Classical Logic = Fibred MLL

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Classical Logic = Fibred MLL

Boolean tautologies are characterizable

- bureaucratically: as theorems derivable in some proof system
- semantically: as valid (universally true) propositions.

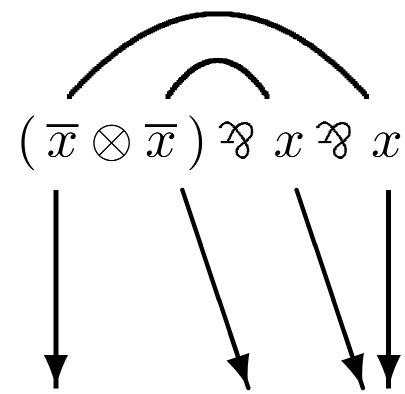
MLL (multiplicative linear logic) theorems are characterizable

- bureaucratically: as theorems derivable in some proof system
- combinatorially: via proof nets.

MAIN RESULT: Combinatorial characterization of Boolean tautologies.

Theorem A is a Boolean tautology iff there exists an MLL theorem B and a wea-con (weakening and contracting) function from the leaves of B to the leaves of A.

$$\text{Boolean proof net on } A \;=\; \underbrace{\text{MLL net on } B \;+\; \text{wea-con function} \begin{array}{c} B \\ \downarrow \\ A \end{array}}_{\text{“fibred MLL net”}}$$



MLL fibration of Peirce’s law $B:$ $(\overline{x} \otimes \overline{x}) \parr x \parr x$

“fibration” $f:$

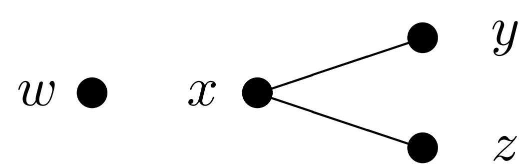
Peirce’s law $A:$ $((\overline{x} \vee y) \wedge \overline{x}) \vee x$ (base)

Graph $G(A)$ of a formula A:

Vertices: leaves of A

Edges: pairs of leaves whose least common ancestor is $\wedge$.

$$w \vee (x \wedge (y \vee z)) \qquad \mapsto$$



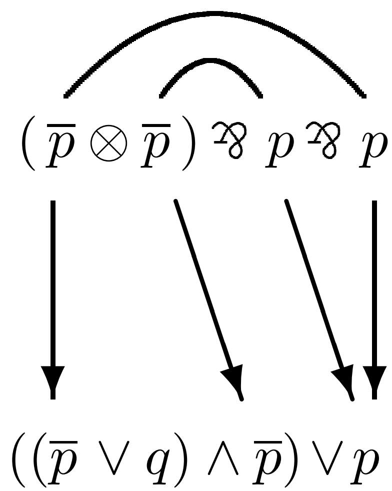
3 maximal cliques (max-cliques): $\{w\}$, $\{x, y\}$, $\{x, z\}$
(the co-clauses of the DNF expansion of A).

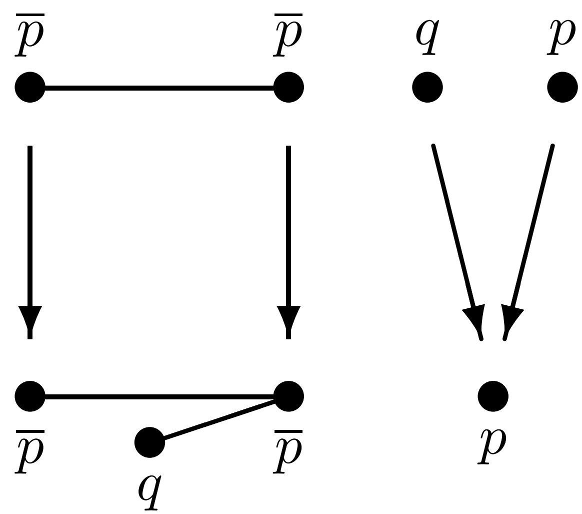
f : leaves$(A) \rightarrow$ leaves(B) is ***wea-con*** when it:

(1) maps max-cliques of $G(A)$ to max-cliques of $G(B)$

(2) preserves labels (literals)

Peirce's Law $((p \Rightarrow q) \Rightarrow p) \Rightarrow p \quad = \quad ((\overline{p} \vee q) \wedge \overline{p}) \vee p$





Semantics of classical proofs, e.g. MLL + contr + weak

Gentzen-style (sequents)

$$\frac{}{p, \overline{p}}\ \text{ax}$$

$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \wedge B}\ \wedge \qquad \frac{\Gamma}{\Gamma, A}\ \text{weak}$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \qquad \frac{\Gamma, A, A}{\Gamma, A}\ \text{contr}$$

Hilbert-style (terms)

$$\overset{\text{axiom}}{(p_1 \vee \overline{p_1}) \wedge \ldots \wedge (p_n \vee \overline{p_n})}$$

$$A \wedge (B \vee C) \xrightarrow{\text{lin. dist.}} (A \wedge B) \vee C$$

$$A \vee A \xrightarrow{\text{contr}} A$$

$$A \xrightarrow{\text{weak}} A \vee B$$

Significance for CL, Classical Logic

Girard's decomposition: $\text{CL} = \underbrace{\text{MLL} + \text{Additives}}_{\text{MALL}} + \text{Exponentials}$

Combinatorics: Very hairy.

