# Origins of the Calculus of Binary Relations

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**Abstract**

*The calculus of binary relations was introduced by De Morgan in 1860, and was subsequently greatly developed by Peirce and Schröder. Half a century later Tarski, Jónsson, Lyndon, and Monk further developed the calculus from the perspective of modern model theory.*

## 1 The Calculus

The origins of the calculus of binary relations go back to 1860 in a paper by Augustus De Morgan, *On the Syllogism: IV; and on the Logic of Relations* [DM60]. De Morgan begins his paper by categorizing Aristotle as "rather too much the expositor of common language, too little the expositor of common thought." Aristotle had denied, in the 4th century BC, that every binary relation has a converse. His example was that "the rudder of the ship" lacked the converse notion "the ship of the rudder." Now De Morgan was always on the lookout for logical fallacies. Though the main targets of his "Budget of Paradoxes" were the circle squarers and cube duplicators, in this instance he was not abashed to challenge the authority of Aristotle 22 centuries later with the argument, "Surely the question 'What ship does this rudder belong to?' must sometimes have been heard in an Athenian dockyard."

De Morgan went on to list the connectives of his calculus. However his connectives are best understood in terms of the connectives we use today, so before listing De Morgan's connectives let us move forward a little in time to Peirce.

The calculus of binary relations as we understand it today has three premises. First it is a logic. Second, it consists of two components, a logical or static component and a relative or dynamic component. Third, the two components parallel one another in that all the logical symbols have their relative counterparts.

Whereas De Morgan's account of the calculus only vaguely hinted at these facets, Peirce understood and described them very clearly [Pei33]. Peirce saw each of the two components, logical and relative, as a fully fledged logic in its own right, consisting of two constants *false* and *true*, one unary operation of *negation*, and two binary operations *disjunction* and *conjunction*.

In modern notation we may tabulate the resulting ten constants and operations thus.

$$\begin{array}{llllll} \textit{Logical}: & 0 & 1 & a^{-} & a+b & ab \\ \textit{Relative}: & 0' & 1' & a^{\smile} & a \dagger b & a;b \end{array}$$

The logical symbols consist of the static constants 0 and 1, complement $a^{-}$, sum $a+b$, and product $ab$. The relative symbols consist of the constants 0' and 1', converse $a^{\smile}$, relative sum $a \dagger b$, and composition $a;b$.

Dedekind in 1888 [Ded01] defined the notion of natural number in terms of the transitive closure of the successor relation. This was a second order definition that could be made first order in two ways. In 1889 Peano found one of those ways, namely his famous first-order axiomatization of the natural numbers. In 1895 Schröder gave the other, namely (following Peirce) to treat binary relations as first-order individuals and to define transitive closure as just another operation. Schröder accordingly added four unary operations abstractly expressing "Induktion oder Rekursion": transitive closure $a_{00}$, reflexive transitive closure $a_0$ (which nowadays we would notate respectively $a^+$ and $a^*$), and their respective De Morgan duals $a_{11}$ and $a_1$ (for which there is no modern notation).

Following Ward and Dilworth [WD39], Birkhoff in 1948 [Bir48] added two more operations, $a\backslash b$ and $b/a$,

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called respectively *right* and *left residuals* and interpretable as relative implications $a \rightarrow b$ and $b \leftarrow a$.

*Interpretation.* A binary relation is a set of pairs of elements assumed to be drawn from an indeterminate but fixed set $X$. The logical operations treat a binary relation purely as a set, ignoring the nature of its elements. 0 denotes the empty relation while 1 denoted (prior to the 1950's)[1] the complete relation $X^2$. Complement $a^-$ is taken relative to 1. Sum and product are union and intersection.

The relative operations take into account that a binary relation consists of pairs. 0' consists of all pairs $(x, y)$ for which $x \neq y$, while 1' is its complement, the pairs $(x, x)$, constituting the diagonal or identity relation. The converse $a^\smile$ consists of all pairs $(y, x)$ such that $(x, y)$ is a pair in $a$. The composition $a;b$ consists of all pairs $(x, z)$ such that there exists $y$ with $(x, y)$ in $a$ and $(y, z)$ in $b$. The relative sum $a \dagger b$ consists of all pairs $(x, z)$ such that for all $y$ either $(x, y)$ is in $a$ or $(y, z)$ is in $b$.

