Rational Mechanics and Natural Mathematics

Vaughan Pratt*
Dept. of Computer Science, Stanford University, Stanford, CA 94305-2140
pratt@cs.stanford.edu

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Abstract

Chu spaces have found applications in computer science, mathematics, and physics. They enjoy a useful categorical duality analogous to that of lattice theory and projective geometry. As natural mathematics Chu spaces borrow ideas from the natural sciences, particularly physics, while as rational mechanics they cast Hamiltonian mechanics in terms of the interaction of body and mind.

This paper addresses the chief stumbling block for Descartes' 17th-century philosophy of mind-body dualism, how can the fundamentally dissimilar mental and physical planes causally interact with each other? We apply Cartesian logic to reject not only divine intervention, preordained synchronization, and the eventual mass retreat to monism, but also an assumption Descartes himself somehow neglected to reject, that causal interaction within these planes is an easier problem than between. We use Chu spaces and residuation to derive all causal interaction, both between and within the two planes, from a uniform and algebraically rich theory of between-plane interaction alone. Lifting the two-valued Boolean logic of binary relations to the complex-valued fuzzy logic of quantum mechanics transforms residuation into a natural generalization of the inner product operation of a Hilbert space and demonstrates that this account of causal interaction is of essentially the same form as the Heisenberg-Schrödinger quantum-mechanical solution to analogous problems of causal interaction in physics.

1 Cartesian Dualism

The Chu construction [Bar79] strikes us as extraordinarily useful, more so with every passing month. Elsewhere we have described the application of Chu spaces to process algebra [GP93], metamathematics [Pra93, Pra94a], and physics [Pra94b]. Here we make a first attempt at applying them to philosophy.

It might seem that traditional philosophical questions would be beyond the scope of TAPSOFT. Bear in mind however that Boolean logic as the basis for

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computer circuits was born of philosophy (and a little statistics). Only slightly more recently, program verification has drawn heavily on more sophisticated logics such as first order, modal, and higher order. Computers being thinking machines, computer science should not neglect the philosophical literature on thinking. It is easy to dismiss "all that stuff" as obsoleted by technology. However good truths, like good wine, must be served at the proper time. We would like to think of our application of Chu spaces to Descartes' inspiring yet short-lived theory of mind-body dualism as a convincing example.

Cartesianism is a "philosophy of everything" founded by René Descartes in the 1630's. Its point of departure was to reject all authority and question everything including the questioner's existence. Descartes resourcefully bootstrapped himself back into existence with an instance of the liar paradox, the absurdity of questioning his own questioning, constructivized as *Cogito, ergo sum.* Emboldened by this success, Descartes posed many more questions whose imaginative answers formed the basis of Cartesianism. This rationalist philosophy flourished for half a century until the march of science contradicted too many of its answers for it to remain a viable grand unified theory of anything. Some of the questions however remain philosophically challenging even today.

A central tenet of Cartesianism is mind-body dualism, the principle that mind and body are the two basic substances of which reality is constituted. Each can exist separately, body as realized in inanimate objects and lower forms of life, mind as realized in abstract concepts and mathematical certainties. According to Descartes the two come together only in humans, where they undergo *causal interaction*, the mind reflecting on sensory perceptions while orchestrating the physical motions of the limbs and other organs of the body.

The crucial problem for the causal interaction theory of mind and body was its mechanism: how did it work?

Descartes hypothesized the pineal gland, near the center of the brain, as the seat of causal interaction. The objection was raised that the mental and physical planes were of such a fundamentally dissimilar character as to preclude any ordinary notion of causal interaction. But the part about a separate yet joint reality of mind and body seemed less objectionable, and various commentators offered their own explanations for the undeniably strong correlations of mental and physical phenomena.

Malebranche insisted that these were only correlations and not true interactions, whose appearance of interaction was arranged in every detail by God by divine intervention on every occasion of correlation, a theory that naturally enough came to be called occasionalism. Spinoza freed God from this demanding schedule by organizing the parallel behavior of mind and matter as a preordained apartheid emanating from God as the source of everything. Leibniz postulated *monads*, cosmic chronometers miraculously keeping perfect time with each other yet not interacting.

These patently untestable answers only served to give dualism a bad name, and it gave way in due course to one or another form of monism: either mind *or* matter but not both as distinct real substances. Berkeley opined that matter did not exist and that the universe consisted solely of ideas. Hobbes ventured the opposite: mind did not exist except as an artifact of matter. Russell [Rus27] embraced *neutral* monism, which reconciled Berkeley's and Hobbes' viewpoints as compatible dual accounts of a common neutral Leibnizian monad.

This much of the history of mind-body dualism will suffice as a convenient point of reference for the sequel. R. Watson's Britannica article [Wat86] is a conveniently accessible starting point for further reading.

