MALL proof nets identify proofs modulo rule commutation

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We show that the proof nets introduced in [4] for MALL (Multiplicative Additive Linear Logic) identify proofs modulo rule commutation: two proofs translate to the same proof net *if and only if* one can be obtained from the other by a succession of rule commutations.

1 Introduction

The proof nets for MALL (Multiplicative Additive Linear Logic [2]) introduced in [4] solved numerous issues with monomial proof nets [3], for example:

- There is a simple (deterministic) translation function from cut-free proofs to proof nets.
- Cut elimination is simply defined and strongly normalising.
- Proof nets form a semi (*i.e.*, unit-free) star-autonomous category with (co)products.

A proof net is a set of *linkings* on a sequent. Each linking is a set of *links* between complementary formula leaves (literal occurrences). Figure 1 illustrates the translation of a proof into a proof net.

In this paper we prove that the translation precisely captures proofs modulo rule commutation: two proofs translate to the same proof net *if and only if* one can be obtained from the other by a succession of rule commutations. A *rule commutation* is a local conversion on a proof that retains the subproofs of its hypotheses, with possible duplication/identification, for example

$$\dfrac{\overline{P^\perp, P} \qquad \dfrac{\overline{Q, Q^\perp}}{Q,\ R\oplus Q^\perp}\oplus_2}{P^\perp,\ P\otimes Q,\ R\oplus Q^\perp}\otimes \qquad\longrightarrow\qquad \dfrac{\dfrac{\overline{P^\perp,P}\qquad \overline{Q,Q^\perp}}{P^\perp,\ P\otimes Q\ ,Q^\perp}\otimes}{P^\perp,\ P\otimes Q,\ R\oplus Q^\perp}\oplus_2$$

in which the lower $\otimes$-rule commutes over the $\oplus$-rule, or

$$\dfrac{\dfrac{\overline{P^\perp,P}}{P^\perp,P\oplus R}\oplus_1 \qquad \dfrac{\overline{Q^\perp,Q}\quad \overline{Q^\perp,Q}}{Q^\perp,\ Q\& Q}\&}{P^\perp,\ (P\oplus R)\otimes Q^\perp,\ Q\& Q}\otimes \qquad\longrightarrow\qquad \dfrac{\dfrac{\dfrac{\overline{P^\perp,P}}{P^\perp,P\oplus R}\oplus_1 \quad \overline{Q^\perp,Q}}{P^\perp,(P\oplus R)\otimes Q^\perp,Q}\otimes \qquad \dfrac{\dfrac{\overline{P^\perp,P}}{P^\perp,P\oplus R}\oplus_1 \quad \overline{Q^\perp,Q}}{P^\perp,(P\oplus R)\otimes Q^\perp,Q}\otimes}{P^\perp,\ P\otimes Q^\perp,\ Q\& Q}\&$$

illustrating duplication (of the $\otimes$-rule and subproof $\dfrac{\overline{P^\perp,P}}{P^\perp,P\oplus R}\oplus_1$) as the $\otimes$-rule commutes over the $\&$-rule.

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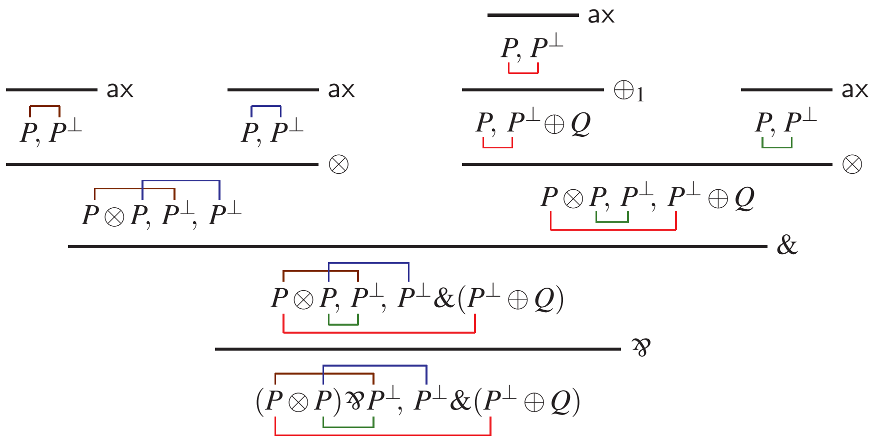

Figure 1: Example of the inductive translation of a MALL proof into a proof net. The concluding proof net has two linkings, one drawn above the sequent, the other below. Each has two links. The proof nets further up in the derivation have one or two linkings, correspondingly above/below the sequent.

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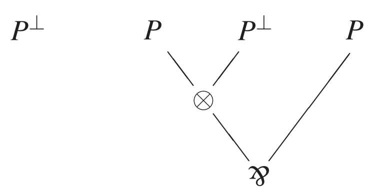
2 MALL

Let MALL denote cut-free multiplicative-additive linear logic without units [2].[1] Formulas are built from literals (propositional variables $P, Q, \ldots$ and their negations $P^\perp$, $Q^\perp, \ldots$) by the binary connectives ***tensor*** $\otimes$, ***par*** $\parr$, ***with*** $\&$ and ***plus*** $\oplus$. Negation $(-)^\perp$ extends to arbitrary formulas with $P^{\perp\perp} = P$ on propositional variables and de Morgan duality: $(A \otimes B)^\perp = A^\perp \parr B^\perp$, $(A \parr B)^\perp = A^\perp \otimes B^\perp$, $(A \oplus B)^\perp = A^\perp \& B^\perp$, and $(A \& B)^\perp = A^\perp \oplus B^\perp$. We identify a formula with its parse tree, labelled with literals on leaves and connectives on internal vertices. A ***sequent*** is a disjoint union of formulas. Thus a sequent is a labelled forest. We write comma for disjoint union. For example,

$$P^\perp,\ (P \otimes P^\perp) \parr P$$

is the labelled forest



Sequents are proved using the following rules:

$$\frac{}{P, P^\perp}\,\mathsf{ax} \qquad \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta}\,\otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr$$

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B}\,\& \qquad \frac{\Gamma, A}{\Gamma, A \oplus B}\,\oplus_1 \qquad \frac{\Gamma, B}{\Gamma, A \oplus B}\,\oplus_2 \qquad \frac{\Gamma \quad \Delta}{\Gamma, \Delta}\,\mathsf{mix} \quad \text{(optional)}$$

[1] We treat cut in Section 6.

$$\dfrac{}{\{\overparen{P,P^{\perp}}\} \;\rhd\; P,P^{\perp}}\,\mathsf{ax} \qquad \dfrac{\theta \;\rhd\; \Gamma, A \quad \theta' \;\rhd\; \Gamma, B}{\theta \cup \theta' \;\rhd\; \Gamma, A \& B}\,\& \qquad \dfrac{\theta \;\rhd\; \Gamma, A \qquad \theta' \;\rhd\; B, \Delta}{\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\} \;\rhd\; \Gamma, A \otimes B, \Delta}\,\otimes$$

$$\dfrac{\theta \;\rhd\; \Gamma, A, B}{\theta \;\rhd\; \Gamma, A ⅋ B}\,⅋ \qquad \dfrac{\theta \;\rhd\; \Gamma, A}{\theta \;\rhd\; \Gamma, A \oplus B}\,\oplus_1 \qquad \dfrac{\theta \;\rhd\; \Gamma, B}{\theta \;\rhd\; \Gamma, A \oplus B}\,\oplus_2 \qquad \dfrac{\theta \;\rhd\; \Gamma \qquad \theta' \;\rhd\; \Delta}{\{\lambda \cup \lambda' : \lambda \in \theta, \lambda' \in \theta'\} \;\rhd\; \Gamma, \Delta}\,\mathsf{mix}$$

Table 1: Alternative but equivalent definition of the function from MALL proofs to linking sets. Here $\theta \;\rhd\; \Gamma$ signifies that θ is a set of linkings on Γ. We use the implicit tracking of formula leaves downwards through rules. The base case is a singleton linking set whose only linking comprises a single link, between P and $P^{\perp}$.

