Shorter proof of universality of Chu spaces

Vaughan Pratt

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The categories $\mathbf{Str}_\kappa$ of κ-ary relational structures and their homomorphisms where κ is any ordinal are universal categories for mathematics to the extent that they realize[1] many familiar categories: groups, lattices, and Boolean algebras when $\kappa = 3$, rings, fields, and categories when $\kappa = 4$, etc.

Elsewhere [LNCS 711, 153-4] we proved that the self-dual category $\mathbf{Chu}_{2^\kappa}$ of Chu spaces over the power set of $\kappa = \{0, 1, \ldots, \kappa - 1\}$ realizes $\mathbf{Str}_\kappa$. Here we streamline that argument.

A κ-ary *relational structure* (A, ρ) consists of a set A, the *carrier*, and a κ-ary relation $\rho \subseteq A^\kappa = (\kappa \to A)$. A *homomorphism* $f : (A, \rho) \to (B, \sigma)$ of such structures is a function $f : A \to B$ such that $t \in \rho$ implies $f \circ t \in \sigma$. These structures and homomorphisms, with the usual composition, form the category $\mathbf{Str}_\kappa$.

A κ-*ruple* (for "relational tuple") *over* A is a binary relation R from κ to A, which we shall regard interchangeably as a set $R \subseteq \kappa \times A$ and a function $r : A \to 2^\kappa$ as appropriate, related by $r(a) = \{i \mid (i, a) \in R\}$, $R = \{(i, a) \mid i \in r(a)\}$. We partially order κ-ruples over A by inclusion, for this purpose representing them as subsets of $\kappa \times A$; we call a superset an *extension*. As a function from κ to A, a κ-tuple over A can be understood as a special case of a κ-ruple over A.

Fact 1. Distinct κ-tuples are incomparable under inclusion.

The composition $r \circ t$ of a ruple r with a tuple t assumes the representations $r : A \to 2^\kappa$, $t : \kappa \to A$. As $r \circ t : \kappa \to 2^\kappa$, this composite is itself a binary relation on κ; $I \leq r \circ t$ ($r \circ t$ is reflexive) just when r extends t. For $r : B \to 2^\kappa$, $f : A \to B$, $t : \kappa \to A$ (the situation treated below), $r \circ f \circ t$ is usefully ambiguous as either the composition of the ruple $f \circ t$ over A with the tuple r over A, or the composition of the ruple t over B with the tuple $r \circ f$ over B.

Define $\hat{\rho}$ as the set $\{r : A \to 2^\kappa \mid \exists t \in \rho \,.\, I \leq r \circ t\}$ of extensions of tuples of ρ, and $\overline{\hat{\rho}} = A^\kappa - \hat{\rho}$ as its complement. We realize (A, ρ) in $\mathbf{Chu}_{2^\kappa}$ as the Chu space $(\overline{\hat{\rho}}, A)$, namely the normal Chu space with point set A whose states are those $r : A \to 2^\kappa$ extending no tuple of ρ.

Theorem 1 *A function $f : A \to B$ is a Chu transform $f : (\overline{\hat{\rho}}, A) \to (\overline{\hat{\sigma}}, B)$ if*

[1] A concrete category D *realizes* a concrete category C when there exists a functor $F : C \to D$ that is full and faithful and which commutes with the respective underlying-set functors of C and D.

and only if it is a homomorphism $f : (A, \rho) \to (B, \sigma)$.

Proof: It suffices to show $\forall r{:}B{\to}2^\kappa[r{\circ}f \in \hat{\rho} \Rightarrow r \in \hat{\sigma}]$ iff $\forall t{:}\kappa{\to}A[t \in \rho \Rightarrow f{\circ}t \in \sigma]$.

(If) Given $r : B \to 2^\kappa$, if $r{\circ}f \in \hat{\rho}$ then there must exist $t \in \rho$ such that $I \leq r{\circ}f{\circ}t$. But then $f{\circ}t \in \sigma$ (f is a homomorphism), whence $r \in \hat{\sigma}$.

(Only if) Given $t \in \rho$, take $r : B \to 2^\kappa$ to be $f{\circ}t$. Then $I \leq r{\circ}f{\circ}t$ (since $r(f(t_i)) = \{j \mid f(t_i) = f(t_j)\}$), so $r{\circ}f \in \hat{\rho}$ (definition of $\hat{\rho}$), whence $r \in \hat{\sigma}$ (f is a Chu transform). But then there must exist some $t' : \kappa \to B$ such that $t' \in \sigma$ and $I \leq r{\circ}t'$. Hence r extends t'. But by Fact 1, $t' = r$, so $f{\circ}t = t'$, whence $f{\circ}t \in \sigma$. ∎