The Second Calculus of Binary Relations

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Abstract

We view the Chu space interpretation of linear logic as an alternative interpretation of the language of the Peirce calculus of binary relations. Chu spaces amount to K-valued binary relations, which for $K = 2^n$ we show generalize n-ary relational structures. We also exhibit a four-stage unique factorization system for Chu transforms that illuminates their operation.

1 Introduction

In 1860 A. De Morgan [DM60] introduced a calculus of binary relations equivalent in expressive power to one whose formulas, written in today's notation, are inequalities $a \leq b$ between terms $a, b, \ldots$ built up from variables with the operations of composition $a;b$, converse $a^{\smile}$, and complement a^{-}. In 1870 C.S. Peirce [Pei33] extended De Morgan's calculus with Boolean connectives $a + b$ and ab, Boolean constants 0 and 1, and an identity $1'$ for composition. In 1895 E. Schröder devoted a book [Sch95] to the calculus, and further extended it with the operations of reflexive transitive closure, a^*, and its De Morgan dual a_1. In full this should be called the De Morgan-Peirce-Schröder-Tarski-Jónsson calculus, taking into account the further model-theoretic contributions of Tarski [Tar41] and Jónsson [JT48, JT52]. However it may reasonably be argued that Peirce did the bulk of the work of bringing the calculus to its modern form, which we recognize by calling it simply the Peirce calculus.

In 1987 J.-Y. Girard [Gir87] introduced linear logic, whose language is strikingly similar to that of the Peirce calculus. Liberally interpreted, linear logic may be regarded as subsuming the Peirce calculus, relevance logic [Dun86], quantales [Mul86], and related logics. But we feel this is too broad, since these interpretations lack the bilinear tensor product characteristic of linear algebra, present in the calculus to be described here. Moreover these nonconstructive interpretations considerably predate linear logic, and are done an injustice by

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sweeping them all under the rubric of linear logic. At the same time the great originality and strength of linear logic are undermined by presenting it as both a nonconstructive and constructive logic, modeled by Girard with respectively phase and coherence spaces. The former is not so novel, as some in the relevance logic community have pointed out, and as others [BvN36] could also. Linear logic is seen in its best light as the realization of the Curry-Howard isomorphism for linear algebra, imaginatively moving logic into new but legitimate territory.

This paper focuses these distinctions more sharply by describing both the Peirce calculus and linear logic as calculi having the same domain, namely binary relations, and essentially the same constants and operations, but with strikingly different interpretations. Furthermore the associated equational logic consists as usual of equations for the former, but isomorphisms for the latter, characteristic of the passage from nonconstructive to constructive logic. We further sharpen this focus by eliminating all other distinctions as far as possible consistent with the substantive details of the two calculi. Key to this passage is the replacement of composition by tensor product, these being mathematics' basic operations of sequential and parallel composition respectively, which for some time now in our own writing about models of behavior [Pra86] we have been calling respectively *concatenation*, or sequence, and *orthocurrence*, or flow.

This juxtaposition of the calculi is achieved by translating Chu's construction of *-autonomous categories [Bar79], ordinarily given in the rarefied atmosphere of commuting diagrams, into the same elementary set-theoretic terms in which the Peirce calculus is customarily described. Although the Chu interpretation of linear logic is conventionally understood via adjunctions in terms of (co)products and tensor products, in our elementary account of this interpretation we shall not even need to define the morphisms that go with the Chu objects, which we leave to the second half of the paper.

Chu's construction was originally studied by P. Chu as a Master's thesis under the supervision of M. Barr, who supplied the basic definition. More recently Barr has commented that "At the time, the formal construction appeared not to have substantial mathematical interest, but it appears to be the most interesting part in the present context." [Bar91]. Certainly within the past four years or so Chu's construction has turned into one of the more popular constructive interpretations of linear logic [Bar91, BG90, LS91], at least relative to the still-small overall interest in the constructive aspects of linear logic.

There is still no consensus on the proper standard model for linear logic, constructive or not. Our concern is with the algebra of Chu spaces. While we feel this is what linear logic should be about, this is a decision we are happy to leave to the linear logic community, Chu spaces being of mathematical interest independently of their relevance to logic.

The second half of the paper treats Chu transforms, which lift the class of Chu spaces to a category thereof. We obtain a four-stage unique factorization property for Chu transforms that illuminates their role as structure-preserving homomorphisms. And we show that n-ary relational structures and their homomorphisms fully and concretely embed in the category of Chu spaces over the set 2^n. We will treat further aspects and applications of Chu spaces, of which there appear to be a good many even at this early stage, in sequels to this paper.

2 The Peirce and Chu Calculi of Binary Relations

2.1 The Common Language

The Peirce calculus amounts to two copies of the logical connectives *or*, *false*, *and*, *true*, *not*, and *implies*, distinguished as the logical and relative (relational) forms of those connectives. To these Schröder [Sch95] added reflexive transitive closure a_0, nowadays a^*, and its De Morgan dual a_1.

Combining the separate involutory logical and relative complements, a^- and $a^{\smile}$, as a single involutory ($a^{\perp\perp} = a$) complement $a^{\smile -} = a^{\perp}$ [Pra92c, p.252] weakens the Boolean structure of the Peirce calculus to that of a De Morgan lattice [Dun86, p.184,p.193], since neither $a+a^{\perp} = 1$ nor $aa^{\perp} = 0$ hold of binary relations. This seems in practice to leave the utility of the Peirce calculus largely unimpaired, whose operations are as follows.

Peirce Language:	*Logical* :	$a+b$	0	ab	1
	Relative :	$a \dagger b$	$0'$	$a;b$	$1'$
	Nonmonotone :	$a^{\perp}$	$a\backslash b$	b/a	$a \rightarrow b$
	Closure :		a^*	a_1	

These are not independent, and a suitable basis is

Peirce Basis:	$a+b$	0
	$a;b$	$1'$
	$a^{\perp}$	$a^{\ddagger}$

where $a^{\ddagger} = a^{\perp *}$, intermediate between a^* and a_1, to go with $A^{\dagger}$ below.

We eliminate the remaining operations from consideration by reducing them to mere abbreviations, definable in terms of the basic operations as follows.

