Orthocurrence as both Interaction and Observation

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Abstract

Orthocurrence or tensor product $A \otimes B$ of systems A and B can be understood symmetrically as an interaction operator expressing a form of conjunction of state predicates, or asymmetrically as one of two information channels: either $A \multimap B^{\perp}$ as a system consisting of A observing states of B (equivalently, conveying information about the states of B to A), or $B \multimap A^{\perp}$ for the same thing in the other direction. We show how the notion of Chu space or couple arises as a natural corollary of this point of view. We conclude with a history of orthocurrence.

This paper is intended to be read in conjunction with the paper by R. Rodriguez and F. Anger in this workshop [RA01]. The focus of the paper is on the intuition underlying, and history of, orthocurrence rather than on its more technically detailed aspects. For more formal definitions the impatient reader might prefer [Pra99]. One goal here is to persuade the reader that couples themselves, as well as their orthocurrence defined as tensor product, are not just artificial constructs but natural consequences of intuitively plausible aspects of interaction and observation.

1 Introduction

If two people pass in the street with no permanent after-effects attributable to that passage, one could say that they did not interact significantly, or that neither observed the other. The former seems to address the physical aspects of their passing (one imagines some sort of perturbation of the state of each party induced by the other) while the latter sounds more psychological or information-theoretic.

To one trained in physics, physical interaction tends to be symmetric, as with Newton's law of motion that every action has an equal and opposite reaction. Observing on the other hand, certainly to one trained in philosophy or logic, seems less symmetric, not least for covert observation.

A viewpoint we have gradually grown into over two decades (recounted in the history section below) is that interaction and observation can be seen as two sides of the same coin via the well-known basic isomorphism

$$A \multimap (B \multimap C) \cong (A \otimes B) \multimap C,$$

referred to in computer science circles as currying.

Here $A \otimes B$ denotes a system of two interacting systems A and B called their *orthocurrence*, whose size is the product of the sizes of A and B. More precisely, we understand the "stuff" of A to be a set $|A|$ whose elements are the *points* of A. As part of the definition of $A \otimes B$ we have $|A \otimes B| = |A| \times |B|$. Orthocurrence is moreover symmetric, as reflected in the isomorphism

$$A \otimes B \cong B \otimes A.$$

The construct $A \multimap B$ on the other hand denotes a system consisting of the possible outcomes of system A observing system B. When A is a simple system in the sense of having just one vantage point with a clear view of B, $A \multimap B$ is isomorphic to B. When A is more complex, the appearance of B from the various vantage points of A is correspondingly more complex.

For example when A consists of two independent simple observers, $A \multimap B$ is isomorphic to the system B^2, namely B as viewed from two independent vantage points. We denote the As in these two examples as respectively 1 and 2 ($= 1 + 1$); thus $1 \multimap B \cong B$ and $2 \multimap B \cong B^2$.

What A sees on any given occasion from point a of A is a point b of B. The system $A \multimap B$ as a whole then consists of the possible views of B. Such a view constitutes a *variable point* of B with $|A|$ as its domain of variation, i.e. a function $f : |A| \to |B|$. Each observation of B by A yields one such function specifying what A saw collectively at that observation. Although $a \in A$ and $f(a) \in B$ are formally speaking both points here, we tend to think of the former as a vantage point and the latter as an observed value.

Note that this is a distributed notion of observation: in a single observation B may look different from different vantage points of A. Furthermore from a fixed vantage point different observations may yield different values of B.

2 States

The basic isomorphism admits of various insights. Let us begin by taking A and B to be two "systems of interest" and C to be merely a certain ambient environment, denoted $\perp$. Then $A \multimap \perp$ is that environment as seen from A. This makes $A \multimap (B \multimap \perp)$ that environment as seen from B as seen from A. The basic isomorphism then tells us that this is isomorphic to $(A \otimes B) \multimap \perp$, namely the environment seen from the system $A \otimes B$ consisting of A interacting with B.

Thinking of the environment as a kind of mirror reflecting whoever looks into it, we abbreviate $A \multimap \perp$ to $A^\perp$, called the *perp* operator. The previous isomorphism then becomes

$$A \multimap B^\perp \cong (A \otimes B)^\perp.$$

$A^\perp$ has the same form as the intuitionistic logic abbreviation $\neg A$ for $A \to 0$, namely intuitionistic negation. However $A^\perp$ is not a negated proposition *per se* but rather a system consisting of a system observing its environment.

