Scheduling Algebra

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Abstract

The goal of this paper is to develop an algebraic theory of process scheduling. We specify a syntax for denoting processes composed of actions with given durations. Subsequently, we propose axioms for transforming any specification term of a scheduling problem into a term of all valid schedules. In particular, we axiomatize an operator that restricts attention to the *efficient* schedules. These schedules turn out to be representable as trees, because in an efficient schedule actions can only start at time zero or when a resource is released, i.e. upon termination of the action binding a required resource. All further delay would be useless. Nevertheless, we do not consider resource constraints explicitly at the time being. We show that a normal form exists for every closed term of the algebra and establish soundness of our axiom system with respect to a schedule semantics, as well as completeness for efficient processes.

Introduction

The goal of this paper is to develop an algebraic theory of scheduling. The ultimate objective is to find an optimal schedule for a given process subject to a set of resource constraints and some goal function. Here we abstract from both resource constraints and goal function, and generate those schedules that are potentially optimal for certain constraints and goal functions, provided the latter does not favor spending idle time. Such schedules are called *efficient*. Actions are assumed to require a specific amount of time for their execution. For simplicity, we only consider specifications in which no action has more than one immediate causal predecessor, i.e. the precedence graph has a multi-tree order.

An example process is given in fig. 1.

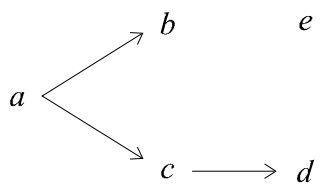

Fig. 1: Example process

The algebraic term for this process is: $a; (b \parallel c; d) \parallel e$

To visualize a schedule, the so-called GANTT chart [Cla35] is used. It depicts every action as a box labeled with its identifier. The length of the box indicates the duration of the action (see fig. 2).

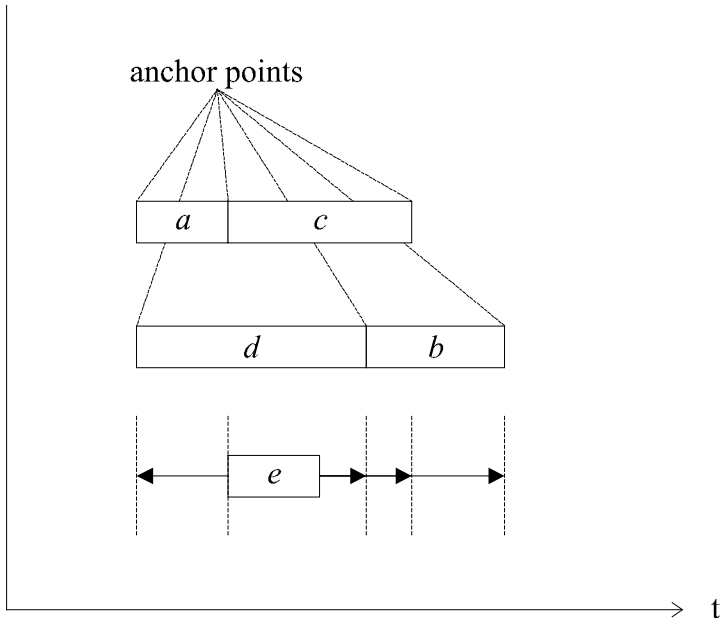

Fig. 2: GANTT diagram

It has already been observed [BEP96] that in an efficient schedule (and hence in an optimal one) each process either starts at time 0 or at the termination time of some other process. So if the schedule for the actions *a* to *d* is already given as in fig. 2, an additional process *e* may start at time 0 if it does not use a resource required by either *a* or *d*. If *e* conflicts with *a* but not with *b*, it may begin execution upon termination of *a* (as drawn in fig. 2). So in total, we have five possibilities for the starting time of *e*:

0 or the termination time of *a*, *b*, *c* or *d*.

We call these points in time ‘anchor points’.

In literature, a lot of algebras of concurrent processes are given, e.g. ACP [BK84], CCS [Mil80] or CSP [Hoa85]. They have been extended to treat time (e.g. ACP [BB91], TCCS [MT90], TCSP [RR88]). But most of them still view actions as instantaneous. Our approach however assigns each action a duration. In addition, the existing algebras do not incorporate a concept of efficiency. Hence, a new algebra is necessary. We call it *scheduling algebra* and develop its theoretical framework in the following sections.

Syntax

Let A be a set of *atomic actions* (external actions), and let T be the set of positive reals including zero. A pair (a, t) of $A \times T$ written as $a(t)$ is called a time-stamped action or activity. It

indicates that action *a* takes *t* units of time (the duration of *a*). The elements *t* of *T* also denote internal time actions which can be seen as silent steps (τ, *t*).

Process terms can be constructed using the constants

ε	denoting the empty process,
$a(t)^+ \in A \times T$	denoting the starting of action *a* with duration *t*,

the unary operators

$a(t)\cdot$	*sequence*, $a(t)\cdot x$ meaning that *x* starts immediately after the execution of *a*, i.e. at time *t*,
$t\cdot$	*time offset* for $t \in T$; in $t\cdot x$ the starting time of the actions in *x* is delayed by *t* time units and an anchor point at time *t* is added,
Δ	arbitrary *delay operator*, meaning that the prefixed process can be postponed indefinitely,
[.]	*delay elimination*, deleting every occurrence of the delay operator from the enclosed term by attaching the processes prefixed by Δ to every possible anchor point,

and the binary operators

$\cup$	*choice*; $x \cup y$ denotes the execution of either *x* or *y* exclusively,
$\curlywedge$	*forking of time* ('Hühnerfuß' or 'chicken claw'); $x \curlywedge y$ means that the initial actions of both *x* and *y* start at the same time.

In addition, we use the following abbreviations:

$a(t);$	(*causal*) *precedence*, a unary operator expressing linear order, so $a(t); x$ says that *x* has to be executed some time (but not necessarily immediately) after completion of *a*. We have: $a(t); x = a(t)\cdot \Delta x$.
$\parallel$	*concurrence*, a binary operator expressing independence of processes. We have: $x \parallel y = \Delta x \curlywedge \Delta y$.

The binding precedence of the binary operators (from loose to tight) is: $\cup$, $\parallel$ and $\curlywedge$. The unary operators bind tightest.

We take *a*, *b* to range over $A \times T \cup A$, *t*, *u*, *v* to range over *T* and *x*, *y*, *z* to range over all process expressions. For convenience, we sometimes abbreviate the expression $a(t)$ by *a*. Moreover, we leave out trailing ε's in terms $a(t)\cdot \varepsilon$ and $t\cdot \varepsilon$.

