The four essential Aristotelian syllogisms, via substitution and symmetry

Vaughan R. Pratt
Stanford University

January 14, 2020

Abstract

There being no limit to the number of categories, there is no limit to the number of Aristotelian syllogisms. Aristotle showed that this potential infinity of syllogisms could be obtained as substitution instances of finitely many syllogistic forms, further reduced by exploiting symmetry, in particular the premises' independence of their order. A consensus subsequently emerged that there were 24 assertoric syllogistic forms. A more modern concern, completeness, showed that there could be no more.

Using no additional principles beyond substitution and symmetry, we further reduce these 24 to four forms. A third principle, contraposition, allows a further reduction to two forms, namely the unconditional form and the conditional form, conditioned on one of the terms being inhabited. We achieve these reductions via a regularly organized proof system in the form of a graph with 24 vertices and three kinds of edges corresponding to the three principles.

1 Dedication

This paper is for Istvan Nemeti and Hajnal Andreka, to appear in this Festschrift in their honour. Before launching into my topic I should offer a few words of appreciation.

I first met Istvan and Hajnal in August 1979 on the steps of the meeting place in Hanover where the 6th Conference on Logic, Methodology, and Philosophy of Science was about to start. I was very keen to meet them as I had recently been reading Henkin, Monk and Tarski's *Cylindric Algebras*, to which they had been added for the second volume to bring the authorship up to five.

Wanting to show my appreciation for cylindric algebras, in place of the customary greeting the first thing I said to them was that when $D = \emptyset \neq V$ (the case of the empty universe), the concrete cylindric algebra 2^{D^V} collapsed in a way that validated *all* sentences of first order logic whence the empty universe did not invalidate any sentence, contrary to the conventional wisdom that some existential sentences valid in all nonempty universes were invalid in the empty universe. (This depends on a quirk of concrete cylindric algebras where interpretations of symbols includes interpretation of variables. Validity in first order logic means truth in all interpretations, so if there are no elements in the domain to assign to variables then every sentence, as well as its negation, is vacuously valid because there cannot be any interpretation to witness falsehood.)

With anyone else such a greeting might have seemed a bit abrupt if not borderline nerdy. However they took it very well and even expressed their appreciation of the point.

Since then I have met them at many conferences related to algebraic logic, sometimes on their account when they have kindly invited me to speak.

A month before I met them I had written an MIT technical report proving that every free separable dynamic algebra was residually finite, from which follows completeness of the Segerberg axioms for propositional dynamic logic. From time to time Istvan encouraged me to publish this report, but as I had subsequently rewritten it and presented it at the 1980 ACM Symposium on Theory of Computation I felt this wasn't necessary.

In connection with that report I have two things to thank Istvan for.

The first is a short paper he wrote a year or so later in which he showed that every free dynamic algebra is representable (as a Kripke structure). He pointed out first that "separable" in my result was redundant in the case of free algebras, and second that subdirect products of Kripke structures are Kripke structures. The handling editor asked me to referee it. On the one hand I felt this was all immediate, on the other it was clearly a much better way to state the result. Much as I wished I'd thought of that, I immediately gave it my blessing and it duly appeered in print.

The second is that during the next decade, from time to time Istvan kept encouraging me to publish my original technical report, on the ground that it contained material that had not made its way into the STOC paper. He persisted, and the upshot was that in 1992 it appeared in volume 50 of *Studia Logica* on pages 571-605.

In recent years I have traveled less and therefore haven't seen as much of Istvan and Andreka. At our most recent encounter, a number of years ago, Istvan had taken an interest in the Bell inequalities for quantum mechanics, which John Bell had worked out in the mid-1960s. Istvan expressed to me his astonishment that classical statistics could be violated in such a counterintuitive way. Although I'd taken four years of physics as an undergraduate in the early 1960s, Bell's theorem was much too new to have had any impact in Australia back then, so this was an area of physics that Istvan was much further along than me. I therefore very much regretted being unable to share his astonishment at the time. Since then, quantum computing has obliged us computer scientists to get caught up with entanglement, of which Bell's inequalities are a natural consequence. While counterintuitive, entanglement is nevertheless an inevitable consequence of superposition, which in turn is a consequence of the solution space of Schrödinger's equation being closed under linear combinations.