The mix-rule is optional and absent by default. Our treatment is valid for MALL with and without mix.

Throughout this document P, Q, R range over propositional variables, $A, B, \ldots$ over formulas, and Γ, Δ, Σ over sequents. Each of the proof rules above yields an implicit tracking of subformula occurrences, mapping the vertices in the hypotheses to the ones in the conclusion. A formula occurrence in the conclusion of a rule ρ is ***generated*** by ρ if it is not in the image of this map.

3 Function from proofs to proof nets

A ***link*** on Γ is a pair (two-element set) of leaves in Γ. A ***linking*** λ on Γ is a set of links on Γ.[2] Every MALL proof Π of Γ defines a set θ_Π of linkings on Γ as follows. Define a ***&-resolution*** R of Π to be any result of deleting one branch above each &-rule of Π. By downwards tracking of formula leaves, the axiom rules of R determine a linking λ_R on Γ. Define $\theta_\Pi = \{\lambda_R : R \text{ is a \&-resolution of } \Pi\}$.

Table 1 defines the same function by induction. See Figure 1 for an example. The fact that this yields the same linking set as the resolution-based function follows from a simple structural induction on proofs. Note that $\otimes$ (resp. &) is *multiplicative* (resp. *additive*): multiply (resp. add) the number of linkings in θ and θ' to obtain the number of linkings on the conclusion.[3]

A linking set is a ***proof net*** if it is the translation of a proof.Mention topological defn; used below TODO

4 Rule commutations

Tables 2, 3 and 4 exhaustively list the rule commutations of MALL. Each commutation may be applied in context, *i.e.*, to any subproof. This collection of rule commutations is not ad hoc: they are generated systematically from a general broad definition of commutation. The general definition is presented in Appendix A. (The definition is more liberal than the one analyzed by Kleene [5] and Curry [1] in the context of sequent calculus [6, Def. 5.2.1].)

We say that a β-rule ***commutes over*** an α-rule, if there is a valid rule commutation where a proof fragment in which the β-rule occurs right below one or more α-rules is replaced by a proof fragment in which this order is reversed. Using either the definition of rule commutation from Appendix A or the enumeration of Tables 2, 3 and 4, it is not hard to check that this happens if and only if one of the

[2] The paper [4] imposed additional conditions in the definition of a linking. We do not need these conditions here.

[3] This observation relies on θ and θ' having no common linking, which follows (by structural induction) from the fact that in any proof net on Γ, every linking touches every formula in Γ (*i.e.*, for every linking λ in the proof net, and every formula(-occurrence) A in Γ, some link of λ contains a leaf of A).

$$
\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_1, A_2, B_1, B_2}}{\Gamma, A_1 \parr A_2, B_1, B_2}\ \parr}{\Gamma, A_1 \parr A_2, B_1 \parr B_2}\ \parr
\quad \overset{\mathsf{C}^{\parr}_{\parr}}{\longleftrightarrow} \quad
\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_1, A_2, B_1, B_2}}{\Gamma, A_1, A_2, B_1 \parr B_2}\ \parr}{\Gamma, A_1 \parr A_2, B_1 \parr B_2}\ \parr
$$

$$
\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_i, B_j}}{\Gamma, A_1 \oplus A_2, B_j}\ \oplus_i}{\Gamma, A_1 \oplus A_2, B_1 \oplus B_2}\ \oplus_j
\quad \overset{\mathsf{C}^{\oplus}_{\oplus}}{\longleftrightarrow} \quad
\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_i, B_j}}{\Gamma, A_i, B_1 \oplus B_2}\ \oplus_j}{\Gamma, A_1 \oplus A_2, B_1 \oplus B_2}\ \oplus_i
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_1} \quad \dfrac{\Pi_3}{B_2, \Sigma}}{A_2, \Delta, B_1 \otimes B_2, \Sigma}\ \otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \otimes B_2, \Sigma}\ \otimes
\quad \overset{\mathsf{C}^{\otimes}_{\otimes}}{\longleftrightarrow} \quad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_1}}{\Gamma, A_1 \otimes A_2, \Delta, B_1}\ \otimes \quad \dfrac{\Pi_3}{B_2, \Sigma}}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \otimes B_2, \Sigma}\ \otimes
$$

$$
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1, B_1} \quad \dfrac{\Pi_2}{\Gamma, A_2, B_1}}{\Gamma, A_1 \& A_2, B_1}\ \& \quad \dfrac{\dfrac{\Pi_3}{\Gamma, A_1, B_2} \quad \dfrac{\Pi_4}{\Gamma, A_2, B_2}}{\Gamma, A_1 \& A_2, B_2}\ \&}{\Gamma, A_1 \& A_2, B_1 \& B_2}\ \&
\quad \overset{\mathsf{C}^{\&}_{\&}}{\longleftrightarrow} \quad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1, B_1} \quad \dfrac{\Pi_3}{\Gamma, A_1, B_2}}{\Gamma, A_1, B_1 \& B_2}\ \& \quad \dfrac{\dfrac{\Pi_2}{\Gamma, A_2, B_1} \quad \dfrac{\Pi_4}{\Gamma, A_2, B_2}}{\Gamma, A_2, B_1 \& B_2}\ \&}{\Gamma, A_1 \& A_2, B_1 \& B_2}\ \&
$$

Table 2: Homogeneous rule commutations. In the last conversion, note the reversal of Π_2 and Π_3.