Example: The expression $a; b \parallel c; d$ denotes two concurrent tasks, one executing first *a* and then *b*, and the other running *c* followed by *d*. Example schedules for the efficient process $[a; b \parallel c; d]$ are shown in fig. 3.

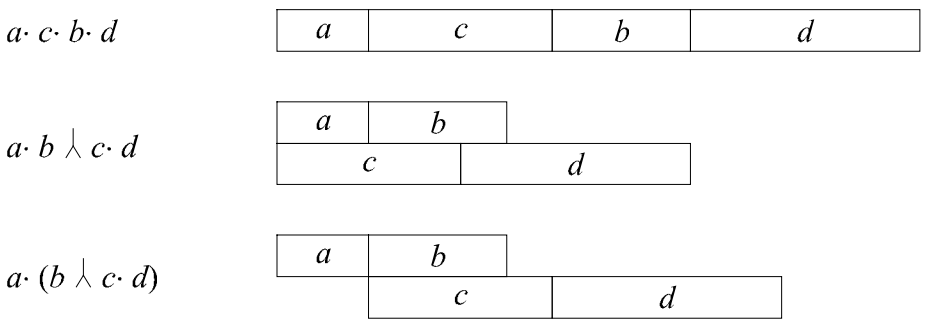

Fig. 3: Example schedules

Axiomatic System

The axioms of scheduling algebra are (+ is the real number addition):

BASIC AXIOMS

$x \cup (y \cup z) = (x \cup y) \cup z$ A1

$x \cup y = y \cup x$ A2

$x \cup x = x$ A3

$(x \curlywedge y) \curlywedge z = x \curlywedge (y \curlywedge z)$ A4

$x \curlywedge y = y \curlywedge x$ A5

ZERO AXIOMS

$0 \cdot x = x$ Z1

$x \curlywedge \varepsilon = x$ Z2

DISTRIBUTIVITY OF CHOICE

$t \cdot (x \cup y) = t \cdot x \cup t \cdot y$ D1

$\Delta (x \cup y) = \Delta x \cup \Delta y$ D2

$x \curlywedge (y \cup z) = x \curlywedge y \cup x \curlywedge z$ D3

NORMAL FORM AXIOMS

$a(t) \cdot x = a(t)^{+} \curlywedge t \cdot x$ N1

$t \cdot x \curlywedge (t + u) \cdot y = t \cdot (x \curlywedge u \cdot y)$ N2

$\Delta x \curlywedge \Delta y = \Delta(x \curlywedge \Delta y \cup y \curlywedge \Delta x)$ N3

ELIMINATION OF DELAY (ANCHOR AXIOMS)

$[x \cup y] = [x] \cup [y]$ AA1

$[a(t)^+ \curlywedge x] = a(t)^+ \curlywedge [x]$ AA2

$[t\cdot x] = t\cdot [x]$ AA3

$[\Delta x] = [x]$ AA4

$[t\cdot x \curlywedge \Delta y] = [t\cdot x \curlywedge y] \cup t\cdot [x \curlywedge \Delta y]$ AA5

$[\varepsilon] = \varepsilon$ AA6

Proposition

The following equation follows from applying N1, D1, D3 and N1 in the given order:

$a\cdot (x \cup y) = a\cdot x \cup a\cdot y$ P1

The distributivity axioms and P1 indicate that we work in a linear-time setting. This is because we treat scheduling as a planning problem where all decisions (choices) are made prior to the execution of the process.

Example

Applying the axioms, the definitions and P1 in a suitable order, we can rewrite:

$[\, a(1); b(3) \parallel c(2); d(1) \,]$

$= [\, \Delta(\, a(1)\cdot \Delta b(3) \,) \curlywedge \Delta(\, c(2)\cdot \Delta d(1) \,) \,]$

$= a^+ \curlywedge 1\,(\, b^+ \curlywedge 3\,(\, c^+ \curlywedge 2\,(\, d^+ \curlywedge 1 \,)\,)\,) \cup a^+ \curlywedge 1\,(\, c^+ \curlywedge 2\,(\, d^+ \curlywedge 1 \,)\,) \cup$
$a^+ \curlywedge 1\,(\, c^+ \curlywedge b^+ \curlywedge 3\,(\, d^+ \curlywedge 1 \,)\,) \cup a^+ \curlywedge 1\,(\, c^+ \curlywedge 2\,(\, b^+ \curlywedge 3\,(\, d^+ \curlywedge 1 \,)\,)\,) \cup$
$a^+ \curlywedge 1\,(\, c^+ \curlywedge 2\,(\, b^+ \curlywedge (\, d^+ \curlywedge 3 \,)\,)\,) \cup a^+ \curlywedge 1\,(\, c^+ \curlywedge 2\,(\, d^+ \curlywedge 1\,(\, b^+ \curlywedge 3 \,)\,)\,) \cup$
$c^+ \curlywedge 2\,(\, a^+ \curlywedge 1\,(\, b^+ \curlywedge 3\,(\, d^+ \curlywedge 1 \,)\,)\,) \cup c^+ \curlywedge 2\,(\, a^+ \curlywedge 1\,(\, d^+ \curlywedge (\, b^+ \curlywedge 3 \,)\,)\,) \cup$
$c^+ \curlywedge 2\,(\, a^+ \curlywedge 1\,(\, d^+ \curlywedge 1\,(\, b^+ \curlywedge 3 \,)\,)\,) \cup c^+ \curlywedge 2\,(\, d^+ \curlywedge a^+ \curlywedge (\, 1\,(\, b^+ \curlywedge 3 \,)\,)\,) \cup$
$c^+ \curlywedge 2\,(\, d^+ \curlywedge 1\,(\, a^+ \curlywedge (\, 1\,(\, b^+ \curlywedge 3 \,)\,)\,)\,) \cup c^+ \curlywedge a^+ \curlywedge 1\,(\, b^+ \curlywedge 3\,(\, d^+ \curlywedge 1 \,)\,) \cup$
$c^+ \curlywedge a^+ \curlywedge 1\,(\, b^+ \curlywedge 1\,(\, d^+ \curlywedge 2 \,)\,) \cup c^+ \curlywedge a^+ \curlywedge 2\,(\, b^+ \curlywedge 3\,(\, d^+ \curlywedge 1 \,)\,) \cup$
$c^+ \curlywedge a^+ \curlywedge 2\,(\, d^+ \curlywedge 1\,(\, b^+ \curlywedge 3 \,)\,) \cup c^+ \curlywedge a^+ \curlywedge 2\,(\, d^+ \curlywedge (\, b^+ \curlywedge 3 \,)\,)$