Peirce Abbreviations:

$$
\begin{array}{rcl@{\qquad}rcl}
ab & = & (a^{\perp}+b^{\perp})^{\perp} & 1 & = & 0^{\perp} \\
a \dagger b & = & (b^{\perp};a^{\perp})^{\perp} & 0' & = & 1'^{\perp} \\
a\backslash b & = & (b^{\perp};a)^{\perp} \ = \ a^{\perp} \dagger b & a \rightarrow b & = & a^{\perp} + b \\
b/a & = & (a;b^{\perp})^{\perp} \ = \ b \dagger a^{\perp} & & & \\
a^* & = & a^{\perp\ddagger} & a_1 & = & a^{\ddagger\perp}
\end{array}
$$

The language of the Chu calculus is that of linear logic, which we give as follows.[1]

Chu	*Additives* :	$A{+}B$	0	$A{\times}B$	1
Language:	*Multiplicatives* :	$A{\oplus}B$	$\perp$	$A{\otimes}B$	$\top$
	Nonmonotone :	$A^\perp$	$A{\multimap}B$		$A{\Rightarrow}B$
	Exponentials :		$!A$	$?A$	

These are intended to correspond with the Peirce connectives tabulated in the corresponding positions. The so-called *residuals* of the Peirce calculus, $a\backslash b$ and b/a, which from the table of abbreviations can be seen to behave like implications, merge in linear logic into the one "linear implication" $A{\multimap}B$. The implications have "currying" in common: $a\backslash(b\backslash c) = (b;a)\backslash c$ and $a{\rightarrow}(b{\rightarrow}c) = (ab){\rightarrow}c$ hold in the Peirce calculus, while $A{\multimap}(B{\multimap}C) \cong (A{\otimes}B){\multimap}C$ and $A{\Rightarrow}(B{\Rightarrow}C) \cong (A{\times}B){\Rightarrow}C$ will be seen to obtain for the Chu calculus. And Girard's dual exponentials are loosely related to Schröder's dual closures, ideally as a sort of "cotransitive closure;" for simplicity we content ourselves below with the naive interpretation of $!A$ as the domain of A.

As with the Peirce calculus, these operations are not independent, and we choose the following basis, matching our choice of basis for the Peirce calculus. For this purpose we take as primitive not $!A$ itself but rather $A^\dagger = !(A^\perp)$, explained below.

Chu	$A{+}B$	0
Basis:	$A{\otimes}B$	$\top$
	$A^\perp$	$A^\dagger$

We can then similarly define the rest of the linear logic operations as follows.

Chu
Abbreviations:

$$A{\times}B = (A^\perp{+}B^\perp)^\perp \qquad 1 = 0^\perp$$
$$A{\oplus}B = (B^\perp{\otimes}A^\perp)^\perp \qquad \perp = \top^\perp$$
$$A{\multimap}B = (B^\perp{\otimes}A)^\perp = A^\perp{\oplus}B \qquad A{\Rightarrow}B = !A{\multimap}B$$
$$!A = A^{\perp\dagger} \qquad ?A = A^{\dagger\perp}$$

Except mainly for notational differences, the identification of $a\backslash b$ and b/a, and the absence of * from $a \rightarrow b$, we have in this way concentrated whatever differences exist between the two calculi into the primitives, whose very different interpretations we now give.

[1]This is the notation now followed by Barr and Seely, and close to their earlier usage [Bar91, See89]. It replaces Girard's idiosyncratic notation $A{\oplus}B$, $A\&B$, and $A ⅋ B$ by respectively $A{+}B$, $A{\times}B$, and $A{\oplus}B$, and interchanges his assignments of 1 and $\top$. For vector spaces, $A \oplus B$ conventionally denotes the *biproduct* $A{\times}B = A{+}B$, but the Chu calculus distinguishes $\times$ and $+$, freeing up $\oplus$ for this other use. Actually Girard's notation goes quite tidily with our choice of primitives, which become $A{\oplus}B$, 0, $A{\otimes}B$, 1. The trouble here is that coproduct and final object have been $+$ and 1 for many decades now, and one needs a better reason than tidiness to make such a sweeping change. The tensor unit, $\top$ in Barr-Seely notation, is often written I, but almost never 1.

2.2 The Peirce and Chu Interpretations

The Peirce calculus is standardly interinterpreted for $a, b, \ldots$ ranging over subsets of X^2 where X is a fixed infinite but otherwise arbitrary set, namely as follows.

Peirce Interpretation:

$$
\begin{array}{rclrcl}
x(a{+}b)y & \Leftrightarrow & xay \vee xby & x0y & \Leftrightarrow & \mathit{false} \\
x(a;b)z & \Leftrightarrow & \exists y[xay \wedge ybz] & x1'y & \Leftrightarrow & x = y \\
x(a^{\perp})y & \Leftrightarrow & \neg(yax) & a^* & = & 1' + a + a;a + \ldots
\end{array}
$$

We may pass from binary relations on a single fixed set X to binary relations from a *domain* X to a *codomain* Y, which are permitted to vary from one relation to the next, provided we make the operations partial. In this extension $a{+}b$ is defined only when a and b have the same domain and codomain, while $a;b$ is defined just when the codomain of a is the domain of b. Furthermore every set X has its own identity $1'_X$, making $a;b$ the composition no longer of a monoid but of a category. Such structures have been called *Schröder categories* [Jón88].

The Chu calculus assumes such a variable domain and codomain at the outset. We define its connectives, acting on binary relations $A, B, \ldots, A_i, \ldots$ as subsets of $X_A{\times}Y_A$, $X_B{\times}Y_B, \ldots$, $X_i{\times}Y_i, \ldots$, as follows.

Chu Interpretation:

$$
\begin{array}{rclrcl}
A{+}B & = & A{\cdot}0 \bowtie B{\cdot}1 & 0 & = & \lceil\emptyset\rceil \\
A{\otimes}B & = & (\bowtie_{x'\in X_B} A{\cdot}x') \bowtie (\bowtie_{x\in X_A} x{\cdot}B) & \top & = & \lceil\{0\}\rceil \\
x(A^{\perp})y & \Leftrightarrow & yax & A^{\dagger} & = & \lceil Y_A\rceil
\end{array}
$$

We write $\lceil X\rceil$ for the membership relation from X to 2^X, which we take to be the Chu representation[2] of the set X. We write $A{\cdot}z$ for the result of renaming each x in the domain of A to (x, z) (without otherwise changing the relation); $z{\cdot}A$ renames each x to (z, x). Lastly we define the *natural join*[3] $A = \bowtie_i A_i$ of a family A_i of relations thus. Define $X = \bigcup_i X_i$, and define $Y = \{y \in \prod_i Y_i \mid \forall ij \forall x{\in}X_i{\cap}X_j[xA_iy_i = xA_jy_j]\}$. (So if the X_i's are disjoint, $Y = \prod_i Y_i$.) Define A from X to Y such that for each $x \in X_i$, $xAy = xA_iy_i$, well-defined in the event that any x appears more than once in this condition (non-disjoint X_i's) because of how we chose Y.

