On the Representation of Abelian Groups as Chu Spaces

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Abstract

We compare two representations of Abelian groups as Chu spaces. The first represents the elements of any ternary relational structure as words on an 8-letter alphabet. The second is based on the group characters

Using group characters as states, the finite cyclic groups of order n are the following square Chu spaces over $\{0, 1/n, 2/n, \ldots, (n-1)/n\}$.

$$Z_2 = \begin{matrix}00\\01\end{matrix} \;/\; 2 \qquad Z_3 = \begin{matrix}000\\012\\021\end{matrix} \;/\; 3 \qquad Z_4 = \begin{matrix}0000\\0123\\0202\\0321\end{matrix} \;/\; 4 \qquad Z_5 = \begin{matrix}00000\\01234\\02413\\03142\\04321\end{matrix} \;/\; 5$$

(By "/ 3" we mean pointwise division by 3, so that the apparent alphabet $\{0, 1, 2\}$ is really $\{0, 1/3, 2/3\}$.)

These are enough examples to make the pattern for Z_n obvious: take the Chu space to be integer multiplication (not the group operation, which is addition) divided elementwise by n. The group operation can then be recovered as coordinatewise addition mod 1 of the representing rows. These Chu spaces are self-dual as a corollary of the commutativity of integer multiplication.

The Klein four group $V_4 = Z_2 + Z_2 = Z_2 \times Z_2$ doesn't fit this pattern: for one thing how would you map its elements to numbers? Nevertheless it has group characters, namely the columns of

$$V_4 = \begin{matrix}0000\\0101\\0011\\0110\end{matrix} \;/\; 2$$

This is a Chu space over $2 = \{0, 1/2\}$. Schizophrenically it is also the 2D vector space over $GF(2)$.

Z_5, Z_6, and Z_7 are respectively the only group, only abelian group, and only group, of that order, already treated above.

At 8 there are Z_8, $Z_2 + Z_4$, and $3.Z_2(Z_2 + Z_2 + Z_2)$. We've seen Z_8, the other two are

$$Z_2 + Z_4 = \begin{array}{c} 00000000 \\ 01230123 \\ 02020202 \\ 03210321 \\ 00002222 \\ 01232301 \\ 02022020 \\ 03212103 \end{array} \ / \ 4 \qquad\qquad 3.Z_2 = \begin{array}{c} 00000000 \\ 01010101 \\ 00110011 \\ 01100110 \\ 00001111 \\ 01011010 \\ 00111100 \\ 01101001 \end{array} \ / \ 2$$

$3.Z_2$ is the 3D vector space over $GF(2)$. We do not know whether $Z_2 + Z_4$ can masquerade as anything else familiar. Z_8 is the 1D vector space over $GF(8)$.

At 9 we have Z_9 (aka the 1D vector space over $GF(9)$) and $2.Z_3$ (aka the 2D vector space over $GF(3)$).

$$2.Z_3 = \begin{array}{c} 000000000 \\ 012012012 \\ 021021021 \\ 000111222 \\ 012120201 \\ 021102210 \\ 000222111 \\ 012201120 \\ 021210102 \end{array} \ / \ 3$$

These are enough examples to permit deducing the general pattern for finite abelian groups, which we will now describe for any direct sum (= direct product) of n cyclic groups of respective orders $c_1, \ldots, c_n$. Writing the elements of the product as n-tuples $(a_1, \ldots, a_n)$, which we use to index both rows and columns of the Chu space, the entry at row $(a_1, \ldots, a_n)$ and column $(b_1, \ldots, b_n)$ is

$$a_1 b_1/c_1 + a_2 b_2/c_2 + \ldots + a_n b_n/c_n (mod 1)$$

For $m = p^k$ where p is prime, Z_m is representable over an alphabet of size p. In general the alphabet needed to represent a finite abelian group is the largest square-free divisor of its order. For example if 2 and 3 are the only prime divisors of the order of G, then G is representable in $\mathbf{Chu}(Set, 6)$, where 6 is understood to be the alphabet $\{0, 1/6, 2/6, \ldots, 5/6\}$.

This much concerns just the groups in isolation. All this structure is basic to the theory of finite abelian groups and as such well-known and elementary.

For groups taken two or more at a time however, we may ask about the continuous functions between their Chu representations. Since distinct groups will in general be represented over distinct alphabets, the question arises as to whether this is even well-defined. The nice theorem (exercise) is that the homomorphisms between the Chu representations of G and G' (in either direction) are representable over the intersection of their respective alphabets (or union when the source and target Chu spaces are counted as part of the homomorphism), and as such are exactly the continuous functions between those representations.