$= a\cdot b\cdot c\cdot d \cup a\cdot c\cdot b\cdot d \cup a\cdot c\cdot d\cdot b \cup c\cdot a\cdot b\cdot d \cup c\cdot a\cdot d\cdot b \cup c\cdot d\cdot a\cdot b \cup$
$a\cdot (\, c\cdot d \curlywedge b \,) \cup a\cdot (\, b\cdot d \curlywedge c \,) \cup c\cdot (\, a \curlywedge d\cdot b \,) \cup a\cdot c\cdot (\, b \curlywedge d \,) \cup c\cdot a\cdot (\, b \curlywedge d \,) \cup$
$a \curlywedge c\cdot (\, b \curlywedge d \,) \cup a \curlywedge c\cdot b\cdot d \cup a \curlywedge c\cdot d\cdot b \cup c \curlywedge a\cdot b\cdot d \cup a\cdot b \curlywedge c\cdot d$

Normal form theorem

All closed terms can be written in the form $\bigcup N$ where

$$
\begin{array}{lll}
N & ::= & a(t)^+ \curlywedge N \\
 & | & t\cdot N \qquad\qquad \text{for } t \neq 0 \\
 & | & \Delta N \\
 & | & t\cdot N \curlywedge \Delta N \qquad \text{for } t \neq 0 \\
 & | & \varepsilon
\end{array}
$$

This normal form theorem is proved by structural induction on closed terms. We show for every operator that the composition of terms in normal form again yields a term that can be expressed in normal form.

The choice operator distributes over every other operator (D1 - D3, P1, AA1) so that it can always be moved to the top level. This and associativity (A1) and commutativity (A2) allow us to denote every term as a union of choice-free subterms. Prefixing a normal term with Δ or $t\cdot$ (for $t\neq 0$) is *qua definitione* normal, as is ε. Z1 takes care of terms $0\cdot N$. Using Z2, the constant $a(t)^+$ can be written as the normal form $a(t)^+ \curlywedge \varepsilon$. For $a(t)\cdot N$, axiom N1 leads to the normal form

$$a(t)\cdot N \;=\; a(t)^+ \curlywedge t\cdot N.$$

Delay elimination is straightforward with AA1 - AA6. It remains to be shown that *forking* of normal terms can be normalized. Let K, L, M and N be normal terms. Then

$$(a(t)^+ \curlywedge M) \curlywedge N \xrightarrow{\text{A4}} a(t)^+ \curlywedge (M \curlywedge N) \xrightarrow{\text{IH}} a(t)^+ \curlywedge N',$$

$$\varepsilon \curlywedge N \xrightarrow{\text{A5,Z2}} N,$$

where IH abbreviates induction hypothesis. For the remaining 3 normal forms we have to prove normalization only for the upper triangle of the following matrix (due to commutativity of forking), in which $t, u>0$:

$\curlywedge$	$u\cdot K$	ΔK	$uK \curlywedge \Delta L$
$t\cdot M$	N2 ($t\leq u$), IH	√	A4, N2 ($t\leq u$), IH
ΔM		N3, IH, IH	A5, A4, N3, IH, IH
$tM \curlywedge \Delta N$			A4, A4, A5, A4, A4, N2 ($t\leq u$), N3, IH, IH, IH

The proof proceeds applying the laws stated modulo commutativity (√ means that the term is already in normal form).

Semantics

As a semantic model, we define schedules similar to the real-time execution sequences in [RR88]. Each expression maps to a set of possible schedules:

$$[\![.]\!]: \mathit{Expr} \to 2^{\mathbb{N}^{A\times T\times T}\times 2^{T}\times 2^{T-\{0\}}}$$

A schedule is a triple $\sigma = \langle \sigma_0; \sigma_1; \sigma_2 \rangle$ where σ_0 is a multi-set over triples (action, duration, starting time), σ_1 is a set of starting and finishing times and σ_2 a set of anchor points (or finishing times), excluding 0. Because 0 is always an anchor point, there is no need to record it explicitly; suppressing it turns out to have technical advantages. Typically, we leave out the brackets delimiting σ_0, σ_1 and σ_2, i.e.

$$[\![\ a(1)\cdot\ b(2) \between a(1)\ ]\!] \ = \ \{\ \langle\ (a, 1, 0), (a, 1, 0), (b, 2, 1)\ ;\ 0, 1, 3\ ;\ 1, 3\ \rangle\ \}.$$

This schedule is interpreted as follows:

- $a(1)$ is started twice at time 0,
- $b(2)$ at time 1 (immediately after a),
- the starting times are 0 and 1 (for a and b respectively)
- and the anchor points are 1 and 3 (the finishing times of a and b respectively).

Because Z1 yields $a(1)\cdot\ b(2) = a(1)\cdot\ b(2)\cdot\ 0$ and the latter term has a starting time 3 (for the internal action of duration 0), 3 can be regarded as a potential starting time of the process $a(1)\cdot\ b(2)$ as well. For this reason we include in σ_1 not only the starting times, but also the finishing times of actions.

For two schedules σ and ρ, we write $\sigma\cup\rho$ for $\langle\ \sigma_0\cup\rho_0\ ;\ \sigma_1\cup\rho_1\ ;\ \sigma_2\cup\rho_2\ \rangle$.