The Chu interpretation of 0 is the unique 0×1 relation, while $\top$ denotes the 1×2 relation $\begin{pmatrix}0 & 1\end{pmatrix}$. $A^{\dagger}$ denotes the $Y_A \times 2^{Y_A}$ relation $yA^{\dagger}Z = y{\in}Z$. $A^{\perp}$ is converse. $A{+}B$ is the $(X_A{+}X_B) \times (Y_A{\times}Y_B)$ relation (the 0 and 1 implement the disjoint union $X_A{+}X_B$) satisfying $(x, 0)(A{+}B)(y, y') = xAy$ and

[2] This generalizes immediately to $\lceil(X, \leq)\rceil$ for any poset, by interpreting 2^X to consist of just the order ideals of X rather than all subsets. This is the usual open-set representation of a poset as a topological space, a set then being just a discrete or unordered poset.

[3] The join operation comes from database theory. We take X to be the attributes or columns of the relation. In database terms $A \subseteq X{\times}Y$ is an X-ary relation (i.e. X is the set of attributes or columns) on the domain $\{0, 1\}$ consisting of a set Y (the rows) of records, each of which is an X-tuple of bits. This is the transpose of the usual view of A as an $X{\times}Y$ matrix.

$(x,1)(A{+}B)(y,y') = xBy'$, which we illustrate as follows.

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 1 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1 & 0 & 0\end{pmatrix}$$

The rows of $A{+}B$ are those of A followed by those of B, in order, while its columns are indexed in order by (y_0, y_0'), (y_1, y_0'), (y_0, y_1'), (y_1, y_1'), (y_0, y_2'), (y_1, y_2') where $y_i \in Y_A$, $y_i' \in Y_B$.

This leaves just $A{\otimes}B$, whose properties we summarize as follows.

Theorem 1 *$A{\otimes}B$ has domain $X_A{\times}X_B$ and codomain the set of all pairs of functions $(f : X_A \to Y_B, g : X_B \to Y_A)$ satisfying $xAg(x') = x'Bf(x)$, with $(x,x')(A{\otimes}B)(f,g) = xAg(x')$ $(= x'Bf(x))$.*

Proof: The domain of $\bowtie_{x'\in X_B} A{\cdot}x'$ can be seen to be $X_A{\times}X_B$, while by disjointness of $A{\cdot}x'$ as x' varies, the codomain is $\prod_{x'\in X_B} Y_A = Y_A^{X_B}$, i.e. the set of functions $g : X_B \to Y_A$. And the resulting relation A' is defined by $(x,x')A'g = xAg(x')$. Likewise $\bowtie_{x\in X_A} x{\cdot}B$ has the same domain, $X_A \times X_B$, has codomain $Y_B^{X_A}$, i.e. functions $f : X_A \to Y_B$, and is the relation B' defined by $(x,x')B'f = x'Bf(x)$. Hence the join of these two joins also has domain $X_A{\times}X_B$, while its codomain is that subset of the product $Y_A^{X_B}{\times}Y_B^{X_A}$ consisting of those pairs (f,g) such that for all (x,x') in $X_A \times X_B$, $(x,x')A'g = (x,x')B'f$, that is, $xAg(x') = x'Bf(x)$, this then being the value of $(x,x')(A{\otimes}B)(f,g)$. ∎

Corollary 2 *The domain of $A{\multimap}B$ is the set of all pairs of functions $(f : X_A \to X_B, g : Y_B \to Y_A)$ such that for all $x \in X_A$, $y' \in Y_B$, $f(x)By' = xAg(y')$* (cf. the definition of Chu transform in the section of that name, also cf. continuous functions of topological spaces where the Y's are taken to consist of open sets).

As a proposition, the first join repeats A "at X_B different locations," with a fresh set of variables of A for each location, while the second repeats B at locations X_A, with a fresh set of variables of B for each location, such that the two sets of repetitions use the same set $X_A \times X_B$ of variables. The join of the two is the conjunction of these two conditions, expressing the notion of *bilinearity* characteristic of tensor product. Although there is nothing "linear" about binary relations, the "linear" in linear logic expresses the thought that the essence of linear algebra resides in this property rather than in anything to do with the structure of fields [LS91].

We may view the relation A from X to Y as denoting the Boolean proposition P whose set of variables is X and each of whose assignments $s : X \to 2$ of truth values to those variables satisfies P just when there exists $y \in Y$ such that $\forall x \in X[s(x) = xAy]$. The y's in Y then correspond to satisfying assignments, or equivalently to clauses of the DNF form of the proposition. In this view, join is exactly the notion of conjunction of Boolean propositions.

We illustrate this definition of $A{\otimes}B$ with the following example.

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\\ 1 & 1 & 0 & 0\end{pmatrix}$$

The rows of $A{\otimes}B$ in order are (x_0, x'_0), (x_0, x'_1), (x_1, x'_0) (x_1, x'_1). Its columns in order are

$$\begin{array}{ll}(\{x_0{\mapsto}y'_0, x_1{\mapsto}y'_1\}, & \{x'_0{\mapsto}y_0, x'_1{\mapsto}y_1\}),\\ (\{x_0{\mapsto}y'_1, x_1{\mapsto}y'_0\}, & \{x'_0{\mapsto}y_1, x'_1{\mapsto}y_0\}),\\ (\{x_0{\mapsto}y'_2, x_1{\mapsto}y'_1\}, & \{x'_0{\mapsto}y_0, x'_1{\mapsto}y_1\}),\\ (\{x_0{\mapsto}y'_1, x_1{\mapsto}y'_2\}, & \{x'_0{\mapsto}y_1, x'_1{\mapsto}y_0\}),\end{array}$$

these four being the only compatible pairs of (f, g)'s out of the $3^2 \times 2^2 = 36$ possibilities. For example the first column is indexed by the given $(f : X \to Y', g : X' \to Y)$ specifying that the entry in the first row of that column should be $x_0 A g(x'_0) = x_0 A y_0$ $(= x'_0 B f(x_0) = x'_0 B y'_0) = 1$.

These operations on relations are somewhat more intricate than those of the Peirce calculus. The idea however is that one should not work directly with the interpretation but rather indirectly with its logical properties, which is also the idea behind the Peirce calculus. Since the respective logics are of comparable complexity, the gains are potentially greater with Chu logic than with Peirce logic (provided one never has to resort to the explicit interpretation), since more complex machinery is being manipulated at no additional cost in logical complexity. While more complex does not always mean more powerful, in this case a small increase in complexity turns out to lead to a considerable increase in power.