2 The main theorem

Whereas in logic the conclusion of an argument traditionally follows its premises. mathematics reverses that order by enunciating the theorem before proving it. The advantage of the latter order is psychological: if the theorem sounds interesting it motivates the reader to follow the argument, while if it does not it frees the reader to move on to other things.

With that advantage in mind, we begin by exhibiting an organization of the 24 assertoric Aristotelian syllogisms into the four groups shown in Table 1.

General	Particular	
1. AAA-1	6. OAO-3	11. AOO-2
2. EAE-1	7. IAI-3	12. EIO-2 (10)
3. EAE-2	8. AII-3	13. EIO-1 (9)
4. AEE-2	9. EIO-3	14. AII-1 (8)
5. AEE-4	10. EIO-4	15. IAI-4 (7)
∃M	∃S	
16. AAI-3	19. EAO-1	22. EAO-2 (19)
17. EAO-3	20. AAI-1	23. AEO-2
18. EAO-4	21. AAI-4	24. AEO-4

Table 1. The 24 syllogisms, in four classes

We then further organize those syllogisms into just two groups, namely the two connected components of the graph in Figure 1.

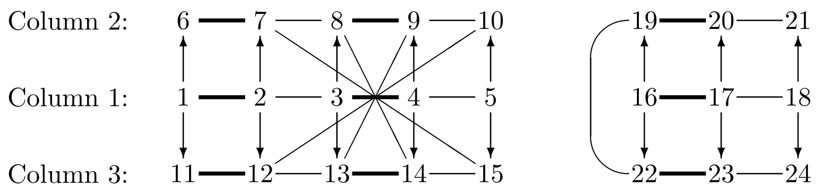

Figure 1. The connections between the 24 valid syllogisms.

Theorem 1. *The edges of the graph of Figure 1 constitute a sound and complete proof system for the 24 assertoric Aristotelian syllogisms.*

The 14 thin undirected edges, 9 thick undirected edges, and 16 directed edges correspond to the customary derivation principles of conversion, obversion, and contraposition respectively. Omission of the 16 directed edges disconnects the graph into four connected components corresponding to the grouping in Table 1. The remainder of this paper provides some historical background, establishes concepts and terminology, and proves the theorem.

3 Aristotle

Aristotle is said to have been born in 384 BCE, which would make 2017 CE the year of his 2400th birthday. Archimedes was supposedly born 97 years later, similarly making 2014 CE the year of his 2300th birthday.[1] While Archimedes is generally considered the greatest mathematician of antiquity in the western world, Aristotle is with equal justification considered its broadest philosopher.

Aristotle made profound contributions to speculative, natural, and practical philosophy that deeply influenced the next 22 centuries of philosophers in Europe and the Middle East. Today we view his natural philosophy as science and his practical philosophy as government, politics, economics, and ethics. This leaves his speculative philosophy as his topics most characteristic of what we think of as philosophy today. During the following two millennia ideas foreign to the ancient Greeks augmented academic philosophy, which while not taking the Greek gods seriously did take monotheism seriously starting in the middle of the first millennium. Aristotle's closest concept to anything redolent of religion was his addition of the Aether to the traditional elements of earth, air, fire, and water, namely as a substance in the heavens that was neither hot nor cold nor wet nor dry but instead divine.

The most technical part of Aristotle's speculative philosophy is what later was organized as his *Organon*.[1] At its core is his *Prior Analytics* which treats the syllogism, the oldest systematic form of logic to have survived to the present day.

4 From Aristotle to the 19th Century

There is a long but rather unclear (by mathematical standards) development of our understanding of Aristotle's syllogisms between 350 BCE and the 18th century. The subject came into sharper focus with writings of Augustus De Morgan [3] at the University of London and Sir William Hamilton at the University of Edinburgh in the 1840s. Both had developed their own accounts of the subject independently. Hamilton accused De Morgan of plagiarizing his account, which led to a public battle that raged in the *Atheneum*.