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_i, B_1, B_2}}{\Gamma, A_1 \oplus A_2, B_1, B_2}\,\oplus_i}{\Gamma, A_1 \oplus A_2, B_1 \parr B_2}\,\parr \qquad \begin{array}{c}\mathsf{C}^{\oplus}_{\parr}\\ \longrightarrow \\ \longleftarrow \\ \mathsf{C}^{\parr}_{\oplus}\end{array} \qquad \dfrac{\dfrac{\dfrac{\Pi}{\Gamma, A_i, B_1, B_2}}{\Gamma, A_i, B_1 \parr B_2}\,\parr}{\Gamma, A_1 \oplus A_2, B_1 \oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_i, B_1}}{\Gamma, A_1 \oplus A_2, B_1}\,\oplus_i \qquad \dfrac{\dfrac{\Pi_2}{\Gamma, A_i, B_2}}{\Gamma, A_1 \oplus A_2, B_2}\,\oplus_i}{\Gamma, A_1 \oplus A_2, B_1 \& B_2}\,\& \qquad \begin{array}{c}\mathsf{C}^{\oplus}_{\&}\\ \longrightarrow \\ \longleftarrow \\ \mathsf{C}^{\&}_{\oplus}\end{array} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_i, B_1} \quad \dfrac{\Pi_2}{\Gamma, A_i, B_2}}{\Gamma, A_i, B_1 \& B_2}\,\&}{\Gamma, A_1 \oplus A_2, B_1 \& B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1, A_2, B_1}}{\Gamma, A_1 \parr A_2, B_1}\,\parr \qquad \dfrac{\dfrac{\Pi_2}{\Gamma, A_1, A_2, B_2}}{\Gamma, A_1 \parr A_2, B_2}\,\parr}{\Gamma, A_1 \parr A_2, B_1 \& B_2}\,\& \qquad \begin{array}{c}\mathsf{C}^{\parr}_{\&}\\ \longrightarrow \\ \longleftarrow \\ \mathsf{C}^{\&}_{\parr}\end{array} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1, A_2, B_1} \quad \dfrac{\Pi_2}{\Gamma, A_1, A_2, B_2}}{\Gamma, A_1, A_2, B_1 \& B_2}\,\&}{\Gamma, A_1 \parr A_2, B_1 \& B_2}\,\parr$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \qquad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_i}}{A_2, \Delta, B_1 \oplus B_2}\,\oplus_i}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \oplus B_2}\,\otimes \qquad \begin{array}{c}\mathsf{C}^{\oplus}_{\otimes}\\ \longrightarrow \\ \longleftarrow \\ \mathsf{C}^{\otimes}_{\oplus}\end{array} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_i}}{\Gamma, A_1 \otimes A_2, \Delta, B_i}\,\otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \oplus B_2}\,\oplus_i$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \qquad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_1, B_2}}{A_2, \Delta, B_1 \parr B_2}\,\parr}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \parr B_2}\,\otimes \qquad \begin{array}{c}\mathsf{C}^{\parr}_{\otimes}\\ \longrightarrow \\ \longleftarrow \\ \mathsf{C}^{\otimes}_{\parr}\end{array} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_1, B_2}}{\Gamma, A_1 \otimes A_2, \Delta, B_1, B_2}\,\otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \parr B_2}\,\parr$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \qquad \dfrac{\dfrac{\Pi_2}{A_2, \Delta, B_1} \quad \dfrac{\Pi_3}{A_2, \Delta, B_2}}{A_2, \Delta, B_1 \& B_2}\,\&}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \& B_2}\,\otimes \qquad \begin{array}{c}\mathsf{C}^{\&}_{\otimes}\\ \longrightarrow \\ \longleftarrow \\ \mathsf{C}^{\otimes}_{\&}\end{array} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_2}{A_2, \Delta, B_1}}{\Gamma, A_1 \otimes A_2, \Delta, B_1}\,\otimes \qquad \dfrac{\dfrac{\Pi_1}{\Gamma, A_1} \quad \dfrac{\Pi_3}{A_2, \Delta, B_2}}{\Gamma, A_1 \otimes A_2, \Delta, B_2}\,\otimes}{\Gamma, A_1 \otimes A_2, \Delta, B_1 \& B_2}\,\&$$

Table 3: Heterogeneous rule commutations. The last three commutations have symmetric variants, obtained by switching $A_2 \otimes A_1$ for $A_1 \otimes A_2$ and exchanging hypotheses of rules from left to right, correspondingly. (The hypotheses are not ordered; however, we apply the convention that a hypothesis that contributes to one side of a $\otimes$ or $\&$ connective is drawn on that side.) Note that there are two copies of the subproof Π_1 on the right side of the final conversion.

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta} \quad \dfrac{\Pi_3}{\Sigma}}{\Delta,\Sigma}\,\mathsf{mix}}{\Gamma,\Delta,\Sigma}\,\mathsf{mix}
\qquad \overset{\mathsf{C}^{\mathsf{mix}}_{\mathsf{mix}}}{\longleftrightarrow} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta}}{\Gamma,\Delta}\,\mathsf{mix} \qquad \dfrac{\Pi_3}{\Sigma}}{\Gamma,\Delta,\Sigma}\,\mathsf{mix}
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_1} \quad \dfrac{\Pi_3}{B_2,\Sigma}}{\Delta,B_1\otimes B_2,\Sigma}\,\otimes}{\Gamma,\Delta,B_1\otimes B_2,\Sigma}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\otimes}}{\overset{\mathsf{C}^{\otimes}_{\mathsf{mix}}}{\overset{\longrightarrow}{\longleftarrow}}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_1}}{\Gamma,\Delta,B_1}\,\mathsf{mix} \qquad \dfrac{\Pi_3}{B_2,\Sigma}}{\Gamma,\Delta,B_1\otimes B_2,\Sigma}\,\otimes
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_i}}{\Delta,B_1\oplus B_2}\,\oplus_i}{\Gamma,\Delta,B_1\oplus B_2}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\oplus}}{\overset{\mathsf{C}^{\oplus}_{\mathsf{mix}}}{\overset{\longrightarrow}{\longleftarrow}}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_i}}{\Gamma,\Delta,B_i}\,\mathsf{mix}}{\Gamma,\Delta,B_1\oplus B_2}\,\oplus_i
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_1,B_2}}{\Delta,B_1\parr B_2}\,\parr}{\Gamma,\Delta,B_1\parr B_2}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\parr}}{\overset{\mathsf{C}^{\parr}_{\mathsf{mix}}}{\overset{\longrightarrow}{\longleftarrow}}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_1,B_2}}{\Gamma,\Delta,B_1,B_2}\,\mathsf{mix}}{\Gamma,\Delta,B_1\parr B_2}\,\parr
$$

$$
\dfrac{\dfrac{\Pi_1}{\Gamma} \qquad \dfrac{\dfrac{\Pi_2}{\Delta,B_1} \quad \dfrac{\Pi_3}{\Delta,B_2}}{\Delta,B_1\&B_2}\,\&}{\Gamma,\Delta,B_1\&B_2}\,\mathsf{mix}
\qquad \underset{\mathsf{C}^{\mathsf{mix}}_{\&}}{\overset{\mathsf{C}^{\&}_{\mathsf{mix}}}{\overset{\longrightarrow}{\longleftarrow}}} \qquad
\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_2}{\Delta,B_1}}{\Gamma,\Delta,B_1}\,\mathsf{mix} \qquad \dfrac{\dfrac{\Pi_1}{\Gamma} \quad \dfrac{\Pi_3}{\Delta,B_2}}{\Gamma,\Delta,B_2}\,\mathsf{mix}}{\Gamma,\Delta,B_1\&B_2}\,\&
$$

Table 4: Mix rule commutations. The second conversion also has a symmetric variant, in which, at the right-hand side, the mix rule applies to the hypothesis contributing to the right argument of the tensor. Since sequents are unordered, we do not need symmetric variants obtained by exchanging the hypotheses of the mix rule. Our general definition of rule commutation in Appendix A also allows a version of $\mathsf{C}^{\mathsf{mix}}_{\mathsf{mix}}$ with three applications of mix, two above and one below. However, this conversion can be generated by the simpler one listed above and therefore is not listed explicitly.