The transitive closure $a^+$ of $a$ is the least relation including both $a^+;a^+$ (transitivity) and $a$, while $a^* = a_0 = 1' + a^+$. Dually $a_{11}$ is the greatest relation included in both $a_{11} \dagger a_{11}$ and $a$, while $a_1 = 0' a_{11}$.

The right residual $a \backslash b$ or $a \rightarrow b$ is elegantly expressed as $a^{\smile -} \dagger b$, or dually as $(a^\smile; b^-)^-$. It can be defined as the set of all pairs $(y, z)$ such that for all $x$, $(x, y)$ in $a$ implies $(x, z)$ in $b$. It is also the greatest $c$ such that $a; c \subseteq b$. The converse notion is the left residual $b/a$, or $b \leftarrow a$, which clearly will be $b \dagger a^{\smile -}$ or $(b^-; a^\smile)^-$, but which can also be considered as the "converse De Morgan dual" of the right residual, namely $b/a = (a^\smile \backslash b^\smile)^\smile$. Equivalently $b/a$ consists of all pairs $(x, y)$ such that for all $z$, $(y, z)$ in $a$ implies $(x, z)$ in $b$, and is the greatest $c$ such that $c; a \subseteq b$. Setting $b$ to 0' leads to the definition of converse in terms of either residual, namely $a^\smile = a^- \backslash 0' = 0'/a^-$.

## 2 A Brief Chronology

As noted above De Morgan was the founder of the calculus, writing a single paper on it in 1860 [DM60]. Though he did not develop an equational calculus he nevertheless completely identified the essence of residuation as we shall see later.

Peirce turned to the subject in 1870, writing several papers and a good many unpublished manuscripts during the next twenty years [Pei33]. Peirce found most of the interesting equational laws of relation algebra.

The third volume of Schröder's series of books on algebraic logic, an 800-page tome [Sch95], was devoted to the calculus. Schröder added to it the operations of transitive closure, reflexive transitive closure, and their De Morgan duals.

Bertrand Russell took up the subject in one of his earliest papers [Rus00] with the explanation "the logic of Peano is hardly complete without an explicit introduction to relations... the definition of *function* is not possible except through knowing a new primitive idea, that of *relation*." But while acknowledging the contributions of Peirce and Schröder Russell viewed them as "difficult and complicated to so great a degree as to doubt their utility" and with the enthusiasm of youth felt both obliged and qualified to begin the subject anew.

The subject then fell into neglect, to be revived forty years later by Tarski in 1941 [Tar41]. Tarski's school, firmly grounded in model theory, indeed perhaps the leading source of model theoretic ideas of its day, then proceeded to apply those ideas to gain a deeper understanding of the calculus than the algebraic logicians of the 19th century could reasonably have been expected to hope for.

The importance we attach today to the niceties of typing, in particular that of domain and codomain of functions and relations, were lost on all early students of binary relations save Russell, who referred [Rus00] to the *domain* of a relation $R$ and notated it $\rho R$. He also notated the dual notion $\breve{\rho} R$; although he did not give it a name then, subsequently in *Principia Mathematica* [RW35, p.247] it was named *converse domain*, in close agreement with our modern terminology of codomain. Russell and Whitehead also proposed the term *field* to describe the domain when it was equal to the codomain.

## 3 Composition as Conjunction

Relational composition or relative product has been viewed as the relative analogue of logical or static conjunction for a long time. Actually it is not even necessary to keep the static conjunction: it is perfectly reasonable to have composition (or concatenation) as the *only* conjunction. This describes for example J. Lambek's 1958 calculus of sentence structure [Lam58], which can be viewed equivalently as either a proof system or (up to weak generative capacity as shown by

[1] The custom now is to take it to be the maximal binary relation among all those being combined with these operations, i.e. the union of all such. This is necessarily an equivalence relation on $X^2$ but need not be the complete relation.

Gaifman around 1960) a context-free grammar, with theoremhood corresponding to derivability. Kleene's 1957 calculus of regular expressions can be similarly viewed as a monotone logic in which composition or concatenation is the only conjunction, $a + b$ is disjunction, and star $a^*$ makes it a sort of "propositional Datalog".

But this view of composition/concatenation as a form of conjunction predates even Peirce and would appear to be due to De Morgan in 1860 [DM60]. The following footnote appears exactly one-third of the way through De Morgan's "On the Syllogism IV" (p.221 in Heath's anthology "On the Syllogism" [Hea66]). Here De Morgan argues that, allowing for the obvious differences, composition $L;M$ of relations $L$ and $M$ resembles conjunction $XY$ of "terms" (predicates) $X$ and $Y$. Indeed he notates composition $LM$ the better to suggest conjunction—the $L;M$ notation which is now in almost universal use, and is in (fortuitous?) agreement with Algol 60 and dynamic logic [Pra76], was introduced later by Peirce.