The thesis of this paper is that mind-body dualism can be made to work via a theory that we greatly prefer to its monist competitors. Reflecting an era of reduced expectations for the superiority of humans, we have implemented causal interaction not with the pineal gland but with machinery freely available to all classical entities, whether newt, pet rock, electron, or theorem (but not quantum mechanical wavefunction, which is sibling to if not an actual instance of our machinery).

2 Dualism via Chu Spaces

We propose to reduce complex mind-body interaction to the elementary interactions of their constituents. Events of the body interact with states of the mind. This interaction has two dual forms. A physical event a in the body A *impresses* its occurrence on a mental state x of the mind X, written $a \dashv x$. Dually, in state x the mind *infers* the prior occurrence of event a, written $x \models a$. States may be understood as corresponding more or less to the possible worlds of a Kripke structure, and events to propositions that may or may not hold in different worlds of that structure.

With regard to orientation, impression is *causal* and its direction is that of time. Inference is *logical*, and logic swims upstream against time. Prolog's backward-chaining strategy dualizes this by viewing logic as primary and time as swimming upstream against logic, but this amounts to the same thing. The basic idea is that time and logic flow in opposite directions.

Can a body meet a body? Only indirectly. All direct interaction in our account of Cartesian dualism is between mind and body. Any hypothesized interaction of two events is an inference from respective interactions between each of those events and all possible states of the mind. Dually, any claimed interaction of two states is inferred from their respective interactions with all possible events of the body.

The general nature of these inferences depends on the set K of values that events can impress on states. The simplest nontrivial case is $K = 2 = \{0, 1\}$, permitting the simple recording of respectively nonoccurrence or occurrence of a given event in a given state. In this case body-body and mind-mind interactions are computed via a process called residuation. Specifically, event a *necessarily precedes* event b when every state x witnessing the occurrence of b also witnesses a. This inferred relationship is calculated formally by left residuation, which we describe in detail later. The dual calculation, right residuation, *permits* a transition from state x to state y when every event a impressing itself on x does so also on y. That is, any transition is permitted just so long as it forgets no event. These simple-minded criteria are the appropriate ones for the small set $K = 2$.

For $K = 3$ more complex rules for inferring necessary precedence and possible transition obtain, including the possibility of forgetting (to be written up). At $K = 8$ we have groups and semigroups, the latter embedding all *abstract* category theory [PT80]. For K the set (not field) of complex numbers, right and left residuation are naturally taken to be the respective products $\langle\varphi|\psi\rangle$ or $\varphi^*\psi$ and $|\psi\rangle\langle\varphi|$ or $\psi\varphi^*$, corresponding to respectively inner product and its dual outer product in a Hilbert space.

This conveys the flavor of our proposal. We now equip these general ideas with enough algebraic structure and properties to make the proposal interesting, useful, and we hope convincing.

The following analogy serves to fix ideas. The numbers ± 1 are connected in two ways, algebraic and geometric. The algebraic connection is via the operation of negation, an involution $(- - x = x)$ that connects them logically by interchanging them. The geometric connection is via the interval $[-1, 1]$ of reals lying between these numbers, a closed convex space connecting them topologically. We refer to these connections as respectively the *duality* and *interaction* of -1 and 1. The connections themselves might respectively be understood as mental and physical, but this takes us beyond our present story.

We regard each point of the interval as a weighted sum of the endpoints, assuming nonnegative weights p, q normalized via $p + q = 1$, making each point the quantity $p - q$. An important property of interaction is that it includes the endpoints, namely as the special cases where one of p or q is zero. An important property of duality is that it extends to interaction, namely via the calculation $q - p = -(p - q)$.

We shall arrange for Cartesian dualism to enjoy the same two basic connections and the two associated properties, with mind and body in place of -1 and 1 respectively. Ideally the duality would be a negation-like involution that interchanges their roles; no information is lost in this transformation, and the original mind or body is recovered when the transformation is repeated. And ideally the interaction would turn out to be the long-sought solution to dualism's main conceptual hurdle. Chu spaces achieve both of these in a very satisfactory way.

The counterparts to ± 1 in our Chu space formulation of Cartesian dualism are the respective categories **Set** and $\mathbf{Set}^{\mathrm{op}}$. That is, at 1 we place the class of all sets, each understood as a pure body. At -1 we place what would appear at first sight to be the same sets, which we propose to construe as pure minds.

Our first distinction between body and mind will be the trivial one of using different variables to range over these sets: A, B over bodies, X, Y over minds. The second distinction will be in how the two kinds of sets transform into each other. Later we make a third distinction within the objects themselves by realizing the two kinds as Chu spaces with dual form factors: sets tall and thin, antisets short and wide.

Bodies transform with functions. We turn the class of bodies into **Set** by first superimposing on it the graph whose edges comprise all functions, with each function $f : A \to B$ connecting the set A to the set B. We then promote this graph to a category by equipping it with the standard composition rule for functions, as an instance of composition of binary relations, along with an identity function $1_A : A \to A$ at every set A.