No faithful MALL proof nets until Hughes, van Glabbeek 2003.

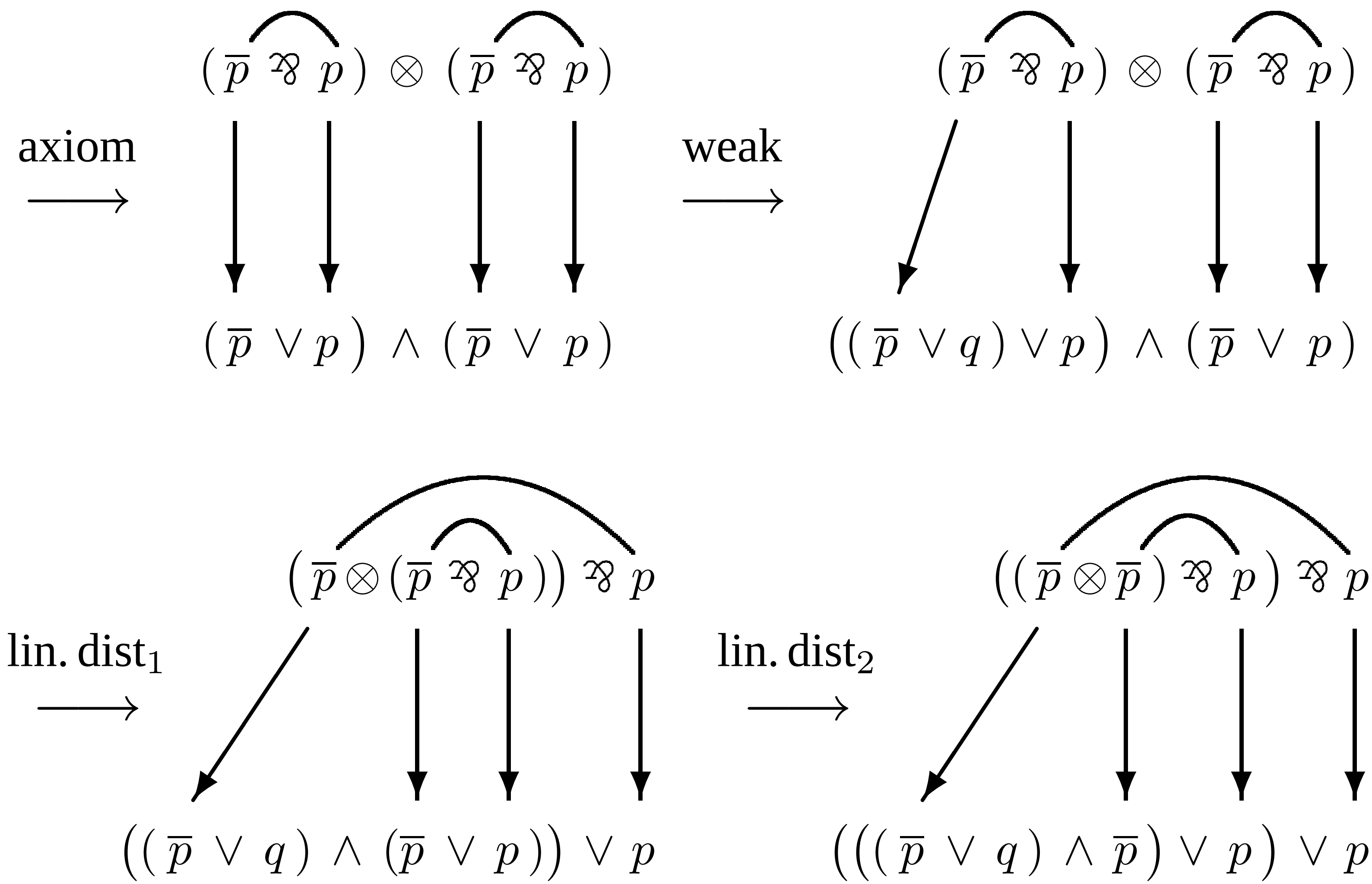
And even those proof nets remained hairy.