The offset of a schedule by a fixed amount of time is defined by:

$$\sigma + u \ = \ \langle\ \{\!|\ (a, t, p+u)\ |\ (a, t, p)\in \sigma_0\ |\!\}\ ;\ \{\ p+u\ |\ p\in \sigma_1\ \}\ ;\ \{\ p+u\ |\ p\in \sigma_2\ \}\ \rangle.$$

The general definition of the semantics is as follows:

$$\begin{aligned}
[\![\ \varepsilon\ ]\!] &= \{\ \langle\ \varnothing\ ;\ 0\ ;\ \varnothing\ \rangle\ \} \\
[\![\ a(t)^{+}\ ]\!] &= \{\ \langle\ (a, t, 0)\ ;\ 0\ ;\ \varnothing\ \rangle\ \} \\
[\![\ a(t)\cdot\ P\ ]\!] &= \{\ \langle\ (a, t, 0)\ ;\ 0\ ;\ \{t\}-\{0\}\ \rangle \cup (\sigma+t)\ |\ \sigma\in [\![P]\!]\ \} \\
[\![\ t\cdot\ P\ ]\!] &= \{\ \langle\ \varnothing\ ;\ 0\ ;\ \{t\}-\{0\}\ \rangle \cup (\sigma+t)\ |\ \sigma\in [\![P]\!]\ \} \\
[\![\ \Delta P\ ]\!] &= \{\ \langle\ \varnothing\ ;\ 0\ ;\ \varnothing\ \rangle \cup (\sigma+u)\ |\ \sigma\in [\![P]\!], u\in T\} \\
[\![\ [P]\ ]\!] &= \{\ \sigma\in [\![P]\!]\ |\ \sigma_1 \subset \sigma_2\cup\{0\}\ \} \\
[\![\ P\cup Q\ ]\!] &= [\![P]\!] \cup [\![Q]\!] \\
[\![\ P \between Q\ ]\!] &= \{\ \sigma\cup\rho\ |\ \sigma\in [\![P]\!], \rho\in [\![Q]\!]\ \}
\end{aligned}$$

Note that this definition guarantees that $0 \in \sigma_1$, $0 \notin \sigma_2$ and $\sigma_2 \subset \sigma_1$, for all schedules σ.

The role of the elimination operator [.]

A term not containing [.] represents a set of schedules, both efficient and inefficient ones. Delay elimination restricts attention to the efficient schedules, thereby eliminating all schedules known not to be optimal under any goal function or constraints. On the syntactical level, this corresponds to the elimination of Δ's, which is achieved by the operator [.]. One might ask why this is not done implicitly, i.e. by giving every term p the semantics of $[p]$. The answer is that the following implication does not hold:

$$[\![\, [x] \,]\!] = [\![\, [y] \,]\!] \;\Rightarrow\; [\![\, [x \,\|\, z] \,]\!] = [\![\, [y \,\|\, z] \,]\!].$$

A counterexample to the implication can be constructed easily by setting:

$$x = a(1)\cdot b(1),$$
$$y = a(1)\cdot \Delta\, b(1),$$
$$z = c(1).$$

We then get:

$$[\![\, [a(1)\cdot b(1)] \,]\!] = \{\, \langle\, (a, 1, 0), (b, 1, 1)\, ;\, 0, 1, 2\, ;\, 1, 2 \,\rangle \,\} = [\![\, [a(1)\cdot \Delta\, b(1)] \,]\!],$$

but:

$$\langle\, (a, 1, 0), (b, 1, 2), (c, 1, 1)\, ;\, 0, 1, 2, 3\, ;\; 1, 2, 3 \,\rangle \in [\![\, [a(1)\cdot \Delta\, b(1) \,\|\, c(1)] \,]\!],$$
$$\langle\, (a, 1, 0), (b, 1, 2), (c, 1, 1)\, ;\, 0, 1, 2, 3\, ;\; 1, 2, 3 \,\rangle \notin [\![\, [a(1)\cdot b(1) \,\|\, c(1)] \,]\!].$$

Correctness

Axioms A1 to A3 are correct w.r.t. our schedule semantics because set union is associative, commutative and idempotent. Associativity and commutativity of multi-set union yield A4 and A5. Z1 and Z2 are explained by 0 and the empty set being the neutral elements of addition and (multi-)set union respectively, taking into account that 0 is already in σ_1 for every schedule σ. Correctness of D1 can be derived as follows:

$$\begin{aligned}
[\![\, t\cdot (x \cup y) \,]\!] &= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\}-\{0\} \,\rangle \cup (\sigma + t) \mid \sigma \in [\![x \cup y]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\}-\{0\} \,\rangle \cup (\sigma + t) \mid \sigma \in [\![x]\!] \cup [\![y]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\}-\{0\} \,\rangle \cup (\sigma + t) \mid \sigma \in [\![x]\!] \,\} \;\cup \\
&\quad \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\}-\{0\} \,\rangle \cup (\sigma + t) \mid \sigma \in [\![y]\!] \,\} \\
&= [\![\, t\cdot x \,]\!] \cup [\![\, t\cdot y \,]\!] \\
&= [\![\, t\cdot x \cup t\cdot y \,]\!]
\end{aligned}$$

The proofs of D2 and D3 proceed analogously. For N1 - N3 and AA1 - AA6, we have:

$$
\begin{aligned}
[\![\, a(t)\cdot x \,]\!] &= \{\, \langle\, (a, t, 0)\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\sigma + t) \mid \sigma \in [\![\, x \,]\!] \,\} \\
&= \{\, \langle\, (a, t, 0)\, ;\, 0\, ;\, \varnothing\, \rangle \cup \big(\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\sigma + t) \,\big) \mid \sigma \in [\![\, x \,]\!] \,\} \\
&= \{\, \langle\, (a, t, 0)\, ;\, 0\, ;\, \varnothing\, \rangle \cup \rho \mid \rho \in [\![\, t\cdot x \,]\!] \,\} \\
&= [\![\, a(t)^{+} \between t\cdot x \,]\!]
\end{aligned}
$$

$$
\begin{aligned}
&[\![\, t\cdot x \between (t + u)\cdot y \,]\!] \\
&= \{\, \sigma \cup \rho \mid \sigma \in [\![\, t\cdot x \,]\!], \rho \in [\![\, (t+u)\cdot y \,]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\pi + t) \cup \langle\, \varnothing\, ;\, 0\, ;\, \{t+u\} - \{0\}\, \rangle \cup (\omega + t + u) \\
&\quad \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!] \,\} \\
&=^{1} \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\pi + t) \cup \langle\, \varnothing\, ;\, t\, ;\, \{t+u\} - \{0\}\, \rangle \cup (\omega + t + u) \\
&\quad \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!] \,\} \\
&=^{2} \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\pi + t) \cup \langle\, \varnothing\, ;\, t\, ;\, \{t+u\} - \{t\}\, \rangle \cup (\omega + t + u) \\
&\quad \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\pi + t) \cup (\, \langle\, \varnothing\, ;\, 0\, ;\, \{u\} - \{0\}\, \rangle + t\,) \cup (\omega + t + u) \\
&\quad \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup \big(\, (\, \pi \cup \langle\, \varnothing\, ;\, 0\, ;\, \{u\} - \{0\}\, \rangle \cup (\omega + u)\,) + t \,\big) \\
&\quad \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup \big(\, (\, \pi \cup \rho\,) + t \,\big) \mid \pi \in [\![\, x \,]\!], \rho \in [\![\, u\cdot y \,]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \{t\} - \{0\}\, \rangle \cup (\sigma + t) \mid \sigma \in [\![\, x \between u\cdot y \,]\!] \,\} \\
&= [\![\, t\cdot (x \between u\cdot y) \,]\!]
\end{aligned}
$$