Minds transform with *antifunctions*. An antifunction $g^\perp : X \to Y$ is a binary relation from X to Y whose converse is a function $g : Y \to X$. Adopting the composition rule for binary relations as with **Set** then yields a category dual to **Set**, one that is equivalent, in fact isomorphic, to $\mathbf{Set}^{\mathrm{op}}$ (the result of merely reversing all the edges of **Set**), which we simply identify with $\mathbf{Set}^{\mathrm{op}}$.

These graphs are not isomorphic, even without their respective compositions. A quick way to tell them apart is to look for a vertex whose only edge to it is a self-loop. This vertex occurs only in **Set**, namely as the empty set. Or look for a vertex whose only edge *from* it is a self-loop; this too is the empty set, but in $\mathbf{Set}^{\mathrm{op}}$. The reader will think of other tests.[1]

We now argue that sets are physical and antisets mental. Since the only difference is in how they transform, any distinction between mental and physical must be either dynamic in the sense of being transformational, or algebraic in the sense that structure regulates transformation. We present both types of argument (which themselves can be understood as respectively operational hence mental and denotational hence physical).

Functions identify and adjoin. The function $F : A \to B$ identifies just when it fails to be injective: $f(a) = f(b)$ means that f identifies a and b. It adjoins just when it fails to be surjective: $f : A \to B$ first transforms A *onto* $f(A)$, then adjoins to it $B - f(A)$ to become *into*.

Antifunctions copy and delete. The antifunction $g^\perp : X \to Y$ makes multiple copies just when its converse $g : Y \to X$ fails to be injective: $g(y) = g(y')$ means that $g^\perp$ sends copies of $g(y)$ to both y and y', *inter alia*. It deletes just when g fails to be surjective: $g^\perp : Y \to X$ deletes exactly $Y - g(X)$.

Identifying and adjoining are canonically denotational tasks that mathematicians are accustomed to performing on their spaces, groups, and other algebraic objects. This is the realm of the physical.

Copying and deleting are canonically operational tasks that logicians and computer scientists are accustomed to performing on their proofs, spreadsheets, and other symbolic objects. This is the realm of the mental.

In additional to these transformational arguments we can contrast the discrete or dust-like physical structure of sets with the rigidly intermeshed mental structure of Boolean algebras.

A set is an algebra with no language at all, and no equational theory beyond the equational tautologies $x = x$. There is therefore no mental plane to speak of in sets, making them the most physical of all the objects of traditional concrete (set-based) mathematics, if not of all category theory (and perhaps even there, cf. [RW94]).

[1] Example: look for any vertex having exactly one edge to it from each vertex, and infinitely many edges out. There are lots of these in **Set**, namely the many singletons, all isomorphic, but none in $\mathbf{Set}^{\mathrm{op}}$.

$\mathbf{Set}^{\mathrm{op}}$ is equivalent to the category of complete atomic Boolean algebras (CABA's). But the free CABA generated by the set X is the power set 2^{2^X}. Hence the Boolean operations of each arity X, X empty, finite, or infinite, consist of *all* functions from 2^X to 2. This is the maximum possible language compatible with CABA homomorphisms; not only is every arity represented but every operation of that arity.[2] Furthermore the equational theory of CABA's is maximally consistent in the sense that no new equation can be added without collapsing the entire algebra to a singleton. A CABA as the ultimate know-it-all is as mental as any object of traditional concrete mathematics can be.

We have thus established that the two isolated points $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ represent respectively the physical and the mental. We now proceed with the promised construction. At this point the situation is as for ± 1 on their own: we have two isolated graphs, and we seek a duality and an interaction.

The duality analogous to negation is simply the converse operation for binary relations, which evidently interchanges $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$.

The interaction analogous to the interval $[-1, 1]$, which includes the points it connects as part of the interval, consists of all Chu spaces and a graph superimposed on them, which includes as subgraphs $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$. That is, the interaction consists of adding further vertices and edges, in addition to those already present, to populate an interval from $\mathbf{Set}^{\mathrm{op}}$ to $\mathbf{Set}$.

A *Chu space* $\mathcal{A} = (A, X, \models)$ *over* a set K consists of a set A of *points*, an antiset X of *states*, and an $X \times A$ matrix $\models$ with entries drawn from K.[3] These provide the vertices of the interval.

This ontogeny of the Chu space recapitulates the phylogeny we are working towards. A and X are respectively the body or object and mind or menu of the space, $\models$ is their interaction, and matrix transposition is the duality interchanging mind and body to yield the dual Chu space $\mathcal{A}^{\perp} = (X, A, \models^{\smile})$.

Points have *necessary* existence, all being present simultaneously in the physical object A. States are *possible*, making a Chu space a kind of a Kripke structure [Gup93]: only one state at a time may be chosen from the menu X of alternatives.

Lafont and Streicher [LS91] were the first to single out Chu spaces as a case of the more general Chu construction $\mathbf{Chu}(V, k)$ [Bar79, Bar91], namely $V = \mathbf{Set}$, worthy of separate attention as a natural model of linear logic [Gir87] embedding topological spaces, vector spaces, and coherent spaces. They referred to these objects as games, understanding $\models$ as the payoff matrix of a von-Neumann-Morgenstern two-person game.