$\beta \backslash \alpha$	mix	$\otimes$	$\oplus$	$\parr$	&
mix	✓	✓	✓	✓	✓
$\otimes$	✓	✓	✓	✓	✓
$\oplus$	✓	✓	✓	✓	✓
$\parr$	○	○	✓	✓	✓
&	•	•	○	○	○

Table 5: Rule commutations. The check marks flag pairs $\frac{\alpha}{\beta}$ where a (lower) β-rule always commutes over an α-rule. The marks ○ indicate situations where β-rules commute over α-rules only under certain syntactic restrictions, which can be found by studying the results of commuting α- over β-rules. The • denotes commutation under certain syntactic restrictions.

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following cases applies (*cf.* Table 5):

- $\beta \in \{\otimes, \oplus, \mathsf{mix}\}$;
- $\beta = \parr$ and $\alpha \neq \otimes, \mathsf{mix}$;
- $\beta = \parr$, $\alpha = \otimes$ or mix, and both arguments of the formula generated by the $\parr$-rule occur in the same hypothesis of the α-rule;
- $\beta = \&$, $\alpha \neq \otimes, \mathsf{mix}$, the β-rule generates a formula $B_1 \& B_2$, and its right hypothesis is an α-rule analogous to its left hypothesis (just having B_2 in place of B_1, or vice versa); or
- $\beta = \&$, $\alpha = \otimes$ or mix, the β-rule generates a formula $B_1 \& B_2$, its right hypothesis is an α-rule analogous to its left hypothesis (just having B_2 in place of B_1, or vice versa), and the subproofs of Π of the two hypotheses of these two α-rules that do not include the B_i are identical.

This, in turn, yields exactly the rule commutations of Tables 2–4.

5 Rule commutation theorem

In this section we prove that the kernel of our function from MALL proofs to proof nets coincides precisely with equivalence modulo rule commutations:

THEOREM 1 *Two MALL proofs translate to the same proof net if and only if they can be converted into each other by a series of rule commutations.*

Proof. If Π' can be obtained from Π by commuting rule occurrences, then Π and Π' translate to the same linking set: taking a &-resolution on either side of a commutation (Tables 2–4) induces essentially the same &-resolutions (or deletions) of the subproofs Π_i. For example, in the last commutation in Table 3, if we choose *right* for the distinguished &-rule, we delete subproof Π_2 from both sides, and induce corresponding &-resolutions of Π_1 and Π_3. The converse is proved in the remainder of the section. □

Given a set of linkings Λ on a sequent Γ, let $\Gamma \upharpoonright \Lambda$ be obtained from the forest Γ by deleting all vertices that are not below a leaf of Γ that occurs in Λ (*i.e.*, in a link in a linking of Λ). A &-vertex w in Γ is ***toggled*** by Λ if both arguments of w occur in $\Gamma \upharpoonright \Lambda$. A link a ***depends*** on w in Λ if there exists $\lambda, \lambda' \in \Lambda$ such that $a \in \lambda$, $a \notin \lambda'$, and w is the only & toggled by $\{\lambda, \lambda'\}$. Construct the ***graph*** $\mathscr{G}_\Lambda$ [4] from $\Gamma \upharpoonright \Lambda$ by adding the edges of $\bigcup_{\lambda \in \Lambda} \lambda$, as well as all ***jump*** edges from leaves ℓ and ℓ' to any &-vertex on which the link $\{\ell,\ell'\}$ depends in Λ. Below we will need the following properties of a proof net θ on a MALL sequent Γ, established in [4].

$$\textit{Any set of two linkings in } \theta \textit{ toggles a } \&\textit{-vertex of } \Gamma. \tag{1}$$

$$\textit{Each root vertex (formula occurrence) in } \Gamma \textit{ occurs in } \mathscr{G}_\theta. \tag{2}$$

$$\begin{array}{l}\textit{For every } \lambda \in \theta \textit{ and each root } \&\textit{-vertex } w \textit{ in } \Gamma \\ \textit{there is a } \lambda' \in \theta \textit{ such that } w \textit{ is the only } \& \textit{ toggled by } \{\lambda,\lambda'\}.\end{array} \tag{3}$$

A formula occurrence $A = A_1 \alpha A_2$ in a MALL sequent Γ ***separates*** a proof net θ on Γ if (i) $\alpha \in \{\parr, \&\}$, (ii) $\alpha = \oplus$ and one of the A_i does not occur in $\mathscr{G}_\theta$, or (iii) $\alpha = \otimes$ and $\mathscr{G}_\theta$ has no cycle through this $\otimes$.

LEMMA 1 *If the last rule of a proof Π of Γ generates A, then A separates the proof net associated with Π.*

Proof. The only non-trivial case is $\alpha = \otimes$. Let Γ_1 and Γ_2 be the hypotheses of the last rule ρ of Π, let Π_i be the branch of Π above ρ proving Γ_i, let θ be the proof net associated with Π and θ_i the one associated with Π_i. $\mathscr{G}_\theta$ could have a cycle through α only when in θ a link a in Γ_1 depends on a &-vertex w in Γ_2 (or vice versa). In that case there exists $\lambda,\lambda' \in \theta$ such that $a \in \lambda$, $a \notin \lambda'$, and w is the only & toggled by $\{\lambda,\lambda'\}$. Hence there must be λ_1,λ_1' in θ_1 and λ_2,λ_2' in θ_2 such that $a \in \lambda_1$, $a \notin \lambda_1'$ and w is the only & toggled by $\{\lambda_1 \cup \lambda_2, \lambda_1' \cup \lambda_2'\}$. However, by (1) there must be another &-vertex of Λ that is toggled by $\{\lambda_1 \cup \lambda_2, \lambda_1' \cup \lambda_2'\}$, namely one occurring in Γ_1 that is toggled by $\{\lambda_1,\lambda_1'\}$. □

LEMMA 2 *If a formula occurrence $A = A_1 \alpha A_2$ in a MALL sequent Γ,A separates a proof net θ of Γ,A for which $\mathscr{G}_\theta$ is connected, then there is at most one instance σ of an α-rule that could generate A in the last step of a proof Π of Γ,A with proof net θ.*

Proof.

- Case $\alpha = \parr$: the hypothesis of σ must be Γ,A_1,A_2.
- Case $\alpha = \&$: the hypotheses of σ must be Γ,A_1 and Γ,A_2.
- Case $\alpha = \oplus$: exactly one of the A_i, say A_d, is in $\mathscr{G}_\theta$ (2). Hence the hypothesis of σ must be Γ,A_d.
- Case $\alpha = \otimes$: let Γ,A_1,A_2 be the sequent resulting from deleting the $\otimes$ in A from Γ,A. Since A separates θ, and $\mathscr{G}_\theta$ is connected, the restriction of $\mathscr{G}_\theta$ to Γ,A_1,A_2 falls apart in two disconnected components, one on a sequent Γ_1,A_1, the other on a sequent Γ_2,A_2, where $\Gamma_1 \cup \Gamma_2 = \Gamma$. Now, also using (2), the hypotheses of σ must be Γ_1,A_1 and Γ_2,A_2. □

In each case the proof nets on the hypotheses of σ, induced by the branches of Π that prove these hypotheses, are completely determined by θ.