> A mathematician may raise a moment's question as to whether $L$ and $M$ are properly said to be *compounded* in the sense in which $X$ and $Y$ are said to be compounded in the term $XY$. In the phrase *brother of parent*, are *brother* and *parent* compounded in the same manner as *white* and *ball* in the term *white ball*. I hold the affirmative, so far as concerns the distinction between *composition* and *aggregation*: not denying the essential distinction between *relation* and *attribute*. According to the conceptions by which *man* and *brute* are aggregated in *animal*, while *animal* and *reason* are compounded in *man*, one primary feature of the distinction is that an impossible component puts the compound out of existence, an impossible aggregant does not put the aggregate out of existence. In this particular the compound relation 'L of M' classes with the compound term 'both $X$ and $Y$.'

(De Morgan uses "aggregation" and "compounding" to mean respectively disjunction and conjunction.)

The last two sentences assert that just as the conjunction $X0$ vanishes so does the composition $L;0$, whereas *aggregating* 0 to either $X$ or $L$ (with aggregation defined as union in both cases) makes neither vanish, i.e. $X + 0$ and $L + 0$ need not be 0.

Although De Morgan greatly admired Boole's calculus, he never became more proficient with it than to appreciate the idea behind $A \vee B = A + B - AB$. If he had pursued it as far as Jevons and Peirce did he might have noticed that his analogy in the above quote was the zeroary case of the more general analogy that both conjunction and composition distribute over union. Had he further known of the interdependence of distributivity and residuation he would have taken this yet further to point out that his "Theorem K" (which in more modern terminology asserts that composition is residuated, i.e. is the partially invertible multiplication of what one might call a semifield) held also for conjunction, suitably phrased.

De Morgan then notes some differences, e.g. whereas conjunction $Xx$ of a term $X$ with its "contrary" (Boolean complement) $x$ is 0, composition $L;l$ of a relation $L$ with its complement $l$ need not be 0.

In place of a separate complement operation De Morgan used the case of the letter to indicate its sign. When negation can appear only at atoms we know that every operation must have its De Morgan dual.

De Morgan certainly knew this but seems not to have liked the De Morgan dual of composition, perhaps for lack of a reasonable way of expressing it in English. The English for $L;M$ is "$L$ of an $M$", as in "Alice is an enemy of a son of Bob", that is, Alice(enemy;son)Bob. The De Morgan dual of composition is Peirce's "relative sum" $L \mathbin{\text{⨥}} M$ (Schröder's notation), the English for which is best put as "$L$ of all non-$M$" ("Alice is an enemy of all non-sons of Bob") or equally well as "non-$L$ only of $M$" ("Alice is a non-enemy only of sons of Bob"). This composes one instance of negation with either of the more natural English constructs "$L$ of all $M$" and "$L$ only of $M$".

So what De Morgan supplied in lieu of an explicit De Morgan dual of composition were these two constructs, which he notated as respectively $LM'$ and $L_{,}M$. Note that $LM' = l_{,}m$. Hence either one of $LM'$ and $L_{,}M$ can perform the function of a De Morgan dual of composition. (There is an additional fringe benefit of this notation. In "pushing complement through" the composition, the complement only has to be pushed further down in one argument, and moreover you can choose which argument by choosing one of $LM'$ and $L_{,}M$.)

Given that only one such construct was needed, one might ask why De Morgan defined both. Perhaps for the aforementioned choice. His discussion however suggests that he felt he was really only defining one construct, the prime, which served to change the default existential quantifier on the input or output of the primed relation into a universal quantifier when it was in the superscript or subscript position respectively. De Morgan did not however explain what $M'$ meant on its own.

## 4 Residuation and Theorem K

At the beginning of his 1941 article on the calculus Tarski renders the following judgement of De Morgan.

> De Morgan cannot be regarded as the creator of the modern theory of relations, since he did not possess an adequate apparatus for treating the subject in which he was interested, and was apparently unable to create such an apparatus. His investigations on relations show a lack of clarity and rigor which perhaps accounts for the neglect into which they fell in the following years.

The chronology does not bear out this last assertion. Peirce began his studies of the calculus only a decade after De Morgan's paper, with a warm acknowledgement of De Morgan as the founder of the subject, and proceeded to write vigorously about it for more than a decade. Then Schröder took up the refrain and wrote an entire book about the calculus [Sch95], volume 3 in his series on algebraic logic. It was only in the 20th century that the subject fell asleep for forty years.

