Structure from sorts, properties, and composition
A minimalist approach to topoalgebraic structure

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Background

Presheaves = unary algebras, etc.

Chu spaces $\mathrm{Chu}(Set, K)$

Self-dual, "universal" - great many natural concrete subcategories

e.g. $\mathrm{Grp} \subseteq \mathrm{Chu}(\mathrm{Set}, 8)$, $\mathrm{Rel}_n \subseteq \mathrm{Chu}(\mathrm{Set}, 2^n)$..

Isbell envelope $E(C)$

Self-dual, generalizes $\mathrm{Chu}(\mathrm{Set}, 1)$ but not $\mathrm{Chu}(\mathrm{Set}, 2)$

Communes: $\mathrm{Comm}(\mathrm{Set}, K)$

common generalization of all of the above

Slogan: communes : chu spaces :: presheaves : sets

described at CT'2004

Not self-dual in general

Chu spaces

Let K be a set.

A **Chu space** (A, r, X) over K consists of

sets A and X

a function $r : A \times X \to K$

A **Chu morphism** (f, g) from (A, r, X) to (B, s, Y) is an adjoint pair of functions.

Adjointness: $\forall a \in A, y \in Y[s(f(a), y) = r(a, g(y))]$

Chu category $\mathrm{Chu}(Set, K)$: the category of Chu spaces over K and their morphisms.

Communes

Let $K : L \nrightarrow J$ be a profunctor (distributor, bimodule, module, relator). Write $S = \mathrm{ob}(J)(\textit{sorts}), P = \mathrm{ob}(L)$ (properties).

A **commune** (A, ρ, X) over K consists of

presheaves $A : J^{\mathrm{op}} \to \mathrm{Set}$ and $X : L \to \mathrm{Set}$

a natural transformation $\rho : A \times X \to K$

where $A \times X : L \nrightarrow J$ (i.e. $: J^{\mathrm{op}} \times L \to \mathrm{Set}$)

satisfies $(A \times X)(s, p) = A_s \times X_p$

(or $A : 1 \nrightarrow J$, $X : L \nrightarrow 1$, $\rho : AX \to K$, tnx Rich W, Jeff E)

A commune morphism (f, g) from (A, ρ, X) to (B, σ, Y) is an adjoint pair of natural transformations $f_s : A_s \to B_s$, $g^p : Y^p \to X^p$.

Adjointness: $\forall a \in A_s, y \in Y_p[\sigma_s^p(f_s(a), y) = r_s^p(a, g^p(y))]$

Community $\mathrm{Comm}(\mathrm{Set}, K)$ = the category of communes over K and their morphisms

Canonical examples

Presheaves $\mathrm{Set}^{J^{\mathrm{op}}}$: Take $L = 0$, $K =! : 0 \nrightarrow J$.
Then $\mathrm{Comm}(\mathrm{Set}, ! : 0 \nrightarrow J) = \mathrm{Set}^{J^{\mathrm{op}}}$.

Chu spaces $\mathrm{Chu}(\mathrm{Set}, K)$: Take $J = 1$, $L = 1$, $K = K(s, p)$.
Then $\mathrm{Comm}(\mathrm{Set}, K : L \nrightarrow J) = \mathrm{Chu}(\mathrm{Set}, K(s, p))$
(So $s = 1$ (tensor unit, generator), $p = \bot$ (cogenerator).)

Isbell envelope $E(C)$: Take $J = L = C$.
Then $\mathrm{Comm}(\mathrm{Set}, \mathrm{Hom}_C : C \nrightarrow C) = E(C)$
(Note: for $K \geq 2$ $Chu(\mathrm{Set}, K)(p, s) = \emptyset$ hence not $E(C)$ for any C.)

Consistent, extensional, discrete

A commune (A, ρ, X) over $K : L \nrightarrow J$ is

Consistent: for all $p \in P$ $(= \mathrm{ob}(L))$, X^p is nonempty.

(Consistent communes allow L to exert influence. When P is empty communes are vacuously consistent.)

Extensional: for all $x, x' \in X^p$, if for all $a \in A_s$ $\rho^p_s(a, x) = \rho^p_s(a, x')$ then $x = x'$.

(No repeated columns. By analogy with topological spaces.)

Discrete: Extensional and X is maximal.

(Discrete does not imply consistent, examples later.)

Minor enhancements to Chu(Set, K)

Sierpinski example:

Goal ($K = 2$): Force states (opens) to contain whole space and empty space
Modify $\mathrm{Chu}(\mathrm{Set}, 2)$ by taking $\perp$ to be Sierpinski space
by adding two morphisms to $p = \perp$
J stays 1 ($|J(s, s)| = 1$), $K(s, p)$ stays 2, L becomes Sierpinski space ($|L(p, p)| = 3$ – no longer rigid).

Labeling example:

Σ-labelled Chu spaces: comma cat $(\Pi_1 \downarrow \Sigma)$
where $\Pi_1 : \mathrm{Chu}(\mathrm{Set}, K) \to \mathrm{Set}$ is $\Pi_1(A, r, X) = A$, $\Sigma : 1 \to \mathrm{Set}$.
Alternatively modify $\mathrm{Chu}(\mathrm{Set}, K)$ by making Σ copies of s

Acyclic graph examples

In the following examples, take the two sorts to be V and E as usual for graphs, with arrows $s^*, t^* : V \rightarrow E$.
That is, $S = \{V, E\}$, $J = \{S, \{s^*, t^*\}\}$ (four morphisms).