For Π a proof, let $\mathscr{G}_\Pi$ abbreviate $\mathscr{G}_{\theta_\Pi}$. We shall prove the following four lemmas by simultaneous structural induction.

LEMMA 3 *Let Π be a proof of a MALL sequent $\Delta,A_1 \otimes A_2,\Sigma$ such that in $\mathscr{G}_\Pi$ any path between (vertices in) Δ,A_1 and A_2,Σ passes through the indicated occurrence of $\otimes$. Then Π can, by means of rule commutations, be converted into a proof Π' whose last step is the $\otimes$-rule with hypotheses Δ,A_1 and A_2,Σ.*

LEMMA 4 *Let Π be a proof of a MALL sequent Γ whose proof net θ is separated by a formula occurrence A in Γ. Then, by means of a series of rule commutations, Π can be converted into a proof Π'' of Γ that generates A in its last step.*

LEMMA 5 *Let Π be a proof of a MALL sequent Δ,Σ for nonempty sequents Δ and Σ, such that in $\mathscr{G}_\Pi$ there is no path between (vertices in) Δ and Σ. Then Π can, by means of rule commutations, be converted into a proof Π'' whose last step is the* mix*-rule with hypotheses Δ and Σ.*

LEMMA 6 *If two proofs Π and Π' of a MALL sequent Γ translate to the same proof net on Γ, then Π can be converted into Π' by a series of rule commutations.*

Lemma 6 is the converse direction of Theorem 1 that has to be established.

Proof. We prove Lemmas 3–6 by a simultaneous structural induction on Π (or equivalently, on Γ).

Induction base (applies to Lemma 6 only). The induction base is trivial, as a MALL sequent that can be proven in one step has at most one proof, a single application of ax.

Induction step for Lemma 3.

- First consider the case that the last step ρ of Π is an application of mix, say with hypotheses Γ_c and $\Gamma_d, A_1 \otimes A_2$.

  Let Π_d be the branch of Π above ρ proving $\Gamma_d, A_1 \otimes A_2$. Let $\Delta_d := \Delta \cap \Gamma_d$ and $\Sigma_d := \Sigma \cap \Gamma_d$. Since $\mathscr{G}_{\Pi_d}$ is a subgraph of $\mathscr{G}_\Pi$, any path in $\mathscr{G}_{\Pi_d}$ between (vertices in) Δ_d, A_1 and A_2, Σ_d passes through the indicated occurrence of $\otimes$. Hence, by induction, Π_d can, by means of rule commutations, be converted into a proof Π'_d whose last step is the $\otimes$-rule with hypotheses Δ_d, A_1 and A_2, Σ_d.

  Let Π_c be the branch of Π above ρ proving Γ_c. Let $\Delta_c := \Delta \cap \Gamma_c$ and $\Sigma_c := \Sigma \cap \Gamma_c$. Since $\mathscr{G}_{\Pi_c}$ is a subgraph of $\mathscr{G}_\Pi$, there is no path in $\mathscr{G}_{\Pi_c}$ between (vertices in) Δ_c and Σ_c. If Δ_c or Σ_c is empty, let $\Pi'_c := \Pi_c$. Otherwise, by induction, using Claim 5, Π_c can, by means of rule commutations, be converted into a proof Π'_c whose last step is the mix-rule with hypotheses Δ_c and Σ_c.

  Let Π' be the proof obtained from Π by replacing Π_d with Π'_d and Π_c with Π'_c. Let Π'' be the proof with the same 3 or 4 subproofs yielding Δ_c, Σ_c, Δ_d, A_1 and A_2, Σ_d that first combines Δ_c with Δ_d, A_1 into Δ, A_1 using mix (provided Δ_c is nonempty), and likewise combines Σ_c with A_2, Σ_d into A_2, Σ using mix (provided Σ_c is nonempty), and then applies $\otimes$ to yield $\Delta, A_1 \otimes A_2, \Sigma$. By means of a few simple rule commutations, Π' can be converted into Π''.

- Next consider the case that the last step ρ of Π is an application of $\otimes$ generating the same formula $A_1 \otimes A_2$. Let the hypotheses of ρ be Γ_i, A_i for $i = 1,2$.

  Let Π_i be the branch of Π above ρ proving Γ_i, A_i. Let $\Delta_i := \Delta \cap \Gamma_i$ and $\Sigma_i := \Sigma \cap \Gamma_i$. Since $\mathscr{G}_{\Pi_1}$ is a subgraph of $\mathscr{G}_\Pi$, there is no path in $\mathscr{G}_{\Pi_1}$ between (vertices in) Δ_1, A_1 and Σ_1. In case Σ_1 is empty, let $\Pi'_1 := \Pi_1$. Otherwise, by induction, using Claim 5, Π_1 can, by means of rule commutations, be converted into a proof Π'_1 whose last step is the mix-rule with hypotheses Δ_1, A_1 and Σ_1.

  In case Δ_2 is empty, let $\Pi'_2 := \Pi_2$. Otherwise, by means of rule commutations, Π_2 can be converted into a proof Π'_2 whose last step is the mix-rule with hypotheses Δ_2 and A_2, Σ_2.

  Let Π' be the proof obtained from Π by replacing Π_i with Π'_i for $i \in \{1,2\}$. Let Π'' be the proof with the same 2, 3 or 4 subproofs yielding Δ_1, A_1, Σ_1, Δ_2 and A_2, Σ_2 that first combines Δ_1, A_1 with Δ_2 into Δ, A_1 using mix (provided Δ_c is nonempty), and likewise combines Σ_c with A_2, Σ_d into A_2, Σ using mix (provided Σ_c is nonempty), and then applies $\otimes$ to yield $\Delta, A_1 \otimes A_2, \Sigma$. By means of a few simple rule commutations, Π' can be converted into Π''.

In the remaining cases let the last step of Π be a a β-rule ρ, generating the formula $B = B_1 \beta B_2 \neq A = A_1 \otimes A_2$. We treat the case that B occurs in Σ; the other case follows by symmetry. Let $\Sigma = \Sigma', B_1 \beta B_2$.

- Suppose $\beta = \oplus$. Let Π_d be the part of Π above ρ, proving the hypothesis Δ, A, Σ', B_d of ρ (where d is 1 or 2). Since $\mathscr{G}_{\Pi_d}$ is a subgraph of $\mathscr{G}_{\Pi}$, any path in $\mathscr{G}_{\Pi_d}$ between (vertices in) Δ, A_1 and A_2, Σ', B_d passes through the indicated occurrence of $\otimes$. Thus, by induction, by a series of rule commutations Π_d can be be converted into a proof Π'_d of $\Delta, A_1 \otimes A_2, \Sigma', B_d$ whose last step is the $\otimes$-rule with hypotheses Δ, A_1 and A_2, Σ', B_d. Let Π' be the proof obtained from Π by replacing Π_d by Π'_d. In Π', ρ commutes over the $\otimes$-rule generating A, thereby yielding the required proof Π''.