But a more direct vindication of the importance of De Morgan's first and only paper on the calculus is in De Morgan's observation of a certain facet of the calculus that has attracted the attention of essentially all contributors since, namely residuation. De Morgan named the phenomenon "theorem K, in remembrance of the office of that letter in [the syllogisms] Baroko and Bokardo; it is the theorem on which the formation of what I called opponent syllogisms is founded." [2]

Thanks to Ward and Dilworth, who coined the term "residual" [WD39], we understand residuation today as a form of division. In a field we have that $ab = c$ if and only if $a = c/b$, provided $b \neq 0$. With binary relations we have $a;b \leq c$ if and only if $a \leq c/b$ if and only if $b \leq a\backslash c$, without exception. Here $c/b$, the left residual of $c$ by $b$, is the relation consisting of all pairs $(x, y)$ such that for all $z$, $(y, z)$ in $b$ implies $(x, z)$ in $c$. Conversely $a\backslash c$, the right residual of $c$ by $a$, or $a$ under $c$, consists of all pairs $(y, z)$ such that for all $(x, y)$ in $a$, $(x, z)$ is in $c$.

[2] I am indebted to Roger Maddux for drawing Theorem K to my attention.

By contraposition the above pair of equivalences becomes:

$$a;b \leq c \quad \Leftrightarrow \quad a^{\smile};c^{-} \leq b^{-} \tag{1}$$
$$a;b \leq c \quad \Leftrightarrow \quad c^{-};b^{\smile} \leq a^{-} \tag{2}$$

Thus prepared we are now in a good position to follow De Morgan's Theorem K, which reads verbatim as follows (italics are my interpolations).

> If a compound relation be contained in another relation, by the nature of the relations and not by causality of the predicate [*that is, independently of the interpretation of the relations*], the same may be said when either component is converted, and the contrary of the other component and of the compound change places. That is if, be $Z$ what it may, every $L$ of $M$ of $Z$ be an $N$ of $Z$, say $LM))N$, then $L^{-1}n))m$, and $nM^{-1}))l$. [*That was the theorem, the rest is the proof.*] If $LM))N$, then $n))lM'$ and $nM^{-1}))lM'M^{-1}$. But an $l$ of *every* $M$ of an $M^{-1}$ of $Z$ must be an $l$ of $Z$: hence $nM^{-1}))l$. Again, if $LM))N$, then $n))L,m$, whence $L^{-1}n))L^{-1}L,m$. But an $L$ of an $L$ of none but $m$s of $Z$ must be an $m$ of $Z$; whence $L^{-1}n))m$.

Writing $\leq$ for )), $M^{-}$ for $m$, and $M^{\smile}$ for $M^{-1}$, Theorem K becomes

$$L;M \leq N \quad \Rightarrow \quad L^{\smile};N^{-} \leq M^{-} \tag{3}$$
$$L;M \leq N \quad \Rightarrow \quad N^{-};M^{\smile} \leq L^{-} \tag{4}$$

Theorem K can now be seen to assert one direction of the residuation laws (1) and (2).

Was De Morgan aware of the converse direction?

In (1) replace $a$ by its converse and $b$ and $c$ by their negations and simplify using $a^{--} = a$ and $a^{\smile\smile} = a$ to yield:

$$a^{\smile};b^{-} \leq c^{-} \quad \Rightarrow \quad a;c \leq b \tag{5}$$

But this is just the converse direction of (1) with variables $b$ and $c$ interchanged.

So given the involutory nature of negation and converse, of which De Morgan was well aware, the converse of Theorem K amounts to just one of what turns out to be a large number of changes that can be rung by application of the laws of double negation, double converse, and contraposition. De Morgan presumably would only have given this particular permutation if he wished to draw attention to the fact that both directions held.

But it is of less interest whether De Morgan noticed the converse than how much of the residuation principle in the calculus is captured by Theorem K. *In conjunction with the involution laws, Theorem K constitutes a complete characterization of residuation.*

Peirce gives the following method of enumerating the equivalent forms of $a;b \leq c$ in a manuscript titled *On the Logic of Relatives*, [Klo86, p.341]. There are two contraposition principles that can be applied independently, one for each of negation and converse, yielding $c^- \leq a^- \dagger b^-$, $b^\smile;a^\smile \leq c^\smile$, and $c^{\smile -} \leq b^{\smile -} \dagger a^{\smile}$'. Independently we may cyclically permute the variables according to:

$$a;b \leq c \quad \Leftrightarrow \quad c^{\smile -};a \leq b^{\smile -} \tag{6}$$

$$\tag{7}$$

We can derive this permutation from (1) by applying converse to the right hand side. The cyclic permutation being of order three, Peirce obtains a total of 12 equivalent inequalities.