There is a chicken-and-egg question here as to whether Chu spaces are more naturally understood as a game or a player of a game. As players, the spaces $\mathcal{A}$ and $\mathcal{B}$ play the *interaction game* $\mathcal{A} \otimes \mathcal{B}$, their tensor product. This interaction has featured prominently in our own research as an operation we called *orthocurrence* [Pra85, Pra86]. We originally identified orthocurrence as ordinary

[2] One can add further operations, for example modal logic adds $\Diamond$. However CABA homomorphisms respect none of these additional operations whatsoever.

[3] Contrast this with a vector space over a field k, which requires k to be equipped with the four rationals; here K is simply a set with no additional structure.

product in a cartesian closed category of partially ordered multisets (pomsets), but subsequently generalized it to the tensor product of any closed category [CCMP91, Pra93, GP93, Pra94a]. In all cases we took as our basic example the interaction of trains and stations described on the train station wall by the daily schedule. Whereas ordinary product must be capable of being projected *consistently* onto either component, tensor product requires only that each row or column of the resulting rectangular body of the space (how stations appear to conductors, and trains to stationmasters) meet all the constraints imposed on each of the two constituents of the product, the concept of bilinearity. The tensor product constitutes a larger Chu space, which can in turn be a player in a yet larger game.

The representation $\mathcal{A} \otimes \mathcal{B}$ takes the physical viewpoint. The logic of the game may be understood in terms of its dual $(\mathcal{A} \otimes \mathcal{B})^\perp$, which is equivalent to either of $\mathcal{A} \multimap \mathcal{B}^\perp$ or $\mathcal{B} \multimap \mathcal{A}^\perp$. In the former, we take Alice's point of view as our premises and view Bob as the goal. This view dualizes Bob to make his body, which Bob proudly thinks of as his strong points, appear to Alice as Bob's possible Achilles' heels (wrists, etc.). At the same time Bob's mind, which Bob thinks of as his possible options, are seen by Alice as Bob's tricks, all of which she must be simultaneously on her guard against.

A *Chu transform* $(f, g) : (A, X, \models) \to (A', X', \models')$ consists of a function $f : A \to A'$ and an antifunction $g^\perp : X \to X'$, namely the converse of a function $g : X' \to X$, satisfying the *continuity condition* $g(x') \models a \;=\; x' \models' f(a)$ for all $a \in A$ and $x' \in X'$. These provide the edges of the graph on the interval of all Chu spaces running from $\mathbf{Set}^{\mathrm{op}}$ to $\mathbf{Set}$. They compose via $(f', g')(f, g) = (f'f, gg')$ to make the graph a category, denoted $\mathbf{Chu}_K$.

The function f transforms the body of the space denotationally, identifying some points and adjoining others, but neither deleting nor duplicating any. At the same time the antifunction g transforms the mind of the space operationally, i.e. as a symbolic object such as a program or a proof, deleting some states to further constrain the degrees of freedom of the space and copying some as needed so as not to infringe on the degrees of freedom of the newly adjoined points (transformations need only preserve the structure of what they transform and cannot be held responsible for what goes on in the adjoined points). However g never identifies states, which would be logically inconsistent for states having distinct rows, and never adjoins states having new rows, which would be logically unsound (the image could enter a state not permitted its source).

To understand better this last point, let $\mathrm{row} : X \to (A \to K)$ and dually $\mathrm{col} : A \to (X \to K)$ denote the functions satisfying $\mathrm{row}(x)(a) \;=\; x \models a \;=\; \mathrm{col}(a)(x)$. Continuity may then be rephrased in terms of rows: $\mathrm{row}(g(x')) \;=\; \mathrm{row}'(x') \circ f$, verified via $\mathrm{row}(g(x'))(a) \;=\; g(x') \models a \;=\; x' \models' f(a) \;=\; (\mathrm{row}'(x') \circ f)(a)$. That is, every row of B when composed with f must be some row of A, with g a function selecting a suitable row index. When $K = 2$ this is equivalent to requiring that g behave as f^{-1} on rows viewed as characteristic functions of subsets of A'. But then the requirement that every row of $\mathcal{A}'$ be mapped by f^{-1} to some row of $\mathcal{A}$ is recognizable as the condition for a function between topological spaces to be continuous, where rows are understood as open sets.

For technical reasons Chu transforms are usually associated with a fixed K, calling for a distinct category $\mathbf{Chu}_K$ of Chu spaces for each set K. A set theorist should have no difficulty with Chu spaces over different K's transforming into each other, but the resulting category would to begin with lack a tensor unit, an annoying omission when one begins to press the rich algebraic structure of $\mathbf{Chu}_K$ into service.