2.3 K-valued Relations

We now make a small generalization to the notion of binary relation that gives a large increase in the power of the Chu calculus. We allow K-valued binary relations where K is an arbitrary set. Thus instead of xAy either holding or not, it has a value from K. More formally, A is a triple (X, Y, a) where X and Y are sets and $a : X{\times}Y \to K$ is a K-valued function.

We need consider only $\lceil - \rceil$ and $\bowtie$. We generalize $\lceil X \rceil$ from membership of elements of X in elements of 2^X to application of elements of K^X (i.e. functions $f : X \to K$) to elements of X.[4] And the join operation immediately generalizes from $\{0, 1\}$-valued to K-valued relations since nothing in the definition of join depended on that special case.

2.4 Equational Laws

We have $a{+}b = b{+}a$ for the Peirce calculus, and it is natural to expect this for the Chu calculus as well. However on closer inspection we notice that each x in

[4]The generalization to posets $\lceil (X, \leq) \rceil$ is intended only for $K = 2$, though some analogous notion might be possible for larger K, in particular for power sets $K = 2^n$.

the domain of A becomes $(x, 0)$ in $A+B$ but $(x, 1)$ in $B+A$. We may however claim the isomorphism $A+B \cong B+A$, in which $(x, 0)$ in $A+B$ is matched up with $(x, 1)$ in $B+A$. This applies to the other laws of the Chu calculus as well. (A is isomorphic to B when there exist bijections $X_A \cong X_B$, $Y_A \cong Y_B$ of their index sets making their corresponding entries equal.) The full list of isomorphisms (and equalities where possible) we know to hold for extensional T_0 Chu spaces, those with no repeated rows or colums (discussed in more detail in the paragraph following Definition 3 below), is as follows.

$$\begin{array}{lclclclcl} A+(B+C) & \cong & (A+B)+C & A+0 & \cong & A & A+B & \cong & B+A \\ A\otimes(B\otimes C) & \cong & (A\otimes B)\otimes C & A\otimes\top & \cong & A & A\otimes B & \cong & B\otimes A \\ A\otimes(B+C) & \cong & (A\otimes B)+(A\otimes C) & A\otimes 0 & \cong & 0 & A^{\perp\perp} & = & A \end{array}$$

Without attempting to be complete (though this should be pretty close to the expressible consequences of the RA axioms [JT48, JT52]), the Peirce laws include all these less $a;b = b;a$ and $(a+b)^\ddagger = a^\ddagger;b^\ddagger$ (spoiled by the noncommutativity of composition), plus the Boolean properties expressible in the available language, e.g. idempotence and distributivity over each other of ab and $a+b$, along with $a;((a;b)^\perp;a)^\perp = a;b$, and all regular expressions involving * (those not involving * being already covered).

From the Chu laws and the definitions of abbreviations we can derive for example $(A\otimes B)\multimap C = (C^\perp\otimes(A\otimes B))^\perp \cong ((C^\perp\otimes A)\otimes B)^\perp = B\multimap(C^\perp\otimes A)^\perp = B\multimap(A\multimap C)$. We leave $(A\times B)\Rightarrow C \cong A\Rightarrow(B\Rightarrow C)$ as an exercise. We are not aware of any completeness results for the isomorphism theory of the Chu calculus.

An equivalent to $(A+B)^\dagger \cong A^\dagger\otimes B^\dagger$ is $!(A\times B) \cong\, !A\otimes !B$, but this uses the nonprimitive $\times$, our first reason for taking $A^\dagger$ as primitive rather than $!A$, the contravariance of $A^\dagger$ notwithstanding. This law just asserts the previously noted fact that the codomain of a sum is the product of the codomains of the arguments, the naturality of which is our second reason for preferring $A^\dagger$ over $!A$.[5] Abstracting away $A+B$, we obtain the law $\lceil X\times Y\rceil \cong \lceil X\rceil\otimes\lceil Y\rceil$ for sets X, Y. This remains valid when X and Y are generalized to posets, these forming a cartesian closed category. That is, to form the cartesian product of sets and posets when represented as Chu spaces, form their tensor product in the Chu calculus, not their direct product $A\times B$ which yields something different. The product of join-semilattices, when these are represented as Chu spaces over 2 whose rows are closed under finite bitwise OR (union), is however not formed by tensor product; in fact the tensor product of a meet-semilattice with a join-semilattice is a distributive lattice (since tensor product works by conjoining properties), details in a future paper.

[5]It seems plausible to us that both are needed, in that $!A$ may well be more appropriately interpreted as either the symmetric (boson) or antisymmetric (fermion) tensor algebra generated by A, as contemplated in recent unpublished work of Blute, Panangaden, and Seely on " Old Foundations for Linear Logic: Holomorphic Functions in Banach Spaces as Models of Exponential Types," concerning the Fock space interpretation of $!A$, and of Blute on "Modelling linear logic with vector spaces," a forthcoming talk at the Cornell workshop on linear logic, June 1993. We hope to understand this issue better in the near future.

Given that the linear logic primitives are all definable with $\lceil - \rceil$, $\bowtie$, and $A^\perp$, it may be worth investigating taking these as an even simpler basis for linear logic, suitably organized.

Note that $A^\dagger$, $A^{\dagger\dagger}$, $A^{\dagger\dagger\dagger} \ldots$ is $Y_A, K^{Y_A}, K^{K^{Y_A}}, \ldots$, in contrast to $!!A = !A$.

2.5 Historical Notes

In introducing linear logic, Girard proposed phase spaces and coherence spaces [Gir87] as respectively nonconstructive and constructive interpretations, the distinction being whether each sequent $\Gamma \vdash \Delta$ is considered to denote a truth value or a set of proofs of Δ from Γ. But when Girard presented his logic at a category theory conference in Boulder in 1988, M. Barr recognized the suitability for modeling linear logic of his *-autonomous categories in general and his student P. Chu's construction of such in particular [Bar79]. Chu spaces, as the objects of Chu's construction for the category of sets, seem to be a particularly attractive constructive model of linear logic.