## 5 Linear Negation

In 1882, in a Johns Hopkins circular *Remarks on [B.I. Gilman's "On Propositions and the Syllogism"]* [Klo86, p.345], Peirce developed the notion of $a^{\smile -}$ as a negation operator in its own right. Today we can think of this negation as the "perp" or linear negation operation $a^\perp$ of linear logic, which I will use here. Peirce noted the following properties.

$$a;b \leq c \quad \Leftrightarrow \quad c^\perp;a \leq b^\perp \quad \Leftrightarrow \quad b;c^\perp \leq a^\perp \tag{8}$$

$$a \leq b \dagger c \quad \Leftrightarrow \quad c^\perp \leq a^\perp \dagger b \quad \Leftrightarrow \quad b^\perp \leq c \dagger a^\perp \tag{9}$$

$$a \leq b \quad \Leftrightarrow \quad a;b^\perp \leq 0' \quad \Leftrightarrow \quad b^\perp;a \leq 0' \tag{10}$$

$$a \leq b \quad \Leftrightarrow \quad 1' \leq a^\perp \dagger b \quad \Leftrightarrow \quad 1' \leq b \dagger a^\perp \tag{11}$$

$$a;a^\perp \leq 0' \qquad 1' \leq a \dagger a^\perp \tag{12}$$

These will all be familiar to students of linear logic.

Another Peirce law with a connection to linear logic is from *The Logic of Relatives* [Klo86, p.456]. Peirce says "Two formulae so constantly used that hardly anything can be done without them are

$$a;(b \dagger c) \leq a;b \dagger c$$

$$(a \dagger b);c \leq a \dagger b;c$$

These correspond to half of the weak distributivity laws for linear logic studied by Cockett and Seely [CS91].

## 6 Tarski's School

In 1939 Tarski was visiting the US when Germany invaded his native Poland, and he accordingly took up permanent residence in the US, taking a position at UC Berkeley in 1942 where he remained until his death in 1983.

In 1941 Tarski revived the long-dormant calculus with a paper titled *On the Calculus of Relations* [Tar41]. He adhered to the notation of Schröder, identified Peirce as the "creator of the theory of relations," listed a number of axioms for a general calculus, stated without proof that the calculus was undecidable as a corollary of Church's result for the elementary two-sorted theory of relations (relations and related individuals), and asked about the connection between his axiomatized calculus and the theory of relations.

He also suggested simplifying his calculus by restricting it to equations. This program was realized in collaboration with his student Bjarni Jónsson and the resulting list of axioms was announced in 1948 as the defining the variety (equational class) **RA** of *relation algebras* [JT48]. The content of the **RA** axioms may be succinctly summarized thus. A relation algebra is a Boolean algebra with a monoid; the monoid is residuated (on both sides) with respect to the Boolean order (in the sense we have seen above); and there is a converse operation $a^\smile$ permitting residuation to be expressed as $a\backslash b = (a^\smile;b^-)^-$ and $a/b = (a^-;b^\smile)^-$. A stream of papers on **RA** appeared shortly thereafter [Lyn50, CT51, JT52].

Tarski had asked whether **RA** completely axiomatized the equational theory of binary relations. In 1950 Lyndon gave a negative answer to the question; the missing equations are complicated to describe, which one might take as a sort of "pragmatic completeness argument" for **RA**. Lyndon claimed as a corollary that the class **RRA** of *representable* relation algebras, those all of whose symbols save 1 had their standard interpretation in terms of relations over an infinite set $X$, could not be a variety. In 1955 Tarski contradicted Lydon's unfortunate corollary. In 1964 Donald Monk showed that **RRA** had no finite axiomatization.

Tarski was very much a philosopher in his outlook, and saw the calculus not only as a mathematical object of independent interest but as an elegant language within which to do set theory, in the process eliminating the need for variables. (This vision is familiar in computer science with Schönfinkel's combinatory logic and Backus' FPP programming language.) The case and foundations for this vision are collected in Tarski and Givant's book **A Formalization of Set Theory without Variables**, published in 1987.