The structure of $\mathbf{Chu}_K$ is that of linear logic [Gir87], which can be understood as the logic of four key structural properties of $\mathbf{Chu}_K$: it is concrete, complete, closed, and self-dual (which therefore makes it also cocomplete and coconcrete). The associated linear logic connectives are respectively $!A$, $A \oplus B$ (and unit 0), $A \multimap B$ (and left unit 1), and $A^\perp$, which form a complete basis for linear logic. $\mathbf{Chu}_K$ is complete but perhaps for syntactic simplicity linear logic weakens completeness to finite products. Furthermore it is not yet agreed whether induction is a necessary element of concreteness.

Just as $\{-1, 1\} \subseteq [-1, 1]$, so are sets and antisets made part of the category of Chu spaces, as follows. The set A is identified with the Chu space $\mathcal{A} = (A, K^A, \gamma)$ where for each $x : A \to K$, $\gamma(x, a)$ denotes the application $x(a)$. The function $f : A \to A'$ is identified with the pair $(f, f^\perp) : (A, K^A, \gamma) \to (A', K^{A'}, \gamma)$ where $f^\perp : K^{A'} \to K^A$ is defined by $f^\perp(g)(a) = g(f(a))$. When $K = 2$, $f^\perp$ can be seen to be the usual inverse-image function f^{-1}, making this topology's continuity condition as remarked earlier. We call the Chu space $\mathcal{A}$ a *realization*[4] of the set A in $\mathbf{Chu}_2$.

Dually the antiset X is identified with $(K^X, X, \gamma^\smile)$ where $\gamma^\smile$ is converse application, satisfying $\gamma^\smile(x, a) = a(x)$, and the antifunction $g^\perp : X \to X'$ (i.e. the function $g : X' \to X$) is identified with the pair (f, g) where $f : K^X \to K^{X'}$ is defined at each $h : X \to K$ by $f(h)(x') = h(g(x'))$. This constitutes a realization of $\mathbf{Set}^{\mathrm{op}}$ in $\mathbf{Chu}_2$.

Just as the duality of ± 1 extended to $[-1, 1]$, so does the mind-body duality of $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ extend to $\mathbf{Chu}_K$. The dual of $\mathcal{A} = (A, X, \models)$ is $\mathcal{A}^\perp = (X, A, =\!\mid)$, while the dual of the Chu transform (f, g) is (g, f). Moreover the duality of sets and antisets achieved via converse of their transforming binary relations is also achieved via Chu duality for their realizations in $\mathbf{Chu}_K$.

To each finite Chu space $\mathcal{A}$ we associate integers P and Q measuring respectively the *discipline* and *versatility* of $\mathcal{A}$, in terms of the amount by which the space fails to be a set or an antiset respectively. Write $\|A\|$ for the number of *distinct* columns of the matrix, and likewise $\|X\|$ for the number of distinct rows. Let $P = K^{\|A\|} - \|X\|$ and $Q = K^{\|X\|} - \|A\|$, both nonnegative. For $K \geq 2$ these cannot vanish simultaneously or we would have an integer solution to $K^{K^A} = A$. Hence we can safely define nonnegative reals $p = P/(P + Q)$, $q = Q/(P + Q)$ satisfying $p + q = 1$. We take $p - q$ as the location of $\mathcal{A}$ in the interval $[-1, 1]$ itself, giving a sense in which $\mathbf{Chu}_K$ lies between $\mathbf{Set}^{\mathrm{op}}$ and $\mathbf{Set}$.

[4] A *representation* is a full embedding of one category in another, i.e. a full and faithful functor $F : C \to D$. A *realization* is a concrete representation; that is, C and D are concrete categories, meaning they have underlying set functors $U_C : C \to \mathbf{Set}$ and $U_D : D \to \mathbf{Set}$, with which F commutes, $U_D F = U_C$, i.e. the realizing object has the same underlying set as the object it realizes [PT80, p.49].

Notice that this procedure assigns sets and antisets to 1 and -1 respectively, while exactly square Chu spaces are sent to 0.

Although the position of a Chu space in $[-1,1]$ gives some indication of its form factor, these positions turn out not to populate $[-1,1]$ densely. For example at $K=2$ the intervals $[\frac{1}{3},\frac{1}{2})$ and $(\frac{1}{2},\frac{2}{3}]$ contain no Chu spaces, since Chu spaces that are only one away from being square are below $\frac{1}{3}$ or above $\frac{2}{3}$, and indeed the interval is riddled with such holes. One imagines being able to distribute Chu spaces more uniformly along $[-1,1]$ with the help of say $\|A\|/\|X\|$, but in choosing such a formula it would help to have some reason for wanting a dense distribution.

