overline{x} \otimes \overline{x}) \parr x \\ \Big\downarrow{\scriptstyle (\overline{\rho}_{\overline{xf}}\otimes\overline{x}) \parr x} \\ \big((\overline{x} \parr \bot) \otimes \overline{x}\big) \parr x \\ \Big\downarrow{\scriptstyle \big((\overline{x} \parr w_y)\otimes\overline{x}\big) \parr x} \\ \big((\overline{x} \parr y) \otimes \overline{x}\big) \parr x \end{array} \tag{11}$$

A notion of proof net for classical logic is discussed in [Gir91][6] though promptly dismissed by Girard. Proof nets involve almost as much 'syntactic bureaucracy' as sequent calculus, since they have explicit nodes corresponding to the non-logical rules of contraction and weakening. Correspondingly, one is forced to consider equivalences on proof nets [FP04] (corresponding to rule commutations involving contraction and weakening) that suggest proof nets fail to provide a satisfactory notion of abstract normal form for classical proofs.

The paper [Hug04] presents a notion of *combinatorial proof* in which superposition is represented mathematically (as graph homomorphisms), rather than syntactically using weakening and contraction nodes (as in a proof net). For example, the categorical proof of Peirce's law above translates into the following graph homomorphism $h : G \to G'$:



The target (lower) graph G' is a cograph presentation of the Peirce's law formula $\big((\overline{x} \parr y) \otimes \overline{x}\big) \parr x$. The source (upper) graph G is a coloured graph, each colour class (indicated by vertex type, $\circ$ or $\square$) being an 'axiom link'. The homomorphism h is given by the arrows.

Reduction to atomic weakening. As with axiom, cut and identity links, weakening reduces to the atomic case, by coherence. For example,

$$\begin{array}{c} \bot \\ {\scriptstyle p\otimes q}\Bigg\downarrow \\ p \otimes q \end{array} \quad = \quad \begin{array}{c} \bot \\ \big\downarrow{\scriptstyle \overline{c}_\bot} \\ \bot \otimes \bot \\ \big\downarrow{\scriptstyle w_p \otimes w_q} \\ p \otimes q \end{array} \qquad\qquad \begin{array}{c} \bot \\ {\scriptstyle w_{p \parr q}}\Bigg\downarrow \\ p \parr q \end{array} \quad = \quad \begin{array}{c} \bot \\ \big\downarrow{\scriptstyle (\overline{\rho})_\bot^{-1}} \\ \bot \parr \bot \\ \big\downarrow{\scriptstyle w_p \parr w_q} \\ p \parr q \end{array} \tag{12}$$

[6] See [Rob03] for a detailed development, in a two-sided presentation.

and dually,

$$
\begin{array}{c}
p \parr q \\
{\scriptstyle w_{p \parr q}}\Big\downarrow \\
\\
1
\end{array}
=
\begin{array}{c}
p \parr q \\
\Big\downarrow {\scriptstyle \overline{w}_p \parr \overline{w}_q} \\
1 \parr 1 \\
\Big\downarrow {\scriptstyle \overline{c}_1} \\
1
\end{array}
\qquad\qquad
\begin{array}{c}
p \parr q \\
{\scriptstyle \overline{w}_{p \parr q}}\Big\downarrow \\
\\
1
\end{array}
=
\begin{array}{c}
p \parr q \\
\Big\downarrow {\scriptstyle w_p \parr w_q} \\
1 \parr 1 \\
\Big\downarrow {\scriptstyle \rho_1} \\
\bot
\end{array}
\tag{13}
$$

3 Deep inference system SKSg

The correspondence between SKSg and linearly distributive categories with negation plus (co)monoids should be clear from the examples and discussion in the previous section. The correspondence is absolutely precise [7] (*i.e.*, the former is the latter, minus the coherence laws) except for the rules

$$\frac{(\mathsf{f},\mathsf{f})}{\mathsf{f}}= \qquad\qquad \frac{[\mathsf{f},\mathsf{f}]}{\mathsf{t}}=$$

Categorically, this corresponds to having isomorphisms

$$
\begin{array}{rcl}
\bot \otimes \bot & \to & \bot \\
1 \parr 1 & \to & 1
\end{array}
$$

The remainder of this section spells out in detail the sense in which SKSg is categorical classical logic, minus coherence. Define a ***raw MLL category*** as a category $\mathbb{C}$ equipped with:

- five functors,
  - a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$,
  - a functor $\parr : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$,
  - a constant object (nullary functor) $1 \in \mathbb{C}$,
  - a constant object (nullary functor) $\bot \in \mathbb{C}$,
  - a full and faithful[8] functor $(-)^\bot : \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$;
- eight natural isomorphisms[9],

$$
\begin{array}{rcl}
\alpha & : & A \otimes (B \otimes C) \to (A \otimes B) \otimes C \\
\sigma & : & A \otimes B \to B \otimes A \\
\rho & : & A \to A \otimes 1 \\
\lambda & : & A \to 1 \otimes A
\end{array}
\qquad
\begin{array}{rcl}
\overline{\alpha} & : & (A \parr B) \parr C \to A \parr (B \parr C) \\
\overline{\sigma} & : & A \parr B \to B \parr A \\
\overline{\rho} & : & A \parr \bot \to A \\
\overline{\lambda} & : & \bot \parr A \to A
\end{array}
$$