We have been using $A+B$ and $A \otimes B$ in our concurrency work, starting with [Pra86] where they are notated respectively $A \| B$, called *concurrence* (meaning noninteractive asynchronous parallel composition) (p.47), and $A \times B$, *orthocurrence*, meaning flow or mixing of one process through or in another (p.49, also §3), an interactive form of parallel composition. Subsequently it was realized [CCMP91, p.208] that $A \times B$ should have been tensor product $A \otimes B$; the confusion with $A \times B$ occurred because the earlier work was conducted in the category **Pos**, which being cartesian closed identifies the two. The confusion was exposed when the passage from ordered time to real time broke the $A \times B$ definition. By the same token the Chu representation of the direct product of posets (conflict-free schedules) is the tensor product of the Chu representations of those posets, but this does not extend to schedules having conflicts and other forms of causal structure. Our interpretations of concurrence and orthocurrence, which have been evolving over the intervening years, appear to have been moving steadily towards the Chu interpretation. It remains to connect up the Chu interpretation with the yet more general interpretations of [CCMP91], which we expect to be only a matter of details.

With regard to orthocurrence as flow, e.g. of a sequence A of trains through a sequence B of stations, the bilinearity expressed in the equation defining tensor product corresponds to the notion that when we stand on the platform of any station $b \in X_B$ we see the same sequence A of trains, and vice versa when we watch the stations go by from any train $a \in X_A$.

We have been using $A^\perp$ only relatively recently [Pra92b, Pra92a], as the basic link between schedules and automata, and as complementarity in quantum mechanics [Pra93]. Automata express behavior as graphs with states as vertices and events as edges; schedules dualize this by interchanging them, with $A^\perp$ denoting the automaton form of the schedule A. The generalization of the event spaces of [Pra92b, Pra92a] to the Chu spaces of this paper is anticipated by the partial distributive lattices of [Pra93, §5], which are essentially Chu spaces, whose role in this application we defer to a separate paper.

3 Chu Spaces and Transforms

We now imbue a binary relation with a spatial character by taking its domain to be its point set, and its codomain to be its degrees of freedom or *states*, reflected in the following notation.

Definition 3 *A Chu space $A = (P, S, v)$ over a set K consists of a set P of points, a set S of states, and a function $v : P \times S \to K$ assigning a value $v(p, s)$ to each point p in each state s.*

We associate with $v : P \times S \to K$ the functions $v_- : P \to (S \to K)$ and $v^- : S \to (P \to K)$ satisfying $v_-(p)(s) = v^-(s)(p) = v(p, s)$. We abbreviate $v_-(p)$ to v_p, called the *extension* of p, and $v^-(s)$ to v^s, the *extension* of s. We think of v^s as one of the permitted *paintings* of the underlying set P, with K for our palette. Dually v_p is a painting of S, understood as the varying values of one point encountered as one traverses the "possibility space" of alternative paintings of P. We write $V^* = \{v^s \mid s \in S\}$ for the set of extensions of states, through which $S \stackrel{v^-}{\to} K^P$ factors as $S \to V^* \to K^P$. Dually $P \stackrel{v_-}{\to} K^S$ factors through the set $V_* = \{v_p \mid p \in P\}$ of extensions of points as $P \to V_* \to K^S$. When all states have distinct extensions, i.e. v^- is injective ($S \cong V^*$), we call v *extensional* (shorter and more mnemonic than Barr's "right separated" [Bar91]) and say it has *enough points* (to distinguish states). (This situation is very important, allowing us to interpret (P, S) for $S \subseteq K^P$ as a Chu space.) When all points have distinct extensions ($P \cong V_*$) we call A T_0 (by analogy with the topological property of that name), and say it has *enough states* (Barr: left separated). Locales are the prototypical example of a nonextensional but T_0 space [Vic89, p.61].

Logically speaking, points are *necessary* in the sense that they are necessarily all present in the space at the one time. Dually, states are *possible*, in that the space is in one state *or* another, like the possible worlds of a Kripke structure (our conventional understanding of spaces does not permit us to imagine that the whole space is in all states simultaneously).

More states mean more degrees of freedom, corresponding to less structure. At one extreme the extensional space (P, K^P) contains all possible states and hence has the vacuous structure of a set or discrete space, which we shall identify with the set P itself. We view the omitted states, those in $K^P - V^*$, as the *atomic* properties of the space, collectively constituting the theory or *structure* of the space. At the other extreme the space $(P, \emptyset)$ omits all states, which we view as the inconsistent structure on P. A one-state space, S a singleton, is "rigid," every point having a uniquely determined or constant value. The canonical rigid space is $(K, \{0\})$, which we denote $\perp$ (not K, which a moment ago we associated, in its capacity as a set, with the discrete space (K, K^K)).

The dual of $A = (P, S, v)$ is the *state space* $A^\perp = (S, P, v^\smile)$ where $v^\smile(s, p) = v(p, s)$. Duality interchanges points and states, and hence necessity and possibility. This confers on duality one of the qualities of logical negation, another being double negation, $A^{\perp\perp} = A$.

We turn now to the notion of a transform of Chu spaces, foreshadowed in Theorem 1.

Definition 4 *A Chu transform* $(f, g) : (P, S, v) \to (Q, T, w)$ *consists of functions* $f : P \to Q$, $g : T \to S$ *satisfying* $v(p, gt) = w(fp, t)$ *for all* $p \in P$ *and* $t \in T$. *Composition of* $(f', g') : (Q, T, w) \to (R, U, X)$ *with* $(f, g) : (P, S, v) \to (Q, T, w)$ *is defined by* $(f', g')(f, g) = (f'f, gg')$, *satisfying* $v(p, gg'u) = w(fp, g'u) = x(f'fp, u)$ *and hence a Chu transform. Associativity is inherited from that of function composition. The identity transform on* (P, S, v) *is the pair* $(1_P, 1_S)$ *of identity functions on* P, S *respectively. We abbreviate* $(f, g) : (P, S, v) \to (Q, T, w)$ *to* $f : A \to B$ *when unambiguous.*

It is easy to see from the explanation of $A \otimes B$ in the previous section that defining $A \multimap B$ to be $(A \otimes B^{\perp})^{\perp}$ makes it the Chu space whose points are the linear transformations from A to B.

We denote by $\mathbf{Chu}(K)$ the category of Chu spaces and their Chu transforms so composed.[6] $A + B$ and $A \times B$ as defined for the Chu calculus are respectively coproduct and product in this category, which $A \multimap B$ is the internal hom and $A \otimes B$ its associated tensor product.

Taking the duals $A^{\perp} = (S, P, v^{\smile})$ and $B^{\perp} = (T, Q, w^{\smile})$ of the domain and codomain of $(f, g) : (P, S, v) \to (Q, T, w)$ necessarily entails replacing (f, g) by (g, f). To construe this as a Chu transform we must then treat it as $(g, f) : (T, Q, w^{\smile}) \to (S, P, v^{\smile})$. That is, duality reverses the direction of transforms, much as transposing a matrix reverses the direction of the linear transformation it defines.