- Suppose $\beta = \otimes$. Let Π_1 and Π_2 be the branches of Π above ρ, proving the hypothesis $\Delta_1, A, \Sigma_1, B_1$ and Δ_2, Σ_2, B_2 of ρ, respectively. Here $\Delta = \Delta_1 \Delta_2$ and $\Sigma' = \Sigma_1, \Sigma_2$. We assume that A sides with B_1; the other case proceeds symmetrically. Since $\mathscr{G}_{\Pi_1}$ is a subgraph of $\mathscr{G}_{\Pi}$, any path in $\mathscr{G}_{\Pi_1}$ between (vertices in) Δ_1, A_1 and A_2, Σ_1, B_1 passes through the indicated occurrence of $\otimes$. Thus, by induction, by a series of rule commutations Π_1 can be be converted into a proof Π'_1 of $\Delta_1, A_1 \otimes A_2, \Sigma_1, B_1$ whose last step is the $\otimes$-rule with hypotheses Δ_1, A_1 and A_2, Σ_1, B_1.

  In case Δ_2 is empty, let $\Pi'_2 := \Pi_2$. Otherwise, by induction, using Claim 5, Π_2 can be be converted into a proof Π'_2 of Δ_2, Σ_2, B_2 whose last step is the mix-rule with hypotheses Δ_2 and Σ_2, B_2.

  Let Π' be the proof obtained from Π by replacing Π_i by Π'_i, for $i \in \{1,2\}$. Let Π'' be the proof with the same 3 or 4 subproofs yielding Δ_1, A_1, A_2, Σ_1, B_1, Δ_2 and Σ_2, B_2 that first combines Δ_2 with Δ_1, A_1 into Δ, A_1 using mix (provided Δ_2 is nonempty), and likewise combines Σ_2, B_2 with A_2, Σ_1, B_1 into A_2, Σ', B using $\otimes$, and then applies $\otimes$ to yield Δ, A, Σ', B. By means of a few simple rule commutations, Π' can be converted into Π''.

- Let $\beta = ⅋$. Let Π_ρ be the part of Π above ρ. Then Π_ρ proves the hypothesis $\Delta, A, \Sigma', B_1, B_2$ of ρ. Since $\mathscr{G}_{\Pi_\rho}$ is a subgraph of $\mathscr{G}_{\Pi}$, in $\mathscr{G}_{\Pi_\rho}$ any path between (vertices in) Δ, A_1 and A_2, Σ', B_1, B_2 passes through the indicated occurrence of $\otimes$. Hence, by induction, using by Claim 3, Π_ρ can, by means of rule commutations, be converted into a proof Π'_ρ whose last step is the $\otimes$-rule with hypotheses Δ, A_1 and A_2, Σ', B_1, B_2. Let Π' be the proof obtained from Π by replacing Π_ρ by Π'_ρ. In Π' the ⅋-rule ρ commutes over the $\otimes$-rule generating A, thereby yielding the required proof Π''.

- Let $\beta = \&$. The rule ρ has hypotheses Δ, A, Σ', B_1 and Δ, A, Σ', B_2. Let Π_i be the branch of Π above

$$\dfrac{\begin{array}{c}\vdots\,\Pi_1\\ \Delta, A_1 \otimes A_2, \Sigma', B_1\end{array} \qquad \begin{array}{c}\vdots\,\Pi_2\\ \Delta, A_1 \otimes A_2, \Sigma', B_2\end{array}}{\Delta, A_1 \otimes A_2, \Sigma', B_1 \& B_2}\ \&(\rho)$$

  ρ proving Δ, A, Σ', B_i. Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_{\Pi}$, in $\mathscr{G}_{\Pi_i}$ any path between (vertices in) Δ, A_1 and A_2, Σ', B_i passes through the indicated occurrence of $\otimes$. Hence, by induction, using Claim 3, Π_i can, by means of rule commutations, be converted into a proof Π'_i whose last step is the $\otimes$-rule with hypotheses Δ, A_1 and A_2, Σ', B_i. So the left hypotheses of Π'_1 and Π'_2 are both Δ, A_1, and we claim that the proof nets on them induced by the subproofs Π'_{11} and Π'_{21} of Π leading up to these hypotheses must be the same.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\Pi'_{11}\\ \Delta, A_1\end{array} \qquad A_2, \Sigma', B_1}{\Delta, A_1 \otimes A_2, \Sigma', B_1}\otimes \qquad \dfrac{\begin{array}{c}\vdots\,\Pi'_{21}\\ \Delta, A_1\end{array} \qquad A_2, \Sigma', B_2}{\Delta, A_1 \otimes A_2, \Sigma', B_2}\otimes}{\Delta, A_1 \otimes A_2, \Sigma', B_1 \& B_2}\ \&(\rho)$$

For if not, let λ be a linking in the proof net of Π'_{11} but not in the proof net of Π'_{21}. (The symmetric case goes likewise.) Then, using (2) and Table 1, for some linking μ on A_2,Σ',B_1, the linking $\nu := \lambda \cup \mu$ must be in θ. Using (3), let $\nu' \in \theta$ be such that β is the only & toggled by $\{\nu,\nu'\}$. Again using Table 1, $\nu' = \lambda' \cup \mu'$ for some linking λ' in the proof net of Π'_{21}. Since there must be a link $a = \{\ell,\ell'\}$ such that $a \in \lambda$ but $a \notin \lambda'$ (or vice versa), in $\mathscr{G}_\theta$ there is a jump edge from ℓ to β. This contradicts the assumption that in $\mathscr{G}_\theta$ any path between (vertices in) Δ,A_1 and $A_2,\Sigma',B_1 \invamp B_2$ passes through the indicated occurrence of $\otimes$.

Therefore, by induction, using Claim 6, Π'_{11} can be converted into Π'_{21} by a series of rule commutations. Let Π''_2 be obtained from Π'_2 by replacing its subproof Π'_{21} by Π'_{11}, and let Π' be the proof obtained from Π by replacing Π_1 by Π'_1 and Π_2 by Π''_2. In Π', the $\otimes$-rules generating A commute with the &-rule ρ, thereby yielding the required proof Π''. ■

Induction step for Claim 4. Suppose that Π does not generate $A = A_1 \alpha A_2$ in its last step. The case that $\alpha = \otimes$ is implied by Claim 3. Therefore we assume here that $\alpha \in \{\oplus, \invamp, \&\}$.

- First consider the case that the last step of Π is the application of a mix-rule ρ. Then $\Gamma = \Delta, A$ and A occurs in a hypothesis Δ_d, A of ρ (where $\Delta_d \subseteq \Delta$). Let Π_d be the branch of Π above ρ proving Δ_d, A. Its proof net is separated by A in Δ_d, for otherwise the proof net θ of Π would not be separated by A in Γ. Thus, by induction, by a series of rule commutations Π_d can be be converted into a proof Π'_d of Δ_d, A that generates A in its last step. Let Π' be the proof of Γ obtained by replacing Π_d by Π'_d in Π. In Π', ρ commutes over the α-rule generating A, thereby yielding the required proof Π''.