[7] Pedantically, for the correspondence to be *absolutely* precise, since SKSg has n-ary tensor and par operations for $n \geq 1$, denoted $(A_1, \ldots, A_n)$ and $[A_1, \ldots, A_n]$, one would define $A \otimes B \otimes C$ as shorthand for $A \otimes (B \otimes C)$ *etc.*, for the corresponding n-ary (derived) functors.

[8] A functor $F : \mathbb{C} \to \mathbb{D}$ is ***full and faithfull*** if for all objects C, C' in $\mathbb{C}$, the homsets $\mathbb{C}(C, C')$, the homsets $\mathbb{D}(F(C), F(C'))$ are isomorphic.

[9] Given functors $F, G : \mathbb{C} \to \mathbb{D}$, a ***natural transformation*** τ from F to G is a family of morphisms $\tau_C : F(C) \to G(C)$ indexed by objects $C \in \mathbb{C}$, such that for all morphisms $f : C \to C'$ in $\mathbb{C}$, holds $F(f); \tau_{C'} = \tau_C; G(f)$. The transformation is an isomorphism iff each τ_C is an isomorphism. A morphism $f : A \to B$ is an ***isomorphism*** if there exists $f' : B \to A$ with $f; f' = \mathsf{id}_A$ and $f'; f = \mathsf{id}_B$.

- and three natural maps,

$$\begin{aligned}\delta &: A \otimes (B \parr C) \to (A \otimes B) \parr C \\ \gamma &: A \otimes \overline{A} \to \bot \\ \overline{\gamma} &: 1 \to \overline{A} \otimes A\end{aligned}$$

such that for all objects $A, B \in \mathbb{C}$,

$$\begin{aligned}\overline{A \otimes B} &= \overline{A} \parr \overline{B} & \qquad \overline{\overline{A}} &= A \\ \overline{A \parr B} &= \overline{A} \otimes \overline{B} & \qquad \overline{1} &= \bot\end{aligned} \tag{14}$$

Thus a raw MLL category is a symmetric linearly distributive category with negation [CS91, BCST96, CS97], stripped of the coherence laws, and with strict negation, satisfying the equalities in (14), as usual in linear logic [Gir87, Blu91].

A Appendix: System S

The following is quoted from [DHPP99]. System S is based on the natural maps defining a linearly distributive category [CS91]. The linear fragment deep inference system SKSg [Brü04] is an extension of system S with units. System S was technically useful for proving full completeness of Chu spaces for unit-free MLL.

> *The language of MLL consists of finite formulas built up from literals (propositional variables P or $P^\perp$) using connectives* tensor $A \otimes B$ *and* par $A \parr B$*. We expand the abbreviations $(A \otimes B)^\perp$ to $A^\perp \parr B^\perp$, $(A \parr B)^\perp$ to $A^\perp \otimes B^\perp$, $A \multimap B$ to $A^\perp \parr B$, $A^{\perp\perp}$ to A, and $A \otimes B \otimes C$ to $(A \otimes B) \otimes C$.*
>
> *We axiomatize MLL with one axiom schema together with rules for associativity, commutativity, and linear or weak distributivity as follows.*
>
> ***System S:***
>
> $$\begin{array}{lrcl} \mathsf{T}_n & \multicolumn{3}{l}{(L_1^\perp \parr L_1) \otimes \ldots \otimes (L_n^\perp \parr L_n), \;\; n \geq 1} \\ \mathsf{A} & (A \otimes B) \otimes C & \vdash & A \otimes (B \otimes C) \\ \overline{\mathsf{A}} & (A \parr B) \parr C & \vdash & A \parr (B \parr C) \\ \mathsf{C} & A \otimes B & \vdash & B \otimes A \\ \overline{\mathsf{C}} & A \parr B & \vdash & B \parr A \\ \mathsf{D} & (A \parr B) \otimes C & \vdash & A \parr (B \otimes C) \\ \mathsf{E} & A \otimes B & \vdash & A' \otimes B' \\ \overline{\mathsf{E}} & A \parr B & \vdash & A' \parr B' \end{array}$$
>
> *Rules E and $\overline{\mathsf{E}}$ assume $A \vdash A'$ and $B \vdash B'$, i.e. the other rules may be applied not only to formulas but to their subformulas.*

In this passage, the L_i denote arbitrary literals.[10]

[10][DHPP99] refers to system S as "Hilbert-style", which was perhaps an overly liberal use of the adjective.

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