## 7 Notation

The notation of the basic calculus stabilized to its present ten symbols around 1950. Here is a blow by blow account of its prior development.

De Morgan's notation was different in all respects from the modern one. He wrote $LM$ for composition, and indicated negated variables in lower case and converse as $L^{-1}$. As we saw earlier he did not have Peirce's relative sum but had instead two "implications" $LM'$ and $L_{\prime}M$. He wrote $\leq$ as )) but had no symbols for the logical connectives or for any of the constants.

Peirce developed the modern selection of operations and its notation, to within the following details. He followed De Morgan in writing composition as $ab$, the switch to $a;b$ originating with Schröder. He initially wrote the logical connectives with a comma—$a +\!\!, b$ and $a, b$—to indicate idempotence of each, but later dropped the comma from $a + b$. Schröder dropped the comma from $a, b$ and the two logical connectives thereafter settled down to $a+b$ and $ab$, at least for the calculus of relations if not elsewhere in logic. Russell followed Peano in using $a \cup b$ and $a \cap b$ for logical sum and product of relations [Rus00, RW35].

Peirce's choice of logical constants 0,1 and relative constants 0', 1' have been followed faithfully by all, even Russell [Rus00]. (Note the distinction between 0', 1' and $0', 1'$; the latter is incorrect.)

Peirce wrote negation and complement as $\bar{a}$ and $\breve{a}$ respectively, stacked as $\breve{\bar{a}}$. Schröder followed this, but while Russell was happy with $\breve{R}$ he followed Peano in writing the negation of relation $R$ as $-R$ [Rus00]. Tarski stuck at first to the Peirce-Schröder notation but adopted the typographically more convenient notation $a^{\smile}$ in [JT48]. In [JT52] this was extended to negation as $a^{-}$ and $a^{\smile -}$ and has remained that way ever since, the convenience of computerized typesetting for setting $\breve{\bar{a}}$ notwithstanding.

Peirce had a variety of notations for operations similar to relative sum, initially written with various arrangements of superscripts and subscripts. In 1883 he chose $a \dagger b$ for relative sum, and in 1897 switched to what he called a scorpion tail which resembled Schröder's $a \mathbin{+\!\!\!\!\lrcorner} b$ but with the hook much larger and swung somewhat to the right to look like a screw-in curtain hook.

The notations for transitive closure and reflexive transitive closure have settled down to $a^{+}$ and $a^{*}$ respectively in computer science, which however has no notation for their respective De Morgan duals $a_{11}$ and $a_{1}$ in Schröder's notation.

Peirce tried and failed in 1870 to invent residuation as inverse to composition. By 1882 he had the right machinery in the form of the properties of $a^{\smile -}$, but failed to notice that this provided the key to division.

Nevertheless in 1870 he did describe two quotients $a{:}b$ and $\frac{a}{b}$, intended to satisfy respectively

$$(a{:}b)b = a \quad \text{and} \quad b\frac{a}{b} = a.$$

These would have been left and right residual respectively had he realized he should simply use inequality in place of equality!

When Ward and Dilworth introduced residuation in 1939 [WD39] they adopted Peirce's $a{:}b$ notation. Since their monoids were commutative they only needed one residual. Birkhoff [Bir48] extended this notation to the noncommutative case by distinguishing the two residuals by slanting the colon appropriately. The earliest use of the now popular $a/b$ and $a\backslash b$ that I am aware of is Lambek [Lam58]. I do not know who first wrote residuals as implications. In mathematics the corresponding notation for function spaces is of course $a^{b}$, which are typically associated with a symmetric tensor product $a \otimes b$ serving as $a;b$.

## 8 Recommended Reading

To conclude let me strongly recommend Peter Heath's introduction [Hea66] to his collection of De Morgan's half dozen papers all entitled *On the Syllogism.*

I also recommend Peirce, whose eminently readable writings are available in several editions. The most notable of these is a six-volume set entitled *Collected Papers of Charles Sanders Peirce* edited by Hartshorne and Weiss and published by Harvard University Press in the 1930's. Peirce's work on the calculus appears in Volume III, Exact Logic.

More recently there have appeared several volumes of *Writings of Charles S. Peirce: A Chronological Edition* published by Indiana University Press, organized by date of publication (except for unpublished works which appear in order of writing); this is promised eventually to be a sixty-volume set!

Tarski's revival paper [Tar41] is an enjoyable and striking instance of a paper recognizing the exciting possibilities in a long dead subject.

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