In the remaining cases let the last step of Π be the application of a β-rule ρ, generating the formula $B_1 \beta B_2$. Thus $\Gamma = \Delta, A, B_1 \beta B_2$.

- Suppose $\beta \in \{\otimes, \oplus\}$. Then A occurs in a hypothesis Δ_d, A, B_d of ρ (where d is 1 or 2, and of course $\Delta_d = \Delta$ in the case $\beta = \oplus$). Let Π_d be the branch of Π above ρ proving Δ_d, A, B_d. Its proof net is separated by A in Δ_d, A, B_d, for otherwise the proof net θ of Π would not be separated by A in Γ. Thus, by induction, by a series of rule commutations Π_d can be be converted into a proof Π'_d of Δ_d, A, B_d that generates A in its last step. Let Π' be the proof of Γ obtained by replacing Π_d by Π'_d in Π. In Π', ρ commutes over the α-rule generating A, thereby yielding the required proof Π''.
- Let $\beta = \invamp$. Let Π_ρ be the part of Π above ρ. Then Π_ρ proves the hypothesis Δ, A, B_1, B_2 of ρ, and its proof net is separated by A, for otherwise θ would not be separated by A. Thus, by induction, by a series of rule commutations Π_ρ can be be converted into a proof Π'_ρ of Δ, A, B_1, B_2 that generates A in its last step. As above, a rule commutation finishes the proof.
- Let $\beta = \&$. Then ρ has hypotheses Δ, A, B_1 and Δ, A, B_2. Let Π_i be the branch of Π above ρ proving Δ, A, B_i. The proof nets of Π_1 and Π_2 are separated by A in Δ, A, B_i in exactly the same way, *i.e.*, in case $\alpha = \oplus$ choosing the same argument A_d, for otherwise θ would not be separated by A. Thus, by induction, by a series of rule commutations the Π_i can be converted into proofs Π'_i of Δ, A, B_i that generate A in their last steps. The last step of Π'_2 must be analogous to the last step of Π'_1, just having B_2 in place of B_1—the argument is similar to the proof of Lemma 2. Let Π' be the proof of Γ obtained by replacing Π_i by Π'_i in Π, for $i = 1, 2$. In Π', the &-rule ρ commutes over the α-rules generating A, thereby yielding the required proof Π''. ■

Induction step for Claim 5.

- First consider the case that the last step ρ of Π is an application of mix, say with hypotheses Γ_1 and Γ_2. Let Π_i be the branch of Π above ρ, proving Γ_i (for $i = 1, 2$). Since $\mathscr{G}_{\Pi_i}$ is a subgraph of

$\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_i}$ there is no path between $\Delta_i := \Delta \cap \Gamma_i$ and $\Sigma_i := \Sigma \cap \Gamma_i$. In case Δ_i or Σ_i is empty, we let $\Pi'_i := \Pi_i$. Otherwise, by induction Π_i can, by means of rule commutations, be converted into a proof Π'_i whose last step is a mix-rule with hypotheses Δ_i and Σ_i. Let Π' be the proof obtained from Π by replacing Π_i with Π'_i for $i = 1,2$. In Π', ρ commutes over the 0, 1 or 2 mix-rules introduced right above it, thereby yielding the required proof Π''.

In the remaining cases let the last step of Π be the application of a β-rule ρ, generating the formula $B = B_1 \beta B_2$. We treat the case that B occurs in Σ; the other case follows by symmetry. Let $\Sigma = \Sigma', B_1 \beta B_2$.

- Suppose $\beta = \otimes$. The hypotheses of this rule are Δ_i, Σ_i, B_i, for $i \in \{1,2\}$, where $\Delta = \Delta_1, \Delta_2$ and $\Sigma' = \Sigma_1, \Sigma_2$. Let Π_i be the branch of Π proving Δ_i, Σ_i, B_i. Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_i}$ there is no path between Δ_i and Σ_i, B_i. In case Δ_i is empty, we let $\Pi'_i := \Pi_i$. Otherwise, by induction Π_i can, by means of rule commutations, be converted into a proof Π'_i whose last step is a mix-rule with hypotheses Δ_i and Σ_i, B_i. Let Π' be the proof obtained from Π by replacing Π_i with Π'_i for $i = 1,2$. In Π', ρ commutes over the 1 or 2 mix-rules introduced right above it, thereby (possibly using $\mathsf{C}^{\mathsf{mix}}_{\otimes}$ twice and $\mathsf{C}^{\mathsf{mix}}_{\mathsf{mix}}$ once) yielding the required proof Π''.
- Suppose $\beta = \oplus$. The hypothesis of this rule is Δ, Σ', B_d, where d is 1 or 2. Let Π_d be the subproof of Π proving the latter sequent. Since $\mathscr{G}_{\Pi_d}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_d}$ there is no path between Δ and Σ', B_d. By induction Π_d can, by means of rule commutations, be converted into a proof Π'_d whose last step is an application of the mix-rule with hypotheses Δ and Σ', B_d. Let Π' be the proof obtained from Π by replacing Π_d with Π'_d. In Π', ρ commutes over the mix-rule introduced right above it, thereby yielding the required proof Π''.
- Suppose $\beta = \invamp$. The hypothesis of this rule is $\Delta, \Sigma', B_1, B_2$. Let Π_ρ be the subproof of Π proving the latter sequent. Since $\mathscr{G}_{\Pi_\rho}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_\rho}$ there is no path between Δ and Σ', B_1, B_2. By induction Π_ρ can, by means of rule commutations, be converted into a proof Π'_ρ whose last step is a mix-rule with hypotheses Δ and Σ', B_1, B_2. Let Π' be the proof obtained from Π by replacing Π_ρ with Π'_ρ. In Π', ρ commutes over the mix-rule introduced right above it, thereby yielding the required proof Π''.
- Suppose $\beta = \&$. The hypotheses of this rule are Δ, Σ, B_i, for $i \in \{1,2\}$. Let Π_i be the branch of Π proving Δ, Σ, B_i. Since $\mathscr{G}_{\Pi_i}$ is a subgraph of $\mathscr{G}_\Pi$, in $\mathscr{G}_{\Pi_i}$ there is no path between Δ and Σ, B_i. By induction Π_i can, by means of rule commutations, be converted into a proof Π'_i whose last step is a mix-rule with hypotheses Δ and Σ, B_i. So the left hypotheses of Π'_1 and Π'_2 are both Δ, and we claim that the proof nets on them induced by the subproofs Π'_{11} and Π'_{21} of Π leading up to these hypotheses must be the same. The argument goes just as in the proof of Claim 3.

  Therefore, by induction, using Claim 6, Π'_{11} can be converted into Π'_{21} by a series of rule commutations. Let Π''_2 be obtained from Π'_2 by replacing its subproof Π'_{21} by Π'_{11}, and let Π' be the proof obtained from Π by replacing Π_1 by Π'_1 and Π_2 by Π''_2. In Π', the mix-rules generating A commute with the &-rule ρ, thereby yielding the required proof Π''. ■

Induction step for Claim 6. For the induction step, suppose Π and Π' are two proofs of a MALL sequent Γ that have the same proof net θ.