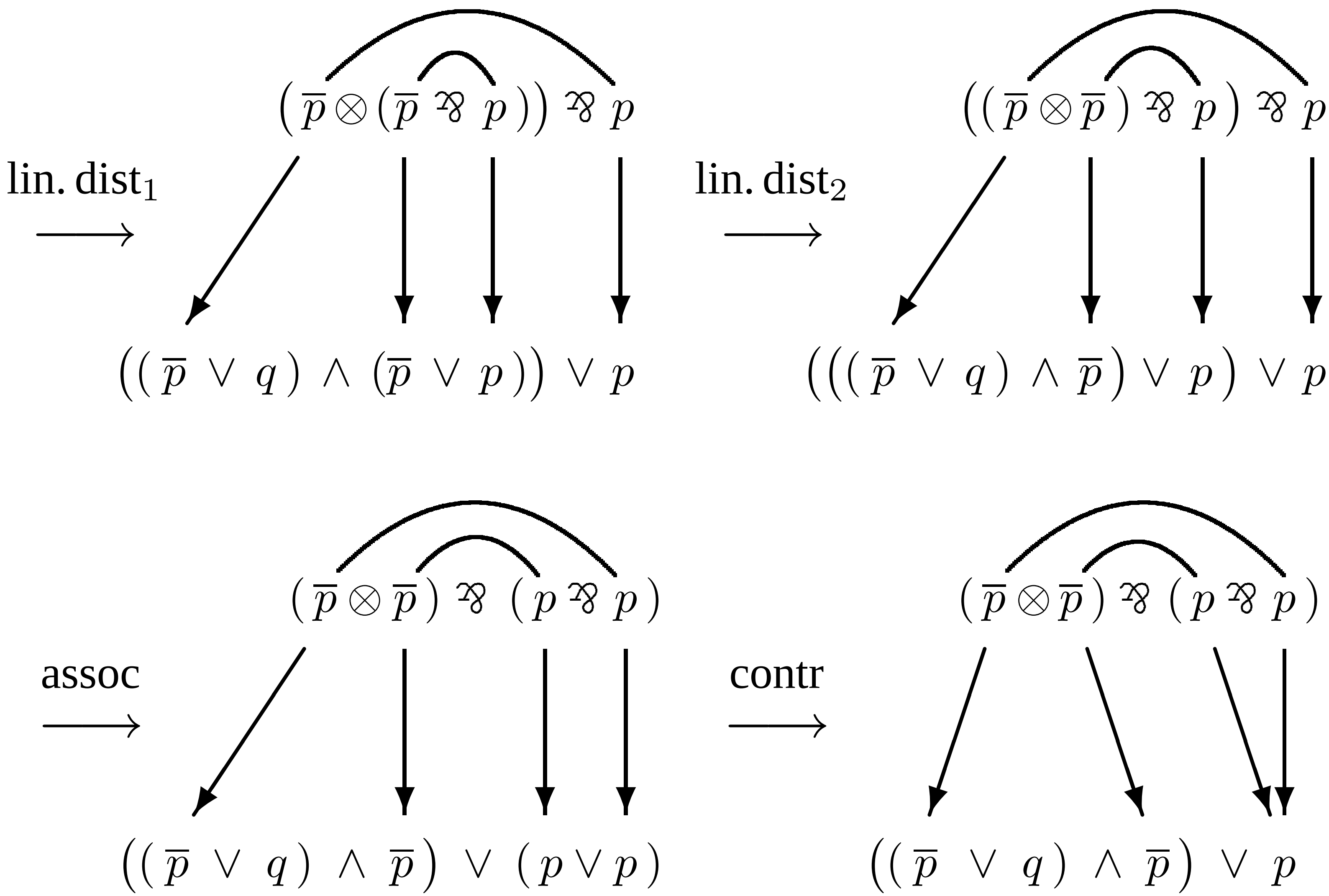
Proposed decomposition: $\text{CL} = \text{MLL} + \text{Fibration}$

Combinatorics: Very simple and natural.

Evidence Peirce envisaged $\text{CL} = \text{MLL} + \text{C} + \text{W}$ in 1882.

Contribution this paper: $\text{C} + \text{W} = \text{Fibration}$ (clique-preserving function).


axiom
$(\overline{p} \parr p) \otimes (\overline{p} \parr p)$
$(\overline{p} \vee p) \wedge (\overline{p} \vee p)$
weak
$(\overline{p} \parr p) \otimes (\overline{p} \parr p)$
$((\overline{p} \vee q) \vee p) \wedge (\overline{p} \vee p)$
lin. dist$_1$
$(\overline{p} \otimes (\overline{p} \parr p)) \parr p$
$((\overline{p} \vee q) \wedge (\overline{p} \vee p)) \vee p$
lin. dist$_2$
$((\overline{p} \otimes \overline{p}) \parr p) \parr p$
$(((\overline{p} \vee q) \wedge \overline{p}) \vee p) \vee p$



$(\overline{p} \otimes (\overline{p} \parr p)) \parr p$
lin. dist_1
$((\overline{p} \vee q) \wedge (\overline{p} \vee p)) \vee p$
$((\overline{p} \otimes \overline{p}) \parr p) \parr p$
lin. dist_2
$(((\overline{p} \vee q) \wedge \overline{p}) \vee p) \vee p$
$(\overline{p} \otimes \overline{p}) \parr (p \parr p)$
assoc
$((\overline{p} \vee q) \wedge \overline{p}) \vee (p \vee p)$
$(\overline{p} \otimes \overline{p}) \parr (p \parr p)$
contr
$((\overline{p} \vee q) \wedge \overline{p}) \vee p$