Take Grph to be $\mathrm{Set}^{J^{\mathrm{op}}}$.

Grph is not spatial. Moreover homorphisms preserve cycles, which the following examples exploit by taking L to be suitable categories of acyclic graphs.

Example 1: unit-length paths

Take L to be the one-object full subcategory of Grph consisting of just the edge and its two vertices. So $S = \{V, E\}$, $P = \{E\}$, $K = \mathrm{Grph}(S, P)$.

Claim. The full subcategory of $\mathrm{Comm}(\mathrm{Set}, K)$ consisting of its discrete consistent objects is equivalent to the full subcategory of Grph consisting of those graphs whose paths are of length at most 1.

Explanation: The states are in bijection with the power set of the isolated vertices. This is because a state "labels" each isolated vertex as either a source or target vertex. However there is only one label for edges, namely 1_E, and this unique label uniquely labels the source and target vertices of the edge.

Although $\mathrm{Comm}(\mathrm{Set}, K)$ contains graphs with paths of length 2, they are not consistent because they cannot be labelled consistently. (This gives the promised example of a discrete object that is not consistent.)

Example 2: path-uniform graphs

Take L to be the full subcategory of Grph whose one object σ is the graph of the successor relation on $\mathbb{Z}$. Vertices are integers, edges are pairs $(i, i+1)$.

$K(V,\sigma) = \mathbb{Z}$, $K(E,\sigma) = \{(i, i+1) | i \in \mathbb{Z}\}$.

A state of an object of $\mathrm{Comm}(\mathrm{Set}, K)$ is a labeling of edges and vertices with integers, such that when an edge is labeled i, its source and target are labeled i and $i+1$ respectively.

Hence every path is labelled with consecutive integers, both on its edges and its vertices, whence the length of a path is determined by the vertex labels of its endpoints.

The consistent discrete objects of $\mathrm{Comm}(\mathrm{Set}, K)$ are therefore the path-uniform graphs, namely those graphs such that for all vertices u, v all paths from u to v have the same length.

Example 3: Bounded-path graphs.

Take L to be the full subcategory of Grph whose one object $<$ is the graph of the strictly-less-than relation on $\mathbb{Z}$. Vertices are integers, edges are the pairs (i, j) such that $i < j$.

Each state labels the vertices with integers in such a way that the integers along any path form a strictly increasing sequence. For each pair of vertices, any state sets an upper bound on the length of all paths from u to v.

The consistent discrete objects of $\mathrm{Comm}(\mathrm{Set}, K)$ are therefore the bounded-path graphs, namely those graphs such that for all vertices u, v there exists an upper bound on the length of paths between them.

Example 4: Countable-chain graphs.

Replace $\mathbb{Z}$ by $\mathbb{Q}$ in the previous example, taking the edges to be the pairs (x, y) such that $x < y$.

In every state, every path is vertex-labeled with distinct rationals. Hence such a path can contain at most countably many vertices.

Obviously many more examples can be constructed by taking L to consist of suitable graphs. For example take it to consist of all finite cycles, one object per length of cycle. This can be seen to be equivalent to Example 2, the graph of the successor relation.

Topoalgebraic categories

Goal: expose simplicity/naturality of communes by defining them with a minimum of machinery.

A **topoalgebraic category** (C, S, P) is a locally small category C equipped with sets S and P of objects.

Let J and L be the full subcategories of C for which $S = \mathrm{ob}(J)$, $P = \mathrm{ob}(L)$.

Let $K : L \nrightarrow J$ be the profunctor for which $K(s, p) = C(s, p)$.

$\lambda u.C(S, u) * C(u, P) : L \nrightarrow J$ is a profunctor.mapping (s, p) to the set $C(s, u) \times C(u, p)$ of composable pairs of morphisms.

Composition is a dinatural transformation from $\lambda u.C(S, u) * C(u, P)$ to $C(S, P)$.

This interprets objects of (C, S, P) as communes and morphisms as commune morphisms.

Interpretation

At a more elementary level, objects u of C are interpreted as multisorted algebraic structures consisting of **elements** $a : s \to u$ of **sort** $s \in S$ and "topologized" with opens or **states** $x : u \to p$ for **property** $p \in P$.

Morphisms $f : u \to v$ of C are interpreted as their left and right actions $(C(S, f), C(f, P))$ on respectively elements of u and states of v.

Morphisms with the same actions are called **equivalent**.

A morphism that is both an element and a state is called a **scalar**, and one that is neither is deemed **ordinary**. Scalars are the elements of K.

Didense, complete

A TAC is **diextensional** when equivalence is identity; **didense** when it is diextensional and for any extension by ordinary morphisms alone every new morphism is equivalent to an old one; and **complete** when for any didense full extension every new object is isomorphic to an old one.

A **community** is a complete didense diextensional TAC.

Abstract communes

topos:presheaf-category :: ?:community

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