First assume that $\mathscr{G}_\theta$ is connected. In that case the last steps of Π and Π' cannot be mix. Let A be the formula occurrence in Γ that is generated by the last step of Π'. By Lemma 1, A separates θ. Hence, using Claim 4, by means of a series of rule commutations, Π can be converted into a proof Π'' of Γ that generates A in its last step. By Lemma 2, the last step σ of Π' is the same as the last step of Π''. Thus each hypothesis Γ_d of σ is proven by a subproof Π'_d of Π', and by a subproof Π''_d of Π''. As Π'_d and Π''_d

have the same proof net, by induction they can be converted into each other by means of a series of rule commutations. If follows that also Π and Π' can be converted into each other by means of a series of rule commutations.

Next assume that $\mathscr{G}_\theta$ is disconnected; let $\Gamma = \Gamma_1, \Gamma_2$ with the Γ_i nonempty sequents, such that in $\mathscr{G}_\theta$ there is no path between (vertices in) Γ_1 and Γ_2. Using Claim 5, Π can, by means of rule commutations, be converted into a proof Π_{mix} whose last step is the mix-rule with hypotheses Γ_i; let Π_i be the branch of Π_{mix} proving Γ_i. Its proof net is simply the restriction of (the linkings in) θ to Γ_i. Likewise, Π' can, by means of rule commutations, be converted into a proof Π'_{mix} whose last step is the mix-rule with hypotheses Γ_i; let Π'_i be the branch of Π_{mix} proving Γ_i. Since Π_i and Π'_i have the same proof net, by induction one can be converted into the other by a series of rule commutations. Consequently, Π can be converted into Π'.

$\square$

6 TODO: cut

A A general concept of rule commutation

In order to properly define rule commutations in a sequent calculus, we consider rules—called ***abstract*** rules—that contain *variables* ranging over formulas and over sequents. The rules for MALL in Section 2 are of this form. Thus, rather than seeing the rule for $\otimes$ as a template, of which there is an instance for each choice of A, B, Γ and Δ, we see it as a single rule containing four variables. When applying such a rule in a proof, formulas and sequents are substituted for the variables of the corresponding type. Here we study rule commutations only for sequent calculi whose abstract rules satisfy two properties: (1) the premises of a rule are free of literals and connectives and thus are built from variables only, and (2) each of these variables occurs exactly once in the conclusion.

This use of variables is a way of formalising the implicit tracking of subformula occurrences described in Section 2 and utilized in Section 3: a subformula occurrence within an occurrence of a formula or sequent substituted for a variable A or Γ appearing in the premises of an abstract rule, tracks to the corresponding subformula occurrence within the occurrence of the same formula or sequent substituted for A or Γ in the conclusion of the rule.

Formally, a ***formula expression*** is built from formula variables, literals and connectives; it is a ***formula*** if it contains no variables. A ***sequent expression*** is a multiset of sequent variables and formula expressions; it is a ***sequent*** if it does not contain any variables. Here a *multiset* of objects from a set S is a function $M : S \to \mathbb{N}$, telling for each object in S how often it occurs in M. An object $x \in S$ with $M(x) > 0$ is called an ***element*** of M. Let $C(M) := \{x \in S \mid M(x) > 0\}$ denote the set of elements of M. In case $M(x) \in \{0,1\}$ for all $x \in S$, the multiset M is usually identified with the set $C(M)$.

An ***abstract rule*** is a pair $\frac{H}{\Gamma}$ of a set H of sequent expressions—the ***premises***—and a single sequent expression Γ—the conclusion. An abstract rule is ***pure*** if it satisfies conditions (1) and (2) above. A ***concrete rule***—simply called ***rule*** outside of this appendix—is a pair $\frac{H}{\Gamma}$ of a *multi*set H of (variable-free) sequents and a single sequent Γ.

A ***substitution*** σ maps formula variables to formula expressions and sequent variables to sequent expressions; it extends to a map from formula expressions to formula expressions and from (sets of) sequent expressions to (multisets of) sequent expressions. A substitution is ***closed*** if it maps formula variables to formulas and sequent variables to sequents. If $\frac{H}{\Gamma}$ is an abstract rule and σ a (closed) substitution, then $\frac{\sigma(H)}{\sigma(\Gamma)}$ is a (closed) ***substitution instance*** of $\frac{H}{\Gamma}$; its ***collapse*** $\frac{C(\sigma(H))}{\sigma(\Gamma)}$ is again an abstract rule.

Given a collection of connectives to determine the valid formulas, a ***sequent calculus***—such as MALL—is given by a set of abstract rules. It induces a set of concrete rules, namely the closed substitution instances of the abstract rules.

A ***proof*** Π in a sequent calculus is a well-founded, upwards branching tree of which the nodes are labelled by sequent expressions and some of the leaves are marked "hypothesis", such that (1) the labels of the hypotheses are free of literals and connectives and thus built from variables only, and (2) if Δ is the label of a node that is not an hypothesis and K is the multiset of labels of the children of this node then $\frac{K}{\Delta}$ is a substitution instance of one of the rules of that sequent calculus. Such a proof derives the abstract rule $\frac{H}{\Gamma}$, where H is the set of labels of the hypotheses, and Γ the label of the root of Π. A proof of a sequent Γ can be regarded as a proof of the abstract rule $\frac{H}{\Gamma}$ with $H = \emptyset$.

It is not hard to show that any abstract rule derivable in a sequent calculus containing pure rules only can be obtained as a collapsed substitution instance of a pure rule derivable in that sequent calculus. Although we do not make use of this insight in our proofs, it helps to motivate the following definitions.

A proof Π deriving a pure rule $\frac{H}{\Gamma}$, together with a substitution σ and proofs of $\sigma(\Delta)$ for each $\Delta \in H$, composes into a proof Π' of $\sigma(\Gamma)$; in case Π' is a subproof of a proof Π'' we say that Π ***occurs in*** Π''. The object obtained from Π by applying σ to all its node labels is called a ***proof fragment*** of Π''.

For α and β two abstract proof rules in a sequent calculus, an ***$\alpha\beta$-proof*** is a proof of a pure rule in which each non-hypothesis node is either the root, and an application of β, or a child of the root, and an application of α. A ***rule commutation*** is the replacement in a proof Π of an $\alpha\beta$-proof occurring in it by a (different) $\beta\alpha$-proof that derives the same rule. Thus, a rule commutation is a local conversion on a proof that retains the subproofs of its hypotheses, with possible duplication/identification.

We leave it to the reader to check that this definition, applied to MALL, generates exactly the rule commutations presented in Section 4.

Based on the above, we say that a concrete β-rule ***commutes over*** a concrete α-rule, if these rules occur in a proof fragment obtained as a substitution instance of an $\alpha\beta$-proof for which there exists a $\beta\alpha$-proof deriving the same rule. This definition of rule commutation is more liberal than the standard definition of rule commutation for a Gentzen sequent calculus [6, Def. 5.2.1], analysed by Kleene [5] and Curry [1]. That definition only covers the case where *each* β rule commutes over *each* α-rule, corresponding with the check marks in Table 5. Moreover, [6] requires—translated in our terminology—the source proof fragment to have two non-leave nodes only (one of β and only one for α), thereby ruling out the commutation of & over any α.

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