This viewpoint is a compromise between those of set theory and category theory. Set theory monistically constructs everything from the single category of pure sets. Category theory pluralistically constructs a plethora of categories. Chu spaces are like sets in that there is only one category $\mathbf{Chu}_K$ of them (modulo the parameter K). $\mathbf{Chu}_K$ is dualistic in that it postulates the two categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$, neither of which is singled out as having priority over the other, and connects them via interaction to form the single much larger category $\mathbf{Chu}_K$. Some impression of its size may be had from the theorem [Pra93, p.153-4] that $\mathbf{Chu}_{2^k}$ realizes the category of *all* $k-ary$ relational structures and their homomorphisms standardly defined. For example Chu_8 realizes the category of ternary relational structures, which in turn realizes the category of groups and group homomorphisms (since its multiplication is the ternary relation $xy=z$), and realization is transitive.

3 The Meaning of Interaction

Thus far we have constructed interaction as no more than a formal notion. We now relate it to our intuitions about causal interaction.

It is ironic that Cartesian philosophy, whose guiding dictum was to question everything, should question causal interaction *between* the mental and physical planes before that *within* the planes. The latter problems must have posed an insufficient challenge to the Cartesians. We argue that the converse is the case: between is actually easier than within!

We interpret interaction as causality. Causality is directional, but the direction depends on whether we have in mind physical or mental causality. We interpret $x \models a$ ambiguously as the time elapsed between the occurrence of the physical a and its impression on the mental state x, and as the truth value of a as a proposition.[5] The former is physical causality or *impression*, flowing forward in time from events to states. The latter is mental causality or *inference*, flowing backwards in time from the thought of a to the inference of a's

[5]The reader may be understandably concerned at this identification of physical events and ostensibly mental propositions. However a Boolean proposition about events in A is of type 2^{2^A} and *each exponentiation* dualizes, whence two of them return us to the physical plane. The truly mental propositions are the constituent descriptive clauses of a physical DNF formula, each describing a possible world.

occurrence. In this way time flows forward (from the usual point of view) while logic flows backward. This is *primary* interaction, and it occurs only *between* the mental and physical plane.

We thus see that the seat of causal interaction in Cartesian duality is not the pineal gland but the identification of impression and inference. We write $x \models a$ as expressing equally the impression of event a on subsequent state x and the deduction by state x of the prior occurrence of event a. The Cartesian dictum *cogito, ergo sum* is the case of this where x is the thinker's state and a the event of his or her existence.

As a proponent of more dynamic logics than traditionally contemplated in logic [Pra76, Pra90a] we point out the atemporal quality of this dictum, a hallmark of classical logic. Examined closely, our analysis shows that Descartes' dictum properly tensed becomes *cogito, ergo eram* (I was), an epitaph both of whose tenses the liar paradox renders true in perpetuity. Our thoughts follow from our events but not conversely and hence may survive them without logical contradiction. A particularly good one may far outlive its source.

We pass now to interaction *within* each plane, whether physical or mental, which we derive as *secondary* interaction from the primary form with the aid of *residuation*, a pair of operations on binary relations that constitute dynamic implications forwards and backwards in time. For $K = 2$, $\dashv$ as a matrix of 0's and 1's is an ordinary binary relation: the event a either is or is not related to state x. This relation is understood ambiguously as a two-valued distance in either time space ($a \dashv x$, physical) or information space ($x \models a$, mental).

Given any two contrary binary relations $R \subseteq U \times V$, $T \subseteq U \times W$, their *right residual* $R\backslash T$ [WD39, Jón82, Pra90b] can be defined equivalently as follows.

(i) As the operation satisfying $R; S \subseteq T$ iff $S \subseteq R\backslash T$. (Think of this as defining division on the left by R, with inequalities where one would expect an equality. The case $R = 0$, all entries 0, requires no special attention.)

(ii) As the largest relation $S \subseteq V \times W$ such that $R; S \subseteq T$.

(iii) As the set of all pairs (v, w) in $V \times W$ such that $uRv \to uTw$ for all $u \in U$.

(iv) As that operation monotone in its right hand argument that satisfies modus ponens, $R; (R\backslash T) \vdash T$, and also $T \vdash R\backslash(R; T)$, where $\vdash$ is read as $\subseteq$. This makes $R; -$ and $R\backslash -$ pseudoinverse operations which when composed either decrease or increase their argument depending on the order of composition.

(v) As the relation $(R^{\smile}; T^{-})^{-}$ where $R^{\smile}$ is converse (transpose) and T^{-} is complement (change all 0's to 1's in the matrix and vice versa). This can be written more neatly as $(T\dagger; R)\dagger$ where $T\dagger$ denotes $T^{-\smile}$. If we think of residuation $R\backslash T$ as a form of implication $R \to T$, and composition as a form of conjunction, and allow for the noncommutativity of relational composition (relative product), then this corresponds to the classical principle $A \to B \;\equiv\; \neg(A \wedge \neg B)$, as well as to linear logic's $A \multimap B \;\equiv\; (A \otimes B^{\perp})^{\perp}$.

It is a straightforward exercise to show the equivalence of these definitions; see [Pra90a] for further discussion.

Definition (v) reveals the contravariance of the operation in R, and its covariance in T, composition being monotone in each argument, a form of bilinearity.