As mentioned in less detail earlier, Chu spaces first arose as the objects of the self-dual symmetric closed monoidal category produced by Chu's construction from a symmetric closed monoidal category $\mathcal{V}$, for the case $\mathcal{V} = \mathbf{Set}$ [Bar79]. Lafont and Streicher have more recently called the objects of this case *games* [LS91]. They observed that vector spaces over a field K may be realized as games over the underlying set of K, and that topological spaces may be realized as games over 2. We observe that Chu transforms of Chu posets $\lceil (X, \leq) \rceil$ realize exactly monotone functions of posets, which generalizes to a comprehensive analysis of Stone duality viewed as a continuum from sets to complete atomic Boolean algebras, to be treated elsewhere.

4 Unique Factorization of Chu Transforms

In this section we develop a factorization yielding a useful insight into what the Chu transform accomplishes, namely the proper management of states or degrees of freedom as the complement of structure, that which is preserved by transformations.

[6] This definition of $\mathbf{Chu}(K)$ takes place in the category $\mathbf{Set}$ of sets and functions, with K as a distinguished set. By generalizing $\mathbf{Set}$ to any symmetric closed monoidal category $\mathcal{V}$ and K to any object K of $\mathcal{V}$, we may correspondingly generalize the above definition to the doubly parametrized category $\mathbf{Chu}(\mathcal{V}, K)$ defined by Barr and Chu [Bar79].

Any function $X \xrightarrow{f} Z$ factors uniquely as $X \xrightarrow{f_1} X/\ker f \xrightarrow{f_2} Y \xrightarrow{f_3} Z$, where f_1 is the quotient of X induced by the kernel of f (a surjection), f_2 is an isomorphism, and Y is a subset of Z. That is, any $f : X \to Z$ may be viewed in exactly one way as transforming X by identifying certain of its elements, renaming the resulting elements to certain elements of Z to form the image $f(X)$, and adjoining $Z - f(X)$. The uniqueness of this factorization depends on the identifications being performed before the additions, since there are many ways of adding multiple copies of $Z - f(X)$ and then identifying them, indeed a proper class of such ways.

Applying this factorization to each of the two components of a Chu transform (f, g) yields $(f_3 f_2 f_1, g_3 g_2 g_1)$. But this now has many possible factorizations, namely the $\binom{6}{3} = 20$ possible merges of $(f_3, 1)(f_2, 1)(f_1, 1)$ with $(1, g_1)(1, g_2)(1, g_3)$.

Each of the two functions (f, g) of a Chu transform factor in this way, with the contravariance of g interchanging the roles of epi and mono for its factors.

We show here a more elaborate unique factorization for Chu transforms (f, g), namely the process described informally as, omit states, then in parallel duplicate states and identify points, and finally add points. This draws distinctions finer than those made by epis and monos, which view both state duplication and point addition as monos, and the other two as both epis. Lacking more abstract terms for these four notions, we adopt our informal names as official and call this the *ODIA* (oh-dear) *factorization* for $\mathbf{Chu}(K)$, properly written $O; (D + I); A$.

We accomplish this factorization in two steps. We refer to the OD half, omission then deletion of states, as an *erasure*, and the IA half, identification then addition of points, as a *move*. We prove unique factorization into an erasure followed by a move. The rest of the ODIA factorization is then an immediate corollary of the surjection-injection factorizations of f and g individually, about which we need say nothing further.

Definition 5 *An erasure is a Chu transform of the form* $(1, g)$*; we let* $\mathcal{E}$ *denote the class of all erasures of* $\mathbf{Chu}(K)$*. Dually a move is of the form* $(f, 1)$*, forming the class* $\mathcal{M}$*.*[7]

An erasure $(1, g) : (P, S, v) \to (P, T, h)$ modifies only states: every state $t \in T$ receives its extension from state $g(t) \in S$. Hence $H^* \subseteq V^*$, whence any state omitted from $A = (P, S, v)$ remains omitted from $H = (P, T, h)$, i.e. *erasure preserves structure.* (Thus we may identify the *omitted* states, namely $K^P - V^*$, with the *structure* of the Chu space (P, S, v).) A state of A may be *duplicated* in H, but we distinguish duplication of existing states from creation of new states.

A move $(f, 1) : (P, T, h) \to (Q, T, w)$ modifies only points, mapping each point $p \in P$ to a point $f(p) \in Q$ having the same extension as p. Since the

[7]Thus $\mathcal{E} \cap \mathcal{M}$ consists of the identities of $\mathbf{Chu}(K)$; when it is necessary that it consist of the isomorphisms, here bijections, it suffices to close $\mathcal{E}$ and $\mathcal{M}$ under composition with bijections. Normally $\mathcal{E}$ and $\mathcal{M}$ consist of respectively epis and monics, our little pun receives some legitimacy from the observation that $(1, g)$ is an epi of $\mathbf{Chu}(K)$, and $(f, 1)$ a monic, just when both f and g are injective.

image $f(H)$ of a move "receives" its states from H, states are neither created nor destroyed in $f(H)$. The "shape" of a state changes concomitantly with that of P under f, but the values of points of P in a given state are not changed as they move to their new locations in Q, all necessary changes in value having been previously accomplished by the erasure.

Bear in mind that the transform $f : A \to B$ maps A *into* B, and that the noncreation of states refers only to A thus transformed, not to the whole target. While the image $f(\mathbb{R})$ of the linear transformation $f : \mathbb{R} \to \mathbb{R}^2$ has at most the degrees of freedom of $\mathbb{R}$, $\mathbb{R}^2$ has visibly more degrees of freedom than $\mathbb{R}$.

Theorem 6 *Every transform factors uniquely and functorially as the composition me of a move m with an erasure e.*

Proof: Uniqueness is immediate: $(f, g) : (P, S, v) \to (Q, T, w)$ must factorize as $(f, 1)(1, g)$. For existence define the intermediate object (P, T, h) as $h(p, t) = v(p, gt)$, making $(1, g)$ a transform. But $v(p, gt) = w(fp, t)$, whence $(f, 1)$ is also a transform.

Functoriality means that any transform of h to h', as a commuting square with sides u, w (Figure 1(a)), factors uniquely as the composition of transforms m to m' with e to e' (Figure 1(b)), mediated by a unique transform v.