For larger A observing with unclouded vision one would expect $A^\perp$ to be also larger, in fact exponentially so as a function of the size of A. For example if $\perp$ has 2 points (the smallest environment with distinguishable features, call them 0 and 1) and A has n independent vantage points from each of which either value of the environment may be seen, then an observation by A of its environment has 2^n possible outcomes.

If however any vantage point a of A is "stuck at" say 0, meaning that a never sees 1, or if some a is not independent of A's other points, meaning that the set of possible views of the environment from the other points depends nontrivially on what the view from a turned out to be, then we may think of A's vision as being clouded or reduced. The fewer combinations A can collectively resolve, the less perfect its vision. Total blindness is the case when $A^\perp$ has only one point.

We thus have two complementary aspects of a system A, size and perspicacity, measuring respectively the stuff and vision of A. We have already formalized stuff as a set of points. We now formalize vision as the set of points of $A^\perp$, called the *states* of A, forming a set X. A state $x \in X$ determines one point of $\perp$ for each point of A, making it a function $x : |A| \to |\perp|$.

So what are the states of $A^\perp$, i.e. the points of $A^{\perp\perp}$? These exist because we have allowed observation of any system B by any system A to form a system $A \multimap B$, forcing the system $(A \multimap \perp) \multimap \perp$ to exist, and we have assumed that all systems have points.

Now $|A^{\perp\perp}|$ must consist of certain functions $g : X \to |\perp|$. What could such a function g be? If g is given $x : |A| \to |\perp|$ and is then asked to produce a point of $\perp$ in a reasonable way, what can it do?

While there are a great many possibilities here, hardly any admit of a uniform definition. One plausible choice would be to allow all functions. The set of states of $A^\perp$ then becomes 2^X, the power set of its set of points. It should now be clear that for systems of the form $A^\perp$, the perp operator amounts to the power set operator. Those systems would turn out, were we to pursue the sequel's line of reasoning for this choice, to be essentially sets transforming via functions, i.e. the category **Set**.

Another uniform choice is all constant functions.

In this case the environment would look the same throughout $A^\perp$, effectively giving $A^\perp$ the observational power of a system with only one point.

We shall instead take what is pretty much the only reasonable choice left, namely those functions that obtain the required value of $\perp$ by applying their argument to some $a \in A$, i.e. those functions g satisfying $\exists a \in |A| \forall x \in |A^\perp|.g(x) = x(a)$.

When there exist two points a_1, a_2 of A from which the environment always looks the same, i.e. for which $x(a_1) = x(a_2)$ for all states x of A, it follows that $\lambda x.x(a_1) = \lambda x.x(a_2)$, resulting in the identification of those two states of $A^\perp$ as defined here. When conversely every pair of points of A is separated by some state x we say that A is separable. In this situation, each point of A is representable faithfully (meaning without confusion) as the function $\lambda x.x(a)$. In this paper, as for [RA01], we shall assume separability.

We may now calculate $A^{\perp\perp}$. Points and states are interchanged a second time. State $\lambda x.x(a)$ becomes a point, from which we can recover a thanks to separability. Point $x : |A| \to |\perp|$ turns back into state $\lambda a.((\lambda x.x(a))(x))$, which we can β-reduce to $\lambda a.x(a)$ and then η-reduce to x to get back to where we started.

Thus we have

$$A^{\perp\perp} = A.$$

That is, the operator $^\perp$ is an involution. Moreover it is identically so rather than merely up to isomorphism, as no information at all need be lost at each transposition.

We observed earlier that $1 \multimap A \cong A$. The case $A = \perp$ tells us that $1^\perp \cong \perp$. Hence $\perp^\perp \cong 1$, i.e. $\perp$ has just one state, making it totally blind. So when the environment observes itself from any given point of itself, all it can see every time is that point.

Returning to the basic (currying) isomorphism, with $C = \perp$, the left hand side is $A \multimap B^\perp$ while the right is $(A \otimes B)^\perp$. By substituting $B^\perp$ for B and reducing $B^{\perp\perp}$ to B, we obtain $A \multimap B \cong (A \otimes B^\perp)^\perp$. Alternatively, by applying perp to both sides, we obtain $A \otimes B \cong (A \multimap B^\perp)^\perp$. This demonstrates that either of interaction and observation can be defined in terms of the other: they are just (quite literally) two sides of the same system (provided we perp B when changing sides).