We therefore call residuation *sesquilinear*, in anticipation of the next section.

Now consider $\Dashv\backslash\Dashv$ in the light of condition (iii). This instance of residuation is a binary relation on X. For all x, y in X, $x(\Dashv\backslash\Dashv)y$ holds just when row x implies (is a subset of) row y for every event, i.e. when $x \to y$ is valid. Now $x \to y$ says that in order to be able to get from x to y, every event a whose occurrence is recorded in x must still be recorded in y. Thus $\Dashv\backslash\Dashv$ consists of those pairs (x, y) which as transitions do not entail taking back the claim that an event has already happened.

This makes $\Dashv\backslash\Dashv$ the natural transition relation on X. This is a partially ordered *automaton*. Elsewhere we have used higher dimensional automata to argue that automata could be reliably paired up as the dual of schedules [Pra92]. We find Chu spaces a very appealing extension of this duality.

The left residual T/S, where $T \subseteq U \times W$, $S \subseteq V \times W$, is the dual of the right. We settle for defining T/S as the set of all pairs (u, v) in $U \times V$ such that $vSw \to uTw$ for all $w \in W$ (cf. (iii)), and ask the reader to infer the other four equivalent formulations corresponding to (i)-(v) above.

The left residual $\Dashv/\Dashv$ is, by dual reasoning to $\Dashv\backslash\Dashv$, that binary relation on A containing (a, b) just when for all $x \in X$, $b \Dashv x$ implies $a \Dashv x$. This makes it the natural temporal precedence relation on events, namely a *schedule* of events, an alternative to automata theory and Kripke structures that has attracted our attention as a reliable model of true concurrency since 1982 [Pra82].

When we unravel the primitive causal links contributing to secondary causal interaction we find that two events, or two states, communicate with each other by interrogating *all* entities of the opposite type. Thus event a deduces that it precedes event b not by broaching the matter with b directly, but instead by consulting the record of every state to see if there is any state volunteering a counterexample. When none is found, the precedence is established. Conversely when a Chu space is in state x and desires to pass to state y, it inquires as to whether this would undo any event that has already occurred. If not then the transition is allowed.

If one truly believed that the universe proceeded via state transitions, this might seem a roundabout and inefficient way of implementing those transitions. However it seems to us, particularly in view of the considerations of the following section, that the more likely possibility is that the universe only *seems* to proceed via state transitions, due perhaps to our ancestors having ill-advisedly chosen monism as the natural world view, perhaps millennia before the rise of Cartesianism, perhaps only some years after its decline. What we conjecture actually happens is that events signal states forward in time, or equivalently that states infer events backwards in time, and the world we imagine we live in is simply what that process looks like to its inhabitants when interpreted monistically.

Why this theory as opposed to any other? Well, certainly no other theory has satisfactorily explained the causal interaction of real mental and physical planes as conceived by Descartes. Whether monism is an equally satisfactory alternative for Descartes' problem is a good question. But for the other applications of Chu spaces considered here, namely concurrency, metamathematics, quantum mechanics, and logic (see below), it seems to us that monism simply cannot compete with dualism.

4 Quantum Mechanics

When time and truth are complex-valued as in quantum mechanics, right residuation is replaced by the sesquilinear operation of inner product $\langle\varphi|\psi\rangle$. This is a complex-valued *correlation* between wavefunctions $\langle\varphi|$ and $|\psi\rangle$, which are given as points of a *Hilbert space*, a metrically complete vector space which is made an inner product space with this operation.

The correspondence with Chu spaces is as follows. Any given choice of basis of Hilbert space defines a set of propositions, one per basis vector. Each coordinate of a given state vector relative to that basis is interpreted as the complex truth value of the corresponding proposition in that state. Relative to that basis, a state vector then corresponds to a row of $\models$, or a column of $=\!|$. Right residuation is defined even for one-state spaces, and is in form the logical counterpart to inner product. The right residual of a one-state space with itself is simply the identity relation on that state, this being the only partial order possible. The inner product of a wavefunction with itself is a scalar, namely its length squared, but quantum mechanics is a projective system where lengths are only physically meaningful in proportion: the length of a single state is no more informative in QM than is the identity partial order on a singleton.

A *mixed state* is a set of pure states and a distribution giving their relative probabilities. Such a distribution can be understood as a quantitative form of disjunction, making a mixed state the quantum mechanical counterpart of a Chu space. Here $\langle\varphi|\psi\rangle$ for mixed states corresponds to the right residual of two Chu spaces. The inner product of a mixed state with itself yields a square matrix of transition probabilities between its constituent pure states. The right residual of a Chu space with itself yields a square matrix of transition *possibilities* when $K = 2$, and a suitably richer relation for larger K, where the possibilities begin to depend on choice of quantale for K, taking us beyond the scope of this paper.

The outer product $|\psi\rangle\langle\varphi|$ produces an operator which transforms Hilbert space. Viewed as a transformation of basis vectors of Hilbert space, such an operator establishes correlations between attributes. The corresponding operation on Chu spaces is left residuation, which likewise produces a (two-valued) correlation between events, which we may identify with attributes.

