Classical Logic = Fibred MLL

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Classical Logic = Fibred MLL

Boolean tautologies are characterizable

- bureaucratically: as theorems derivable in some proof system
- semantically: as valid (universally true) propositions.

MLL (multiplicative linear logic) theorems are characterizable

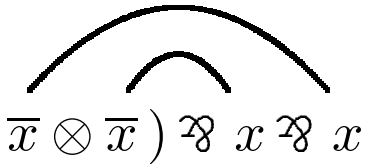
- bureaucratically: as theorems derivable in some proof system
- combinatorially: via proof nets.

MAIN RESULT: Combinatorial characterization of Boolean tautologies.

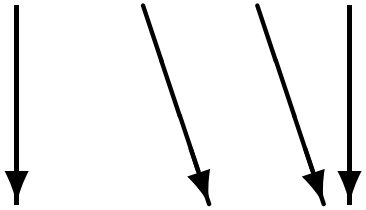
Theorem A is a Boolean tautology iff there exists an MLL theorem B and a wea-con (weakening and contracting) function from the leaves of B to the leaves of A.

$$\text{Boolean proof net on } A \;=\; \underbrace{\text{MLL net on } B \;+\; \text{wea-con function } \begin{array}{c} B \\ \downarrow \\ A \end{array}}_{\text{“fibred MLL net”}}$$

MLL fibration of Peirce's law B :



“fibration” f :



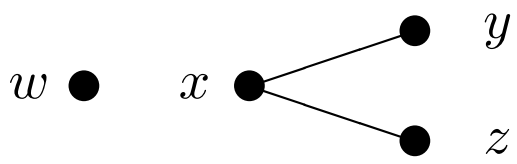
Peirce's law A : $((\overline{x} \vee y) \wedge \overline{x}) \vee x$ (base)

Graph $G(A)$ of a formula A:

Vertices: leaves of A

Edges: pairs of leaves whose least common ancestor is $\wedge$.

$$w \vee (x \wedge (y \vee z)) \qquad \mapsto$$



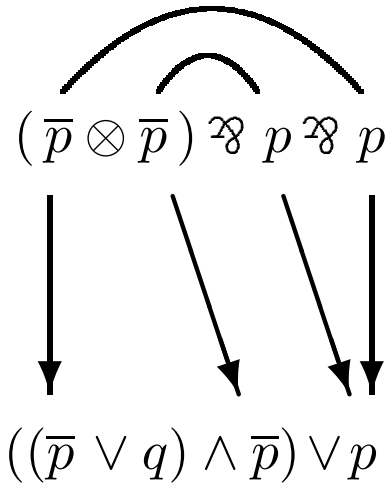
3 maximal cliques (max-cliques): $\{w\}, \{x, y\}, \{x, z\}$
(the co-clauses of the DNF expansion of A).

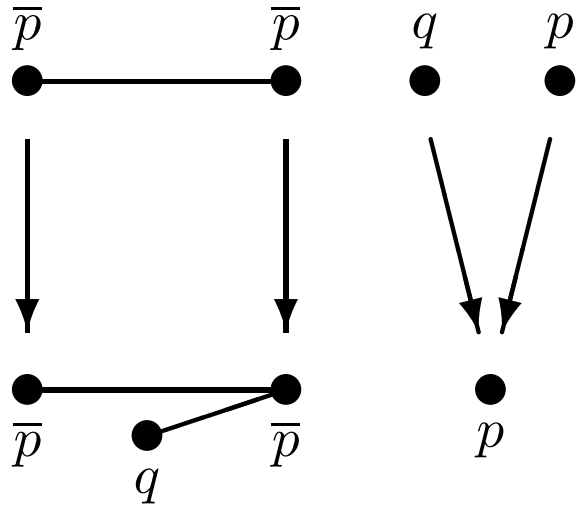
f : leaves$(A) \to$ leaves(B) is ***wea-con*** when it:

(1) maps max-cliques of $G(A)$ to max-cliques of $G(B)$

(2) preserves labels (literals)

Peirce's Law $\quad ((p \Rightarrow q) \Rightarrow p) \Rightarrow p \quad = \quad ((\overline{p} \vee q) \wedge \overline{p}) \vee p$





Semantics of classical proofs, e.g. MLL + contr + weak

Gentzen-style (sequents)

$$\frac{}{p, \overline{p}}\ \text{ax}$$

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \wedge B}\ \wedge \qquad \frac{\Gamma}{\Gamma, A}\ \text{weak}$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \qquad \frac{\Gamma, A, A}{\Gamma, A}\ \text{contr}$$

Hilbert-style (terms)

$$(p_1 \vee \overline{p_1}) \wedge \ldots \wedge (p_n \vee \overline{p_n}) \quad \overset{\text{axiom}}{}$$

$$A \wedge (B \vee C) \overset{\text{lin. dist.}}{\longrightarrow} (A \wedge B) \vee C$$

$$A \vee A \overset{\text{contr}}{\longrightarrow} A$$

$$A \overset{\text{weak}}{\longrightarrow} A \vee B$$

Significance for CL, Classical Logic

Girard's decomposition: CL = $\underbrace{\text{MLL} + \text{Additives}}_{\text{MALL}}$ + Exponentials

Combinatorics: Very hairy.