Via the symmetry isomorphism $A \otimes B \cong B \otimes A$, we also obtain $A \otimes B \cong (B \multimap A^\perp)^\perp$, which we have already seen to be isomorphic to $(A \multimap B^\perp)^\perp$. So $A \multimap B^\perp \cong B \multimap A^\perp$, and therefore (substituting $B^\perp$ for B again) $A \multimap B \cong B^\perp \multimap A^\perp$.

The basic isomorphism also immediately yields associativity of interaction (up to isomorphism). Simply substitute $C^\perp$ for C, replace all instances of $A \multimap B$ by $(A \otimes B^\perp)^\perp$, and cancel all double perps.

At this point the reader should review the foregoing development to assess the extent to which rabbits were pulled out of hats in this theory of interaction and observation. We started with the currying isomorphism as a plausible relationship between these two notions. We postulated a distinguished system constituting the environment. We assumed that systems were distributed in the sense of having multiple vantage points, which did double duty as observed values, thereby allowing us to distinguish size $|A|$ from vision $|A^\perp|$. We did make a choice for the states of $|A^\perp|$, whose effect was to satisfy $A^{\perp\perp} = A$. Everything else in the development was completely forced.

3 Couples

We have now essentially arrived at the notion of Chu space or *couple*[1] almost entirely from reasonable views on the nature of interaction and observation. We may define a couple as a set A of points and a set X of states. In the above a state is a function $x : A \to \Sigma$ (or K, or S [RA01]). This determines a function or *matrix* $r : A \times X \to \Sigma$ definable as $r(a, x) = x(a)$. We then arrive at a more pleasingly symmetric definition of a couple as a matrix (A, r, X). Here A and X are two sets given *a priori* with no structure, and r is a matrix which confers structure simultaneously on both A and X.

The one restriction this approach induces is that no columns are repeated, in which case the couple is called *extensional* (since we were viewing states extensionally as functions). To support perp we also require that no rows be repeated, calling the couple

[1] A term we are currently experimenting with as synonymous with but shorter than Chu space.

separable in that case. The conjunction of "extensional" and "separable" is "biextensional."

We shall call a morphism of couples a continuous function. It would raise no eyebrows at this point were we to simply give the definition of continuity for $f : A \to B$, one statement of which is that the *inverse image* of every state $y : B \to \bot$ of B, defined as $y \circ f : A \to \bot$, is a state of A. However even continuity can be derived from reasonable requirements about interaction and observation rather than by fiat, as follows.

A continuous map is naturally enough a point of $A \multimap B$. This makes it also a state of $A \otimes B^{\bot}$, and it is from the underlying intuition of interaction rather than observation that we shall extract the definition of a map.

Now the stuff of $A \otimes B$ is the rectangle $|A| \times |B|$, whose rows are indexed by $|A|$ and columns by $|B|$. A state of $A \otimes B$ is a function from $|A| \times |B|$ to Σ $(= |\bot|)$. We can think of this state as a solved crossword puzzle [Pra00], one filled in with letters from the alphabet Σ (whence the choice of symbol).

From the common isomorphism of $(A \otimes B)^{\bot}$, $A \multimap B^{\bot}$, and $B \multimap A^{\bot}$, a state of $A \otimes B$ can be understand as a point of either $A \multimap B^{\bot}$ or $B \multimap A^{\bot}$. The meaning of the former reveals it to be a variable state of B, varying over A. From each vantage point a of A we observe not a point of B as with $A \multimap B$ but rather a state of B. Since observation of B from any vantage point should never reveal B to be in any unlicensed state, we demand that each Across word in the puzzle solution be (when transposed) a column of B. The corresponding argument for $B \multimap A^{\bot}$ demands that each Down word in the solution be a column of A. From this perspective the states of A and B provided dictionaries for respectively the Down and Across words allowed in the puzzle.

The states of $A \otimes B$ are then all crossword puzzle solutions meeting the above constraints, i.e. whose horizontal words are (transposed) states (columns) of B *and* whose vertical words are states (columns) of A. The conjunction italicized here is the one promised in the abstract: if we view the criteria for selecting states in each of A and B as predicates on Σ^A and Σ^B respectively, then this combination of conditions on the permitted crossword solutions can be viewed as a form of conjunction of those predicates.

We have now derived, from what is an almost purely intuitive understanding of interaction and observation, everything needed for the definition of orthocurrence on couples as found in [RA01].