$$
\begin{aligned}
&[\![\, \Delta x \between \Delta y \,]\!] \\
&= \{\, \sigma \cup \rho \mid \sigma \in [\![\, \Delta x \,]\!], \rho \in [\![\, \Delta y \,]\!] \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0\, ;\, \varnothing\, \rangle \cup (\pi + t) \cup (\omega + u) \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!], t, u \in T \,\} \\
&=^{1} \{\, \langle\, \varnothing\, ;\, 0, t\, ;\, \varnothing\, \rangle \cup (\pi + t) \cup (\omega + u) \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!], t, u \in T \,\} \\
&= \{\, \langle\, \varnothing\, ;\, 0, t\, ;\, \varnothing\, \rangle \cup (\pi + t) \cup (\omega + u) \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!], t \le u \,\} \cup \\
&\quad \{\, \langle\, \varnothing\, ;\, 0, t\, ;\, \varnothing\, \rangle \cup (\pi + t) \cup (\omega + u) \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!], t \ge u \,\} \\
&=^{3} \{\, \langle\, \varnothing\, ;\, 0\, ;\, \varnothing\, \rangle \cup \big(\, (\, \pi \cup \langle\, \varnothing\, ;\, 0\, ;\, \varnothing\, \rangle \cup (\omega + v)\,) + t \,\big) \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!], t, v \in T \,\} \cup \\
&\quad \{\, \langle\, \varnothing\, ;\, 0\, ;\, \varnothing\, \rangle \cup \big(\, (\, \omega \cup \langle\, \varnothing\, ;\, 0\, ;\, \varnothing\, \rangle \cup (\pi + w)\,) + u \,\big) \mid \pi \in [\![\, x \,]\!], \omega \in [\![\, y \,]\!], u, w \in T \,\}
\end{aligned}
$$

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[1] We always have that $0 \in \pi_1$, hence $t \in \pi_1 + t$.

[2] We leave it to the reader to check that $(\{t\} - \{0\}) \cup (\{t+u\} - \{0\}) = (\{t\} - \{0\}) \cup (\{t+u\} - \{t\})$.

[3] In the first line we write $t + v$ for u. In the second we replace t by $u + w$.

$$= \{ \langle \varnothing ; 0 ; \varnothing \rangle \cup \big((\pi \cup \rho) + t \big) \mid \pi \in [\![x]\!], \rho \in [\![\Delta y]\!], t \in T \} \cup \{ \langle \varnothing ; 0 ; \varnothing \rangle \cup \big((\omega \cup \sigma) + u \big) \mid \sigma \in [\![\Delta x]\!], \omega \in [\![y]\!], u \in T \}$$

$$= \{ \langle \varnothing ; 0 ; \varnothing \rangle \cup (\psi + t) \mid \psi \in [\![x \curlywedge \Delta y]\!], t \in T \} \cup \{ \langle \varnothing ; 0 ; \varnothing \rangle \cup (\zeta + u) \mid \zeta \in [\![\Delta x \curlywedge y]\!], u \in T \}$$

$$= [\![\Delta(x \curlywedge \Delta y)]\!] \cup [\![\Delta(y \curlywedge \Delta x)]\!]$$

$$= [\![\Delta(x \curlywedge \Delta y \cup y \curlywedge \Delta x)]\!]$$

$$[\![[x \cup y]]\!] = \{ \sigma \in [\![x \cup y]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \}$$
$$= \{ \sigma \in [\![x]\!] \cup [\![y]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \}$$
$$= \{ \sigma \in [\![x]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \} \cup \{ \sigma \in [\![y]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \}$$
$$= [\![[x]]\!] \cup [\![[y]]\!]$$
$$= [\![[x] \cup [y]]\!]$$

$$[\![[a(t)^+ \curlywedge x]]\!] = \{ \sigma \in [\![a(t)^+ \curlywedge x]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \}$$
$$= \{ \langle (a, t, 0) ; 0 ; \varnothing \rangle \cup \rho \mid \rho \in [\![x]\!], \rho_1 \cup \{0\} \subset \rho_2 \cup \varnothing \cup \{0\} \}$$
$$= \{ \langle (a, t, 0) ; 0 ; \varnothing \rangle \cup \rho \mid \rho \in [\![x]\!], \rho_1 \subset \rho_2 \cup \{0\} \}$$
$$= \{ \langle (a, t, 0) ; 0 ; \varnothing \rangle \cup \sigma \mid \sigma \in [\![[x]]\!] \}$$
$$= [\![a(t)^+ \curlywedge [x]]\!]$$

$$[\![[t \cdot x]]\!] = \{ \sigma \in [\![t \cdot x]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \}$$
$$= \{ \langle \varnothing ; 0 ; \{t\} - \{0\} \rangle \cup (\rho + t) \mid \rho \in [\![x]\!], (\rho_1 + t) \cup \{0\} \subset (\rho_2 + t) \cup \{t, 0\} \}$$
$$= \{ \langle \varnothing ; 0 ; \{t\} - \{0\} \rangle \cup (\rho + t) \mid \rho \in [\![x]\!], \rho_1 \subset \rho_2 \cup \{0\} \}$$
$$= \{ \langle \varnothing ; 0 ; \{t\} - \{0\} \rangle \cup (\rho + t) \mid \rho \in [\![[x]]\!] \}$$
$$= [\![t \cdot [x]]\!]$$

$$[\![[\Delta x]]\!] = \{ \sigma \in [\![\Delta x]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \}$$
$$= \{ \langle \varnothing ; 0 ; \varnothing \rangle \cup (\rho + u) \mid \rho \in [\![x]\!], u \in T, (\rho_1 + u) \cup \{0\} \subset (\rho_2 + u) \cup \{0\} \}$$
$$=^4 \{ \langle \varnothing ; 0 ; \varnothing \rangle \cup \rho \mid \rho \in [\![x]\!], \rho_1 \subset \rho_2 \cup \{0\} \}$$
$$=^5 \{ \rho \mid \rho \in [\![x]\!], \rho_1 \subset \rho_2 \cup \{0\} \}$$
$$= [\![[x]]\!]$$

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[4] Since $0 \in \rho_1$ and $0 \notin \rho_2$, we have $u \in (\rho_1 + u) \subset (\rho_2 + u) \cup \{0\}$ but $u \notin \rho_2 + u$, hence $u \in \{0\}$.