This perspective leads to the following reconstruction of the emergence of modern quantum mechanics in 1925-26. Classical physics, and the old quantum mechanics, took between-state correlations as basic. Newton's laws, or their expression in terms of Lagrange's equations and the energy-difference Langrangian, were couched in terms of space and time, with velocity v being the derivative of position with respect to time, and momentum being mv. Hamilton made the bold move of taking momentum to be an independent quantity in its own right, observing that two equations per dimension based on a total-energy Hamiltonian yielded an elegantly symmetric reformulation of Langrange's one equation per dimension. From the perspective of classical physics this was no more than an ingeniously symmetric but otherwise unimproved variant of the basic laws of motion.

The new quantum mechanics made Hamilton's "causal interaction" of momentum and position primitive, and derived the classical laws as secondary. Furthermore they used the same logic, only as a complex-valued fuzzy logic rather than a two-valued logic, to achieve this end. This made momentum-space interaction a simple interaction, and the derived momentum-momentum and space-space interactions more complex. These can be understood as having to go both backwards and forwards in time for their complete effect, the basis for Cramer's transactional account of quantum mechanics [Cra86], which Leslie Lamport drew to my attention in 1987.

5 Conclusion

We have advanced a mechanism for the causal interaction of mind and body, and argued that separate additional mechanisms for body-body and mind-mind interaction can be dispensed with; mind-body interaction is all that is needed. This is a very different outcome from that contemplated by 17th century Cartesianists, who took body-body and mind-mind interaction as given and who could find no satisfactory passage from these to mind-body interaction. Even had they found a technically plausible solution to their puzzle, mind-body interaction would presumably still have been regarded as secondary to body-body interaction. We have reversed that priority.

One might not expect mind-body duality as a mere philosophical problem to address any urgent need outside of philosophy. Nevertheless we have offered solutions to the following practical problems that could be construed as particular applications of our general solution to Descartes' mind-body problem, broadly construed to allow scarecrows and everything else to have minds.

What is the conceptual basis of concurrent computation? What is the essence of quantum mechanics? On what foundation should mathematics be based? What is the right logic to reason with?

Concepts for concurrent computation. Our research focus since 1980 has been concurrent computation. Our conclusion is that programmers should be able to move as freely as possible between declarative and imperative modes of thought about the same program. We are now convinced that the duality of schedules and automata, as the realization of the duality of body and mind respectively in the world of programming, provides a better conceptual foundation for concurrent programming than any other model.

Essence of quantum mechanics. We claim that quantum mechanics has not previously been reduced to lay terms by physicists, who have been content to leave the subject as a mysterious jumble of properties of Hilbert space that the working physicist can become acclimatized to and even confident with after sufficient exposure. Mind-body duality and interaction explain respectively complementarity and the inner product in relatively elementary terms making a clear connection with other structures such as the above model of computation and the following foundation for mathematics. The central role of the mental plane in this account of quantum mechanics makes it a *rational* mechanics.

Foundations of mathematics. We implicitly settle for relational structures as the objects of mathematics when we so restrict the models of first-order logic. But this has the unfortunate side effect of excluding some popular mathematical structures, most notably topology, which would appear to require a second order theory. Chu spaces over 2^k realize all k-ary relational structures [Pra93, p.154-3] as well as topological spaces when $K = 2$ [LS91], all as objects of the one category, yielding a novel degree of morphism-sensitive typelessness for foundations. The above connection with quantum mechanics suggests that mathematics based on Chu spaces be thought of as *natural* mathematics, sharing with nature the essential principles of duality and interaction.

Choice of logic. We envision two logics, elementary and transformational. Elementary logic has its usual meaning as the logic of individual objects such as sets, groups, and Boolean algebras. It serves to reason about relationships between elements of such objects. These objects are traditionally understood as relational structures but they can also more generally be understood as Chu spaces as per the preceding paragraph.

Transformational logic bears superficial resemblances to elementary logic but serves to reason about interactions between objects rather than relationships within objects. The structural basis for object interaction is the homomorphism or structure-preserving morphism, from which flows all other interaction structure such as duality, limits, tensor products, homsets, and size (cardinality or concreteness).

The most promising transformational logic seems to us to be Girard's linear logic [Gir87]. $\mathbf{Chu}_K$ is a constructive model of linear logic in the sense that it interprets the sequents of linear logic as sets of proofs rather than as Boolean or intuitionistic truth values. Nonconstructive models of linear logic such as phase spaces seem to us at best a curiosity. As to alternative constructive models, for want of any convincing counterexamples we conjecture mildly that these can all be satisfactorily subsumed by Chu spaces, the case $V = \mathbf{Set}$ of the general Chu construction $\mathbf{Chu}(V, k)$. We have yet to be shown a V that improves on $\mathbf{Set}$ for any significant application of the Chu construction.

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