The one thing remaining is to complete the promised specification of the points of $A \multimap B$. But these are simply the states of $A \otimes B^{\bot}$. The constraint that such a state draw its rows from columns of $B^{\bot}$ becomes that the rows must be rows of B. Since rows are not repeated, this uniquely determines a function $f : |A| \to |B|$.

The dual constraint, that the columns of the state all be columns of A, uniquely determines a function $g : Y \to X$ from the set Y of states of B to the set X of states of A. The fact that both functions are determined by the same state immediately induces the adjointness condition $s(f(a), y) = r(a, g(y))$, since the two sides have in common the entry on the a-th row and y-th column of the state of $A \otimes B^{\bot}$. The adjointness condition can also be understood as the condition for g to be the inverse image function associated to f, justifying "continuous function" as the term for a morphism of couples. The actual continuous function is f, with g serving as witness to its continuity. When A is extensional, f determines g; when not, it is customary to give g explicitly in addition to f.

We leave to the reader to show that, among every class of functions between systems, the class of continuous functions is the least one closed under composition that makes all system states continuous. (Bear in mind that $\bot$ is a system.) An advantage of this characterization is its elegance, a drawback is its third-order quantification over classes of functions.

The couple $|A \multimap B|$ can now be described as a matrix of size $|A \multimap B| \times (|A| \times |B^{\bot}|)$. Regrouping these terms as $(|A| \times |A \multimap B|) \times |B^{\bot}|$, and appealing to the separability of B to replace each column of length $|B^{\bot}|$ in this matrix by a single point of B, we arrive at an $|A| \times |A \multimap B|$ matrix of points of B. This formalizes the remark in the closing paragraph of the introduction, that observation $A \multimap B$ is a distributed notion for which the observed point of B depends on both the vantage point in A and the particular instance of observation.

4 Examples

The purpose of this paper was to provide background for other papers on orthocurrence, such as [Pra86, CCMP91, Pra00, RA01], which contain many good examples. The latter two papers include various cases of orthocurrence of intervals producing Allen-type temporal relations.

There is however one example not treated elsewhere that is worth drawing attention to, that of conflict in orthocurrence. Consider two systems each with two points a and b. Taking $K = 2$, allow all states for each of those systems save that in which both points are 1. (So each system has three states conveniently describable as 00, 01, and 10.) This is known in the concurrency literature as conflict: either of a or b can enter state 1 but not both.

A check of the 2×2 crossword puzzles that disallow 11 as either a row or a column turns up 7 solutions: one with all zeroes, four with a single 1, and two checkerboards. What does this mean?

Consider this applied to trains and stations [GP93], with 1 indicating *at* or $=$ and 0 *not-at* or $\neq$, and railroad track as the common universe of trains and stations. Although neither two trains nor two stations can occupy the same section of track at the same time, a train and a station can. The seven solutions correspond to both trains being out of both stations (a single case), one train being in either station (four cases), and both trains in both stations (two cases).

One important point about this example is that it gives a simple case where tensor product is clearly different from direct product. With trains t, t' and stations s, s', (t, s) in the direct product conflicts not only with (t, s') and (t', s) (as it does for tensor product) but *a fortiori* with (t', s') (since such relationships must be respected at every coordinate of a direct product). It is intuitively clear however that there is no conflict in practice between (t, s) and (t', s'), since the trains are out of each other's way when they are at respective stations that are out of each other's way.

This is surely the simplest conceivable mathematical example (other than those involving empty sets) pointing up in an intuitively clear way the difference between direct product and tensor product. Since direct product and tensor product coincide for both sets and posets, one cannot look there for differences. Furthermore when all one's practical experience of orthocurrence is restricted to posets (as it was when we first started considering it), it is all too easy to jump to the conclusion that orthocurrence should continue to have the characteristics of direct product in other situations, e.g. event structures [NPW81], which combine order and conflict in the one structure. One then mistakenly believes that orthocurrence gives the wrong results for those situations, when in fact the tensor product definition yields exactly the results called for by intuition, and moreover for intuitively satisfying reasons based on a straightforward understanding of interaction and observation.

Now for an example from linear algebra. As noted by Lafont and Streicher [LS91], the vector spaces over a field k may be represented as couples whose points are the points of the space, whose states are its functionals (linear transformations to the one-dimensional space), and whose matrix gives the result of applying a functional to a point (so K is just the underlying set of the field k). The continuous functions between such representing couples are exactly the linear transformations between the vector spaces represented by those couples.