[5] Because always $0 \in \rho_1$.

$[\![\, [t{\cdot}x \between \Delta y] \,]\!]$

$= \{\, \sigma \in [\![\, t{\cdot}x \between \Delta y \,]\!] \mid \sigma_1 \subset \sigma_2 \cup \{0\} \,\}$

$= \{\, \rho \cup \omega \mid \rho \in [\![\, t{\cdot}x \,]\!], \omega \in [\![\, \Delta y \,]\!], \rho_1 \cup \omega_1 \subset \rho_2 \cup \omega_2 \cup \{0\} \,\}$

$= \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\psi + t) \cup (\pi + u)$
$\mid \psi \in [\![\, x \,]\!], \pi \in [\![\, y \,]\!], u \in T, (\psi_1 + t) \cup (\pi_1 + u) \cup \{0\} \subset (\psi_2 + t) \cup (\pi_2 + u) \cup \{t, 0\} \,\}$

$= \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\psi + t) \cup (\pi + u)$
$\mid \psi \in [\![\, x \,]\!], \pi \in [\![\, y \,]\!], (\psi_1 + t) \cup (\pi_1 + u) \subset (\psi_2 + t) \cup (\pi_2 + u) \cup \{t, 0\}, \boldsymbol{u < t} \,\}$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\psi + t) \cup (\pi + u)$
$\mid \psi \in [\![\, x \,]\!], \pi \in [\![\, y \,]\!], (\psi_1 + t) \cup (\pi_1 + u) \subset (\psi_2 + t) \cup (\pi_2 + u) \cup \{t, 0\}, \boldsymbol{u \geq t} \,\}$

$=^{6} \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\psi + t) \cup \pi$
$\mid \psi \in [\![\, x \,]\!], \pi \in [\![\, y \,]\!], (\psi_1 + t) \cup \pi_1 \subset (\psi_2 + t) \cup \pi_2 \cup \{t, 0\} \,\}$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\psi + t) \cup (\pi + t + v)$
$\mid \psi \in [\![\, x \,]\!], \pi \in [\![\, y \,]\!], (\psi_1 + t) \cup (\pi_1 + t + v) \subset (\psi_2 + t) \cup (\pi_2 + t + v) \cup \{t, 0\}, \boldsymbol{v \in T} \,\}$

$= \{\, \omega \cup \pi \mid \omega \in [\![\, t{\cdot}x \,]\!], \pi \in [\![\, y \,]\!], \omega_1 \cup \pi_1 \subset \omega_2 \cup \pi_2 \cup \{0\} \,\}$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup \big(\, (\, \psi \cup (\pi + v) \,) + t \,\big)$
$\mid \psi \in [\![\, x \,]\!], \pi \in [\![\, y \,]\!], (\, \psi_1 \cup (\pi_1 + v) \,) + t \subset \big(\, (\, \psi_2 \cup (\pi_2 + v) \cup \{0\} \,) + t \,\big) \cup \{0\}, \boldsymbol{v \in T} \,\}$

$= \{\, \sigma \mid \sigma \in [\![\, t{\cdot}x \between y \,]\!], \sigma_1 \subset \sigma_2 \cup \{0\} \,\}$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup \big(\, (\, \psi \cup \rho \,) + t \,\big)$
$\mid \psi \in [\![\, x \,]\!], \rho \in [\![\, \Delta y \,]\!], (\psi_1 \cup \rho_1) + t \subset (\, \psi_2 \cup \rho_2 \cup \{0\} \,) + t \,\}$

$= [\![\, [t{\cdot}x \between y] \,]\!]$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup \big(\, (\, \psi \cup \rho \,) + t \,\big)$
$\mid \psi \in [\![\, x \,]\!], \rho \in [\![\, \Delta y \,]\!], \psi_1 \cup \rho_1 \subset (\psi_2 \cup \rho_2) \cup \{0\} \,\}$

$= [\![\, [t{\cdot}x \between y] \,]\!]$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\sigma + t) \mid \sigma \in [\![\, x \between \Delta y \,]\!], \sigma_1 \subset \sigma_2 \cup \{0\} \,\}$

$= [\![\, [t{\cdot}x \between y] \,]\!]$

$\cup \{\, \langle\, \varnothing \,;\, 0 \,;\, \{t\} - \{0\} \,\rangle \cup (\sigma + t) \mid \sigma \in [\![\, [x \between \Delta y] \,]\!] \,\}$

$= [\![\, [t{\cdot}x \between y] \,]\!] \cup [\![\, t{\cdot}\, [x \between \Delta y] \,]\!]$

$= [\![\, [t{\cdot}x \between y] \cup t{\cdot}\, [x \between \Delta y] \,]\!]$

$[\![\, [\varepsilon] \,]\!]$

$= \{\, \langle\, \varnothing \,;\, 0 \,;\, \varnothing \,\rangle \mid \{0\} \subset \varnothing \cup \{0\} \,\} = [\![\, \varepsilon \,]\!]$

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[6] Since $0 \in \pi_1$ and $0 \notin \pi_2$, we have $u \in (\pi_1 + u) \subset (\psi_2 + t) \cup (\pi_2 + u) \cup \{t, 0\}$ but $u \notin (\psi_2 + t) \cup (\pi_2 + u) \cup \{t\}$ provided $u < t$, hence $u = 0$. Otherwise write $u = t + v$.

Completeness w.r.t. Δ-free terms

In this section, we will prove completeness of the axiomatic system for Δ-free terms. The normal form theorem allows restriction to Δ-free normal forms, which, for example, arise from normalizing arbitrary terms of the form [*P*]. Each choice-free subterm of such a term corresponds uniquely to a set of only one schedule.

Lemma 1: Let N be a term of the form

$$N \;::=\; a(t)^+ \curlywedge N \;\mid\; t\cdot N \;\mid\; \varepsilon, \text{ where } t \neq 0.$$

Then $|[\![N]\!]| = 1$.

Proof by structural induction:

$|[\![\varepsilon]\!]| \;=\; |\{ \langle \varnothing ; 0 ; \varnothing \rangle \}| \;=\; 1.$

$[\![t\cdot N]\!] \;=\; \{ \langle \varnothing ; 0 ; \{t\}-\{0\} \rangle \cup (\sigma+t) \mid \sigma \in [\![N]\!] \}$. So if $|[\![N]\!]| = 1$, then $|[\![t\cdot N]\!]| = 1$.