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ A' & \xrightarrow{h'} & C' \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{e} & B & \xrightarrow{e'} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ A' & \xrightarrow{m} & B' & \xrightarrow{m'} & C' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{m} & B' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle e_u}\downarrow & & \downarrow{\scriptstyle e_v} \\ A'' & \xrightarrow{i} & B'' \\ {\scriptstyle m_u}\downarrow & & \downarrow{\scriptstyle m_v} \\ A' & \xrightarrow{m} & B' \end{array}$$

(a) (b) (c) (d)

Figure 1

This can be seen to be equivalent to the requirement that all squares of the form shown in Figure 1(c) have a unique *diagonal fill-in* from B to A'. This is seen by factoring u and v as $m_u e_u$ and $m_v e_v$ respectively as in Figure 1(d). Each of the equal sides $AA'B'$, ABB' of the commuting square has now been $\mathcal{EM}$-factored, as $mm_u e_u$ and $m_v e_v e$ respectively. But $\mathcal{EM}$ factorization is unique (up to a bijection if we have closed $\mathcal{E}$ and $\mathcal{M}$ under composition with bijections), yielding the identity (or a bijection) i from A'' to B''. The sides ABB', $AA'B$ of the square determine $m_v i e_u : A \to B'$, hence $i : A'' \to B''$, hence $i^{-1} : B'' \to A''$, making the diagonal fill-in $m_u i^{-1} e_v : B \to A'$. ∎

Returning to the notion of S as the possible states of (P, S, v), the geometric significance of this unique factorization is that every transform can be viewed as taking place in two stages. First the theory of the space being transformed is strengthened in preparation for the coming move, by erasing suitable states (and permuting and duplicating some of the surviving states), without however moving the points themselves, and without introducing any new states. Then the points are moved in a way that does not modify their assignments in each of the new states.

The $\mathcal{EM}$ factorization thus separates transforms into a purely structure-preserving part followed by a purely point-moving part. This constitutes the dynamic confirmation of our previous static analysis of P and S as the sets of respectively necessary points and possible states. We may think of the omitted states dually as necessary facts. From this perspective, transforms preserve that which is necessary, namely points and facts.

It should be clear from this analysis of the factorization of (f, g) through H that f, g, and h are by no means independent. Indeed either of f or g suffice to determine h. Further, if A is extensional then h (and hence f) determines g, while if B is T_0, h (and hence g) determines f. It follows that our convention of abbreviating (f, g) to $f : A \to B$ is unambiguous when A is extensional.

5 Power of Chu Spaces

We have already mentioned Lafont and Streicher's observation [LS91, p.45] that the category of vector spaces over a field **K** is a full subcategory of $\mathbf{Chu}(K)$, and that the category **Top** of topological spaces is a full subcategory of $\mathbf{Chu}(2)$. We improve on these observations by showing that *every* n-ary relational structure is realizable as an object of $\mathbf{Chu}(2^n)$, giving a very strong sense in which Chu spaces form a universal category.

The earliest instance of a universal category is due to Trnková [Trn66]. The universality of the category of semigroups was established by Hedrlín and Lambek [HL69]. These and a number of other such embeddings all took the form of a full and faithful functor that did not preserve underlying sets, for example representing some finite objects as infinite ones. The advantages accruing from the unifying framework of semigroups are then more than offset by the radically different discipline required to do mathematics in the absence of the expected underlying set.

Pultr and Trnková [PT80] call the kind of *concrete* full embedding we aim for here a *realization*: the functor $F : C \to D$ realizes object A of C when not only is F full and faithful, but $U_D(F(A)) = U_C(A)$, where $U_C : C \to \mathbf{Set}$, $U_D : D \to \mathbf{Set}$ are the respective underlying-set functors. Pultr and Trnková give hardly any realizations, concentrating on weaker forms of full embeddings. In contrast the embedding here is a realization, and a simple one at that.

Here by "A represented as B" we shall mean throughout that the category C_A of all A's *fully* embeds[8] in the category C_B of all B's.

Definition 7 *For any ordinal n, an n-ary relational structure (X, ρ) consists of a set X, the carrier, and an n-ary relation $\rho \subseteq X^n$ on X. A homomorphism $f : (X, \rho) \to (Y, \sigma)$ between two such structures is a function $f : X \to Y$ between*

[8] An embedding is a faithful functor $F : C_A \to C_B$, i.e. for distinct morphisms $f \neq g$ of C_A, $F(f) \neq F(g)$, and is *full* when for all pairs a, b of objects of C_A and all morphisms $g : F(a) \to F(b)$ of C_B, there exists $f : a \to b$ in C_A such that $g = F(f)$.

their underlying sets for which $f\rho \subseteq \sigma$. Here $f\rho$ denotes $\{f\mathbf{a} \mid \mathbf{a} \in \rho\}$, where $\mathbf{a}$ *denotes $(a_0, \ldots, a_{n-1})$ and $f\mathbf{a}$ denotes $(fa_0, \ldots, fa_{n-1})$. We denote by $\mathbf{Str}_n$ the category formed by the n-ary relational structures and their homomorphisms.*

It suffices to treat structures with a single carrier and relation, since k carriers can be combined as their disjoint union, kept track of with k unary relations ($\lceil \log_2 k \rceil$ is enough information-theoretically, but not enough to ensure that homomorphisms respect type). Multiple nonempty relations on a set can be joined to form a single relation on the same set, of arity at most the sum of the arities of its constituent relations. For algebras, structures all of whose $(n+1)$-ary relations are n-ary operations, the join may share the input coordinates of the operations, reducing the total arity to the maximum of the input arities plus the number of operations (including constants).

This notion of homomorphism is standard in the strong sense that *any* class of n-ary relational structures and their homomorphisms constitutes a full subcategory of $\mathbf{Str}_n$. Familiar examples of such categories and their arities include those of semigroups (3), monoids (4), groups (3), rings (4), rings with a multiplicative unit (5), fields (4), lattices (3), lattices with top and bottom (5), Boolean algebras (3), vector spaces (4),[9] directed graphs or binary relations (2), multigraphs (4), posets (2), and categories (4).

Many of these numbers benefit from group structure, for which homomorphisms preserve inverses and identities even when these operations are not given explicitly as part of the relation. Units of monoids, including tops and bottoms of lattices, are not so fortunate and each requires its own unary relation in order to be recognized and preserved by homomorphisms.

The universality achieved here is of a different kind from that achieved by say ZF set theory. Externally a model of ZF is a single object of $\mathbf{Str}_2$ of some cardinality, with membership as its only relation, "internally" coding objects larger than any fixed cardinal including its own. Our universality has no separate notion of an internal world; instead we code our objects purely externally.