Lafont and Streicher also remark that tensor product $\mathcal{U} \otimes \mathcal{V}$ of vector spaces is different from that of couples as applied to the above representatives. While the states are the same for both, vector space tensor product yields more points than couple tensor product. The problem is that while the states of both are closed under linear combinations, as are the points of the vector space tensor product, the points of the couple tensor product are not.

Nevertheless Halmos' following elegant definition of tensor product of vector spaces [Hal74, §25] makes a close connection with the couple definition. "The *tensor product* $\mathcal{U} \otimes \mathcal{V}$ of two finite-dimensional vector spaces $\mathcal{U}$ and $\mathcal{V}$ (over the same field) is the dual of the vector space of all bilinear forms on $\mathcal{U} \oplus \mathcal{V}$."

Note that $\mathcal{U} \oplus \mathcal{V}$ (direct sum of vector spaces) is also the direct product $\mathcal{U} \times \mathcal{V}$ (coproducts and products of vector spaces coincide as so-called biproducts). A bilinear form on $\mathcal{U} \oplus \mathcal{V}$ thus amounts to a matrix indexed by the points of the two spaces. Bilinearity means that every row is a functional on $\mathcal{V}$ and every column a functional on $\mathcal{U}$. But the functionals on a vector space are the columns of the Lafont-Streicher representation of that space. So the bilinear forms Halmos is using are precisely those crossword solutions on $|U| \times |V|$ that restrict themselves to linear words, both across and down.

At this point we now have the couple tensor product. Halmos now passes to the vector space tensor product by observing that the set of states are those of $\mathcal{U} \otimes \mathcal{V}$, and that they form the vector space $(\mathcal{U}\otimes\mathcal{V})^{\perp}$, being closed under linear combinations. All that remains for him therefore is to take the functionals on $(\mathcal{U} \otimes \mathcal{V})^{\perp}$, which then yield the double dual $(\mathcal{U} \otimes \mathcal{V})^{\perp\perp} \cong \mathcal{U} \otimes \mathcal{V}$.

The last step of Halmos' trick is in effect a slick way of closing the points of the couple tensor product under linear combinations. We thus have an embedding of the couple tensor product in the vector space tensor product, whose adjoint (reverse) map is the identity function. This embedding is an instance of a so-called *tensorial strength*, having the form $F(\mathcal{U}) \otimes F(\mathcal{V}) \to F(\mathcal{U} \otimes \mathcal{V})$ where $F : \mathbf{Vct}_k \to \mathbf{Chu}_K$ is the (full and faithful) Lafont-Streicher representation of vector spaces by couples.

5 History

Orthocurrence is an operation of process algebra sibling to concurrence. The essential difference between the two is that whereas concurrence combines events additively, orthocurrence combines them multiplicatively. Both operations can manufacture concurrent behavior from sequential, witness the product of 2-chains in the case of orthocurrence, the topic of this paper.

Mathematically orthocurrence is tensor product. The principal originators of tensor algebra were Gregorio Ricci-Curbastro and his student Tullio Levi-Civita [RLC00], based on Riemann's general metric and Christoffel's curvature tensor, the decisive application being due to Einstein.

A broader notion of tensor product evolved within category theory, where it is allowed to be any weakly associative binary functor. But in practice tensor product is rarely referred to as such unless it has a right adjoint in at least one of its arguments. The first comprehensive treatment of closed categories from this perspective is that of Eilenberg and Kelly [EK65], which studies internal homs both with and without tensor product (doing without being something of a *tour de force*).

The significance of "closed" here, important in understanding the difference between $A \times B$ and $A \otimes B$, is that one need not leave the category in order to have a notion of function space. The elementary notion of a category C takes the space $C(x, y)$ of morphisms from x to y to be a set, the so-called *homset*. The associated homfunctor $\mathrm{Hom}_C : C^{\mathrm{op}} \times C \to \mathbf{Set}$ is defined by $C(w \xrightarrow{f} x, y \xrightarrow{h} z) : C(x, y) \to C(w, z)$, mapping $x \xrightarrow{g} y$ to $w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ (note that whereas this grows the right end of g to the right, a covariant action, it grows the left end to the left, a contravariant action, whence C^{op} rather than merely C). However in category theory sets do not play the fundamental role asssigned them in the traditional development of mathematics, and an internal homfunctor $\mathrm{Hom}_C : C^{\mathrm{op}} \times C \to C$ permits the morphisms from x to y to form an object of C rather than a set.