The case $a(t)^+ \curlywedge N$ proceeds likewise.

Lemma 2: Let N, M be of the form stipulated above, then

$$[\![N]\!] = [\![M]\!] \;\Rightarrow\; \text{A4, A5} \vdash N = M.$$

Proof by structural induction:

$N = a_1(t_1)^+ \curlywedge a_2(t_2)^+ \curlywedge \ldots \curlywedge a_n(t_n)^+ \curlywedge N'$ with $N' = \varepsilon$ or $N' = t\cdot N''$ with $t>0$, and

$M = b_1(u_1)^+ \curlywedge b_2(u_2)^+ \curlywedge \ldots \curlywedge b_m(u_m)^+ \curlywedge M'$ with $M' = \varepsilon$ or $M' = u\cdot M''$ with $u>0$.

Let σ be the unique schedule in $[\![N]\!] = [\![M]\!]$. We have $N' = \varepsilon \Leftrightarrow \sigma_2 = \varnothing \Leftrightarrow M' = \varepsilon$.

In case $N' = M' = \varepsilon$ we have

$\sigma_0 = \{\!| (a_1, t_1, 0), (a_2, t_2, 0), \ldots, (a_n, t_n, 0) |\!\} = \{\!| (b_1, u_1, 0), (b_2, u_2, 0), \ldots, (b_m, u_m, 0) |\!\}$.

Hence $\{\!| a_1(t_1)^+, a_2(t_2)^+, \ldots, a_n(t_n)^+ |\!\} = \{\!| b_1(u_1)^+, b_2(u_2)^+, \ldots, b_m(u_m)^+ |\!\}$, so in particular $n = m$. It follows that A4, A5 $\vdash N = M$.

In case $N' = t\cdot N''$ and $M' = u\cdot M''$ we have

$$[\![N]\!] = \{ \langle (a_1, t_1, 0), (a_2, t_2, 0), \ldots, (a_n, t_n, 0) ; 0 ; t \rangle \cup (\sigma+t) \mid \sigma \in [\![N']\!] \} \text{ and}$$

$$[\![M]\!] = \{ \langle (b_1, u_1, 0), (b_2, u_2, 0), \ldots, (b_m, u_m, 0) ; 0 ; u \rangle \cup (\rho+u) \mid \rho \in [\![M']\!] \}.$$

Again it follows that $\{\!| a_1(t_1)^+, a_2(t_2)^+, \ldots, a_n(t_n)^+ |\!\} = \{\!| b_1(u_1)^+, b_2(u_2)^+, \ldots, b_m(u_m)^+ |\!\}$, so in particular $n = m$. Furthermore, we have $u = t$ because t is the least anchor point in the unique schedule of $[\![N]\!]$, and so is u for $[\![M]\!]$. Finally, $[\![N']\!] = [\![M']\!]$, and therefore by induction A4, A5 $\vdash N' = M'$. Hence A4, A5 $\vdash N = M$.

Theorem: Let K, L be Δ-free normal forms with $[\![K]\!] = [\![L]\!]$ then

$$\text{A1}, \ldots, \text{A5} \vdash K = L.$$

Using A1, A2, A3, it suffices to show that for every choice-free subterm N of K there is a choice-free subterm M of L such that A4, A5 $\vdash N = M$ (and vice versa). By lemma 1, $[\![N]\!] = \{\sigma\}$, so $\sigma \in [\![K]\!] = [\![L]\!]$. Again by lemma 1, there must be a choice-free subterm M of L such that $[\![M]\!] = \{\sigma\} = [\![N]\!]$. Now, by lemma 2, A4, A5 $\vdash N = M$.

Anchor Form

For every t-free term $[P]$, we can eliminate [.] without introducing explicit time offsets.

$$P \text{ is } t\text{-free} \;\Rightarrow\; \exists Q : Q = \lceil P \rceil \text{ where } Q \text{ contains neither [.] nor } t\cdot$$

We call this the *anchor form* of a term because all schedules are given by just anchoring actions to the endpoints of others. The GANTT chart can be drawn easily from this form.

Using a straightforward structural induction, one can establish that for every schedule σ of a t-free term we have

$$\forall t \in \sigma_2 : \exists (a, u, v) \in \sigma_0 : t = u + v \text{ with } u > 0. \qquad (*)$$

Furthermore, employing N2, Z2 and again N2, we obtain

$$t\cdot (u\cdot x \between v\cdot y) \;=\; t\cdot \varepsilon \between (t+u)\cdot x \between (t+v)\cdot y.$$

Every term $[P]$ can be written in a Δ-free normal form, which, using structural induction, Z1 and the law above, can be converted to a sum of terms of the form

$$u_1\cdot a_1(t_1)^+ \between u_2\cdot a_2(t_2)^+ \between \ldots \between u_k\cdot a_k(t_k)^+ \between u_{k+1}\cdot \varepsilon \between u_{k+2}\cdot \varepsilon \between \ldots \between u_n\cdot \varepsilon.$$

Assuming P is t-free, (*) implies that for every $u_i > 0$ there exists a $j \in \{1, \ldots k\}$ such that $u_i = u_j + t_j$ and $t_j > 0$. We will remove step by step the time offsets $u_i\cdot$ in the term above, starting with the largest, until only time offsets $0\cdot$ remain. The intermediate terms encountered during this process will always remain in the form

$$u_1\cdot a_1(t_1)^+ \between u_2\cdot a_2(t_2)^+ \between \ldots \between u_k\cdot a_k(t_k)^+ \between u_{k+1}\cdot N_{k+1} \between u_{k+2}\cdot N_{k+2} \between \ldots \between u_n\cdot N_n \qquad (**)$$

where for every $u_i > 0$ there exists a $j \in \{1, \ldots k\}$ such that $u_i = u_j + t_j$, $t_j > 0$ and the N_i contain neither time offsets nor occurrences of [.] or Δ.