No faithful MALL proof nets until Hughes, van Glabbeek 2003.

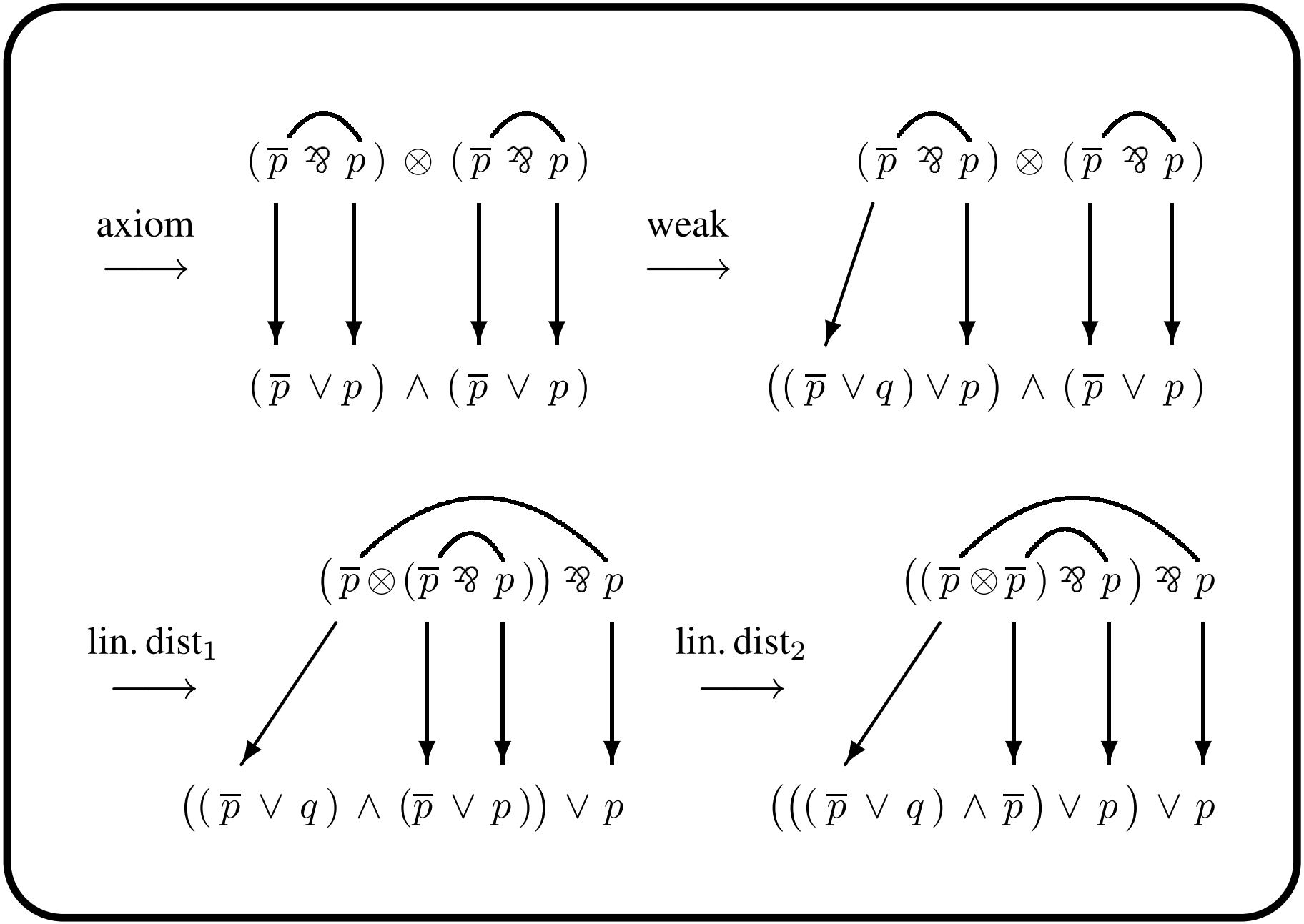
And even those proof nets remained hairy.