Also important is the notion of cartesian closed. This refers to the product to which the internal hom is right adjoint, which is "cartesian" when it is ordinary product, i.e. has projections $x \times y \to x$, $x \times y \to y$.

In the context of process algebra as one way of formalizing reasoning about concurrency, we arrived at the orthocurrence connective while looking for a more algebraic formalization of our predicate calculus solution to Problem 1, Channel with Disconnect, in [Pra85b]. As such the operator first appeared in [Pra85a], and in more detail in [Pra86], where we used it to construct a channel as the orthocurrence of a sequence of messages with a sequence of points through which the messages flow (two points in the simple end-to-end case). At that time we were working with a notion of process defined as a set of pomsets (partially ordered multisets or labeled partial orders), for which ordinary product $A \times B$ seemed the appropriate mathematical formalization.

We later tried extending the partial order notion of time to other metrics satisfying a suitable generalization of the triangle inequality. Our students Ross Casley and Roger Crew pointed out that in some of these other metrics, orthocurrence no longer had sensible projections and hence could not be direct product $A \times B$. However it still retained the properties of tensor product $A \otimes B$, indicating that the passage to other metrics involving passing from the cartesian closed category of posets to closed but not cartesian closed categories of other forms of generalized metric space. We interpreted orthocurrence as flow (of systems past or through each other) and its right adjoint $A \to B$ as observation. These ideas were sketched at a CTCS conference in 1989 and spelled out in [CCMP91].

During the same period Jean-Yves Girard was developing the notion of linear logic [Gir87] as a formalization of structural aspects of proof theory. However despite attending several talks on linear logic starting in 1986, it was not until 1989 that we started to see similarities. The two obvious connections were the additive-multiplicative distinction (*concurrence* vs. *orthocurrence*, *plus* vs. *tensor*, a distinction going as far back as C.S. Peirce's *relative sum* $a + b$ vs. *relative product* $a; b$ [Pei33]), and the product's right adjoint. Girard's influence is felt in our replacement of "flow" by "interaction" in [Pra92c, Pra92a], which seemed at least as suggestive to us of the intuition underlying orthocurrence.

The main difference seemed to be the absence of duality from our process algebra, which seemed more like intuitionistic than classical linear logic. But during 1990-1991 we realized that the absence of duality was a result more of our myopic perception of the limits of the notion of behavior than an intrinsic aspect of process algebra, and that a broader notion of process admitted the same kind of dualization as in classical linear logic [Pra91, Pra92c, Pra92b]. Shortly thereafter, with our Ph.D. student Vineet Gupta, we found [GP93] that couples yielded a delightfully simple yet complete model of what we had been working towards.

Couples go back further than linear logic. In their categorical form they were first proposed by Barr [Bar79, Bar91], and in the set theoretic form followed here by Lafont and Streicher [LS91]. Barr's inspiration for the notion came in turn from work in so-called soft analysis arising out of an idea in Mackey's thesis [Mac45]. Barr defined general V-enriched couples, whose carrier, cocarrier, and alphabet k are objects of a symmetric monoidal closed category V, forming the category $\mathbf{Chu}(V, k)$ studied by Barr's student P. Chu [Bar79, appendix]. Lafont and Streicher treated ordinary couples, the case $V = Set$, i.e. the points form simply a set and likewise the states, under the rubric of games. More recently Barwise and Seligman [BS97] have treated ordinary couples for $k = 2$ under the name of classifications, with tokens, types, and infomorphisms for respectively points, states, and continuous functions.

In 1995, stimulated by several discussions of couples at the WoLLiC conference in Recife, we began to examine whether the so-called proof nets of linear logic could be represented fully and faithfully as dinatural or logical transformations of functors in **Chu**. While we could find dinatural counterexamples, to date we have found no logical counterexamples. Moreover in 1998 with Devarajan, Hughes, and Plotkin we established the result for MLL, the multiplicative fragment of linear logic limited to the connectives $A \otimes B$ and $A^{\perp}$ [DHPP99]. Our research associate Dominic Hughes has since been investigating MALL, MLL plus the additive connective $A \oplus B$, which has turned out to be a far harder problem. A long-running computer search of hundreds of millions of MALL proof structures has so far failed to produce a single counterexample, while however providing useful insight into the question.

Both linear logic and couples are very natural and appealing frameworks for their respective arenas of logic and mathematics. The more precise their match-up therefore, the greater the synergetic boost to the significance of both frameworks. Our fingers are tightly crossed for extending the perfection of the MLL match-up at least to MALL.

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