Let $v > 0$ be the maximum of $\{ u_1, u_2, \ldots u_n \}$, and let $j \in \{1, \ldots k\}$ be such that $v = u_j + t_j$ and $t_j > 0$. Using A4-5, we write all subterms $u_i\cdot a_i(t_i)$ and $u_i\cdot N_i$ with $u_i = v$ as well as $u_j\cdot a_j(t_j)$ at the right of the formula, thus obtaining an expression of the form

$$u'_1\cdot a'_1(t'_1)^+ \between u'_2\cdot a'_2(t'_2)^+ \between \ldots \between u'_l\cdot a'_l(t'_l)^+ \between u'_{l+1}\cdot N'_{l+1} \between u'_{l+2}\cdot N'_{l+2} \between \ldots \between u'_m\cdot N'_m \between K$$

with $K = u_j \cdot a_j(t_j)^+ \between v \cdot M_1 \between v \cdot M_2 \between \ldots \between v \cdot M_h$. Using axioms N2 and Z1, K can be rewritten as $u_j \cdot a_j(t_j)^+ \between v \cdot (M_1 \between M_2 \between \ldots \between M_h)$ and, with N1, as $u_j \cdot a_j(t_j) \cdot (M_1 \between M_2 \between \ldots \between M_h)$. This brings the term again in the form (**) and we can continue with the next-largest member of $\{ u_1, u_2, \ldots u_n \}$. In the end, all remaining u_i's will be 0 and Z1 turns the term into the required form.

Example

Consider the term [*a*; (*b* ‖ *c*) ‖ *d*] where all actions consume one unit of time. Transformation of this term into its anchor form yields 19 possible schedules. They are shown in fig. 4 grouped into the three basic orders "*a*; (*b* ‖ *c*)", "*a*; *b*; *c*" and "*a*; *c*; *b*" with the independent action *d* attached to an anchor point or inserted in front of or in between the other actions.

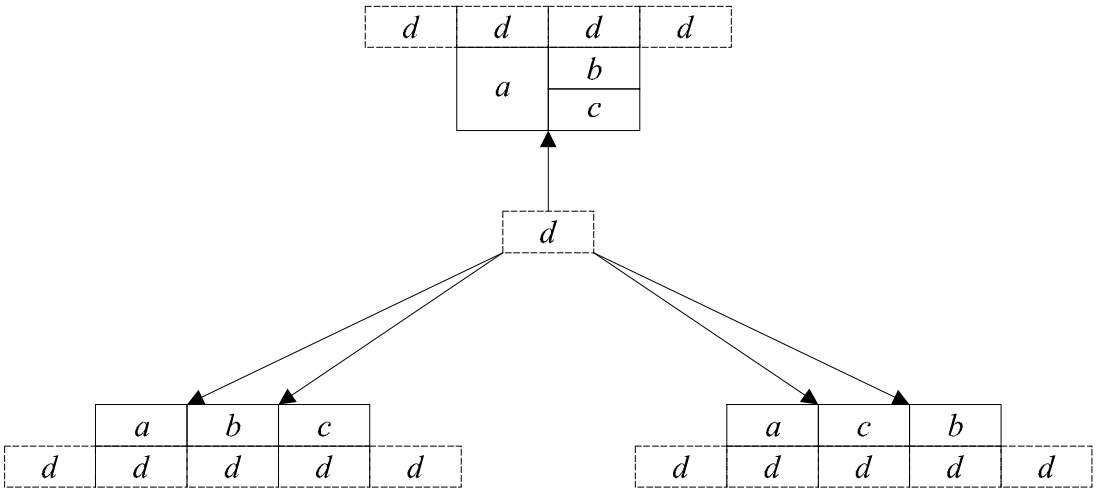

Fig. 4: All schedules

This example might suggest that it is possible to convert a *t*-free term [*P*] into its anchor form without considering the duration of its actions. However, this is not possible in general: the anchor form of the term [*a*(1); *b*(3) ‖ *c*(2); *d*(1)] that has been calculated in a previous example contains the schedule $a(1) \between c(2) \cdot (b(3) \between d(1))$, but not $a(1) \cdot (b(3) \between d(1)) \between c(2)$.

Term Rewriting

We show that modulo associativity and commutativity of choice and Hühnerfuß our axioms, when read from left to right, form a confluent and (strongly) normalizing rewrite system, provided that we add the rewrite rule $[a(t)^+] \to a(t)^+$ (which is derivable from the axiom system). If we drop the rewrite rule A3 (the idempotence of choice), the normal forms are exactly the normal forms in the sense of our normal form theorem, except that all subterms $a(t)^+ \between \varepsilon$ are replaced by $a(t)^+$ (so $a(t)^+$ is a normal form for the rewrite system). In the presence of A3 the same normal forms apply, but without pairs of summands that are equal modulo A4 and A5.

It is easy to check that such normal forms allow no further rewriting; the proof of the normal form theorem (but omitting the reverse application of Z2) shows essentially that all other terms can be rewritten into a normal form. Thus the rewrite system is normalizing. Here we omit the proof that it is even strongly normalizing; however, it involves only standard term rewriting techniques.

Confluence for terms of the form [*P*] follows immediately from the proof of our completeness theorem for such terms (because only the axioms A1-5 are involved in the completeness proof for Δ-free normal forms). Confluence for the entire rewrite system can be established by the Knuth-Bendix method, skipping pairs of overlapping redexes that lay within a term of the form [*P*]. In the following table, we review the relevant pairs of overlapping redexes:

A3 with D1, D2 and D3	trivial
Z1 with D1 and N2 (in two ways)	trivial
Z2 with D3 and Z2	trivial
D1 with N2 (in two ways)	trivial with D3
D2 with N3 (in two ways)	trivial with D3
D3 with D3	trivial
N2 with itself	OK
N3 with itself	OK

An implementation of the system without A3 can be found at the URL

`http://www.uni-koblenz.de/~rittgen/SA.html.`

This applet generates the normal form for the axiom system given any term of your chosing.

Conclusion

In this paper, we have proposed a calculus for durational actions, namely scheduling algebra. The constants and operators of this algebra allow to specify process terms as a multi-tree order over actions with a certain duration. A normal form exists for each term which for Δ-free terms is unique modulo A1-5. Using the axioms of this calculus, the set of all efficient schedules [*P*] can be "computed" for any process *P*. The operator [.] does so by generating a subterm (schedule) for any possible attachment of actions to anchor points, thus eliminating all time delays Δ.

We established soundness of the axiomatic system w.r.t. a schedule semantics and showed that the algebra is complete for Δ-free terms (sets of schedules), such as arise from terms of the form [*P*].

Currently, our calculus is designed for multi-tree precedence. To extend it to arbitrary orders (with more than one predecessor to an action), synchronization must be present in the algebra (as follows from the existence of partial orders which are not “series-parallel” [Gis88]). This is left for future research.

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