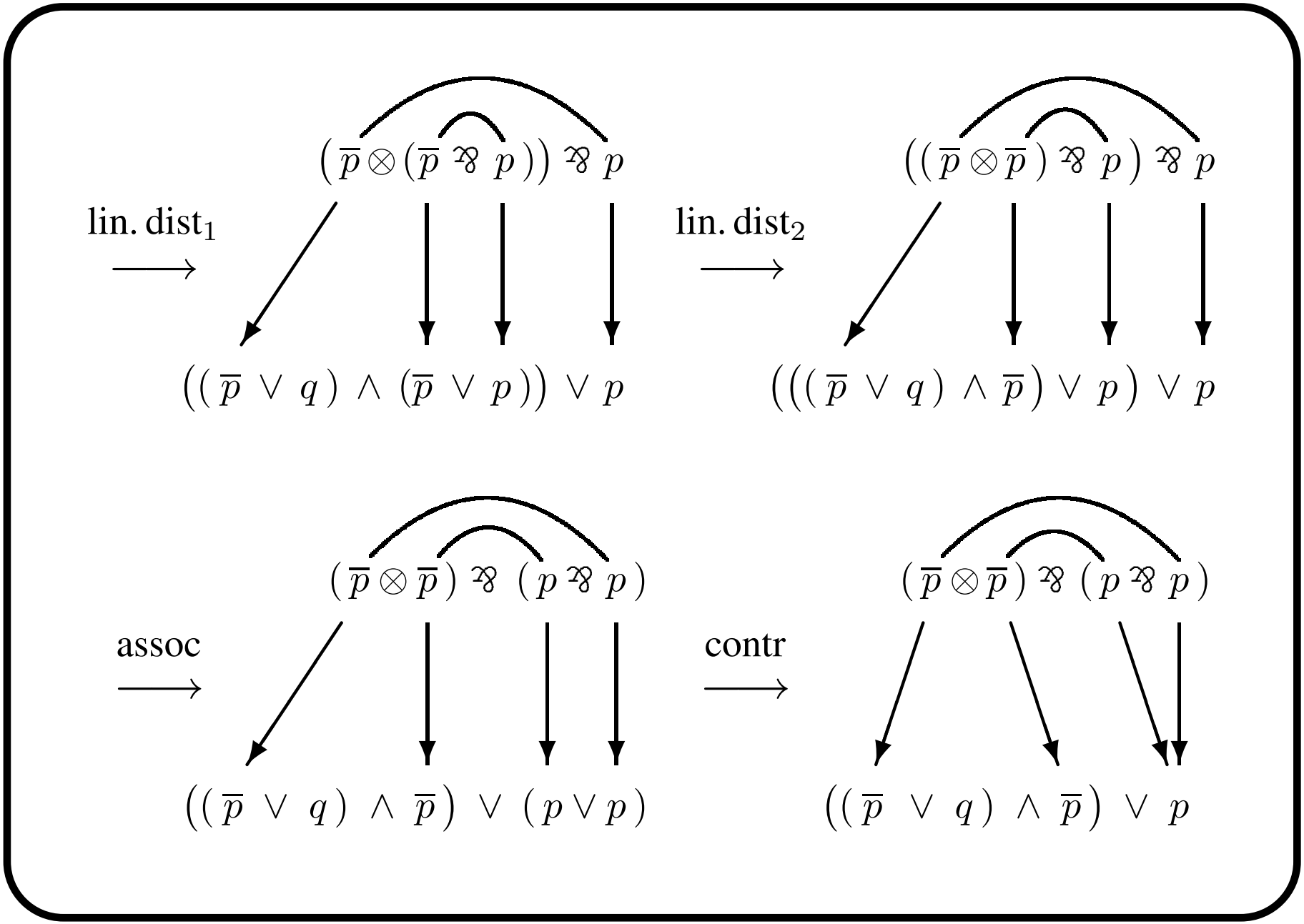
Proposed decomposition: CL = MLL + Fibration

Combinatorics: Very simple and natural.

Evidence Peirce envisaged CL = MLL + C + W in 1882.

Contribution this paper: C + W = Fibration (clique-preserving function).


$(\overline{p} \parr p) \otimes (\overline{p} \parr p)$
axiom
$(\overline{p} \vee p) \wedge (\overline{p} \vee p)$
weak
$((\overline{p} \vee q) \vee p) \wedge (\overline{p} \vee p)$
$(\overline{p} \otimes (\overline{p} \parr p)) \parr p$
lin. dist$_1$
$((\overline{p} \vee q) \wedge (\overline{p} \vee p)) \vee p$
$((\overline{p} \otimes \overline{p}) \parr p) \parr p$
lin. dist$_2$
$(((\overline{p} \vee q) \wedge \overline{p}) \vee p) \vee p$


$(\overline{p} \otimes (\overline{p} \parr p)) \parr p$
lin. dist$_1$
$((\overline{p} \lor q) \land (\overline{p} \lor p)) \lor p$
$((\overline{p} \otimes \overline{p}) \parr p) \parr p$
lin. dist$_2$
$(((\overline{p} \lor q) \land \overline{p}) \lor p) \lor p$
$(\overline{p} \otimes \overline{p}) \parr (p \parr p)$
assoc
$((\overline{p} \lor q) \land \overline{p}) \lor (p \lor p)$
$(\overline{p} \otimes \overline{p}) \parr (p \parr p)$
contr
$((\overline{p} \lor q) \land \overline{p}) \lor p$