We now define the promised functor $F : \mathbf{Str}_n \to \mathbf{Chu}(\mathbf{2}^n)$, namely in definitions 9 and 13, and prove that it is full, faithful, *and concrete*.

The complementarity of constraints and states indicates ρ and $\overline{\rho}$ as the appropriate respective sources of each. We shall define a state to be essentially a subset of $\overline{\rho}$, with however a small but essential refinement. The following lemma obtains from the standard constraint-based definition of homomorphism an equivalent state-based characterization.

Lemma 8 $f\rho \subseteq \sigma \;\Leftrightarrow\; f^{-1}\overline{\sigma} \subseteq \overline{\rho}$. Here $\overline{\rho} = A^n - \rho$ and $\overline{\sigma} = B^n - \sigma$.

Proof:

$$\begin{array}{rcll} f\rho \subseteq \sigma & \Leftrightarrow & \rho \subseteq f^{-1}\sigma & \text{(Definition of } f^{-1}\text{)} \\ & \Leftrightarrow & \overline{f^{-1}\sigma} \subseteq \overline{\rho} & \text{(Complement)} \\ & \Leftrightarrow & f^{-1}\overline{\sigma} \subseteq \overline{\rho} & (f^{-1} \text{ preserves Boolean operations}) \end{array}$$

[9]Treat as partial rings, with uv defined just when u is on a specified axis. This works equally well for homogeneous vector spaces (all over the one field) and heterogeneous, the only nontrivial field endomorphisms being automorphisms.

■

Definition 9 (*F* on objects). *Let* $\mathbf{2}^n$ *denote the set of* n*-bit bit vectors, that is,* n*-tuples over* $\mathbf{2}$*. We define the object part of the functor* $F : \mathbf{Str}_n \to \mathbf{Chu}(\mathbf{2}^n)$ *as taking the* n*-ary relational structure* (A, ρ) *to the Chu space* (A, R, v) *defined as follows. Take* R *to consist of those* n*-tuples* $r \in (2^A)^n$ *of subsets of* A *for which* $\prod_i r_i \subseteq \overline{\rho}$*. Let* $v : A \times R \to 2^n$ *satisfy* $v(a, r)_i = 1$ *if* $a \in r_i$*, and 0 otherwise.*

It might seem that R could be represented more naturally and conveniently as just the power set of $\overline{\rho}$. But observe that a state r as defined here can be recovered from the set $\prod_i r_i$ of its n-tuples just when no component r_i is empty. The definition of $f^{-1} : S \to R$ in Definition 13 below requires each r_i to be available independently even when some are empty.

The crucial test of whether (A, R, v) faithfully represents (A, ρ) is whether ρ can be recovered from (A, R). We show this constructively as follows.

Lemma 10 *For all* $\mathbf{a} \in A^n$*,* $\mathbf{a} \in \rho \;\Leftrightarrow\; \forall r{\in}R\; \exists i{<}n : v(a_i, r)_i = 0$.

Proof:

$$\begin{array}{rlll} \mathbf{a} \in \rho & \Leftrightarrow & \forall r{\in}R : \mathbf{a} \notin \prod_i r_i & \text{(Construction of } R\text{)} \\ & \Leftrightarrow & \forall r{\in}R\; \exists i{<}n : a_i \notin r_i & \text{(Definition of product)} \\ & \Leftrightarrow & \forall r{\in}R\; \exists i{<}n : v(a_i, r)_i = 0 & \text{(Construction of } v\text{)} \end{array}$$

■

Corollary 11 *F is injective on objects.*

Lemma 12 (A, R, v) *is extensional.*

Proof: If $v^r = v^{r'}$ then $\forall i[a \in r_i \Leftrightarrow a \in r'_i]$, so $\forall i : r_i = r'_i$, whence $r = r'$. ■

If we regard R as a subset not of $(2^A)^n$ but of the isomorphic $(2^n)^A$, this makes Lemma 12 clear by qualifying (A, R) as an extensional object of $\mathbf{Chu}(2^n)$. We may view (A, R) for arbitrary $R \subseteq (2^n)^A$ as a generalization of $(A, \overline{\rho})$, which in the case $n = 1$ reduces to ordinary binary relations, which as previously noted capture topological spaces along with other similar structures such as complete lattices etc.

Definition 13 (*F* on maps). *Let* $f : (A, \rho) \to (B, \sigma)$ *be a homomorphism, with* $F(A, \rho) = (A, R, v)$ *and* $F(B, \sigma) = (B, S, w)$ *as per Definition 9. Define* $f^{-1} : (2^n)^B \to (2^n)^A$ *to take* $g : B \to 2^n$ *to* $gf : A \to 2^n$*. Now for all* $s \in S$*,* $\prod_i s_i \subseteq \overline{\sigma}$ *by construction of* S*. Hence* $\prod_i f^{-1} s_i \subseteq \overline{\rho}$*, by Lemma 8. Thus* $f^{-1} s \in R$ *by construction of* R*. We may therefore define* $F(f)$ *as* (f, f^{-1}) *where* $f^{-1} : S \to R$*.*

Theorem 14 *The functor F of Definitions 9 and 13 is concrete, faithful, and full.*

Proof: F is concrete by construction, and *a fortiori* faithful.

For fullness consider any Chu transform $(f, g) : F(A, \rho) \to F(B, \sigma)$ where $F(A, \rho) = (A, R)$ and $F(B, \sigma) = (B, S)$. If $\mathbf{a} \in \rho$, then for every $s \in S$ there exists $i < n$ such that

$$\begin{array}{rrcll} & v(a_i, gs)_i & = & 0 & \text{(Lemma 10 with } r = gs\text{),} \\ \text{whence} & w(fa_i, s)_i & = & 0 & \text{((}f, g\text{) is a Chu transform).} \end{array}$$

Hence by Lemma 10, $f\mathbf{a} \in \sigma$, establishing that f is a homomorphism. And since (A, R, v) is extensional, by Lemma 12, g is determined by f. Hence $F(f) = (f, g)$. ∎

Remarks. (i) Where size matters, R need contain only those states representable as the inverse image of a tuple of singletons. These can be characterized explicitly as those states r with the property that either $r_i = r_j$ or $r_i \cap r_j = \emptyset$ for all $i, j < n$, observing that f^{-1} preserves this property. (ii) Lemma 12 is an inessential bonus. Had Definition 9 produced a nonextensional (A, R, v), we would simply have enforced extensionality, needed for fullness, by identifying those states having the same extension.

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