This little public spat inspired George Boole to write about his idea to found Aristotle's syllogistic on the arithmetic of ordinary polynomials with integer coefficients, reduced to just the multilinear ones with coefficients 0 and 1 via the axiom $x^2 = x$. Boole broached the idea in a short pamphlet in 1847, and developed it into a book in 1854. Neither Boole nor anyone else in that century was able to accept that $x+x = 0$, with the result that William Jevons advocated instead for the theory of complemented distributive lattices favored by Charles Peirce as definitive of Boole's logic.

Starting with Zhegalkin in 1927 and independently[2] Marshall Stone in 1936 in the US, Boole's system was recognized as the theory of Boolean rings as one of several possible finite equational axiomatizations of Boolean algebra, besides

[1] In reckoning these birthdays it should be noted that by convention the Western calendar contains no year zero, instead calling the year that preceded 1 CE the year 1 BCE. There is therefore a parity shift at that boundary.

[2] Soviet Russia's isolation from the west at the time resulted in inadequate recognition of priorities in some cases.

complemented distributive lattices. Thanks to L.E.J. Brouwer and his student Arend Heyting, Heyting algebras that satisfy the Law of the Excluded Middle are another, and recently I wrote about yet another [6]. Modern classical logic takes Boolean algebra as the algebraic basis for zero-order or propositional logic and extends it to one or more domains and predicates thereon, with universally and existentially quantified variables ranging over those domains.

Although almost all logic research today involves some variant of Boole's logic and its extensions, Boole himself envisaged his system not as overthrowing Aristotle's syllogistic but as an algebraic way of formalizing and extending it. That and a certain elegant simplicity to Aristotle's syllogistic has made it an object of continuing study in its own right, with his Square of Opposition as a unifying concept.

5 20th century treatments of the Aristotelian syllogisms

During the 19th century De Morgan, Boole, Peirce, Jevons, Peano, and Frege were instrumental in sharpening mathematical logic. By the early 20th century considerable further progress in various directions had been made by Russell, Hilbert, Łukasiewicz, Lowenheim, Skolem, Brouwer, and others.

During the three decades 1900-1930 the Polish logician Jan Łukasiewicz worked on a wide variety of propositional logics.[4] In 1929 he introduced a parenthesis-free notation now referred to as Polish notation, writing KPQ and CPQ for respectively conjunction and implication between propositions P and Q, and negation of P as NP. As an example of its use he expressed the sentences XaY and XiY as respectively Axy and Ixy, allowing him to write XeY as NIxy and XoY as NAxy. He then derived the properties of Aristotle's Square of Opposition and all 24 of the assertoric syllogisms from the following four axioms.

1.Aaa

2.Iaa

3.$CKAmbAamAab$ (that is, $(A(m,b) \wedge A(a,m)) \rightarrow A(a,b)$)

4.$CKAmbImaIab$

These axioms assert respectively that inclusion is reflexive, all categories are nonempty, inclusion is transitive, and if $m \subseteq b$ and $m \cap a \neq \emptyset$ then $a \cap b \neq \emptyset$.

An obvious concern with this axiomatization is its second axiom, that all categories are nonempty. This disposes of the problem of existential import, which we'll return to later, by brute force. However categories like unicorns and black swans that might be empty, or like two-dimensional cubes and prime square integers that are surely empty, don't fit into Łukasiewicz's ontology.

Furthermore the syllogisms are just a small fragment of what Łukasiewicz's language can express, which therefore amounts to a framework for studying many other things besides the syllogisms rather than a self-contained account of syllogistic reasoning.

In §36 of his 1947 book *Elements of Symbolic Logic* [7] Hans Reichenbach organized the 24 valid syllogisms into the four groups we exhibited in Table 1. His grouping was however not based on the proof theory underlying our Figure 1 but on certain distinguishing characteristics. There are 15 unconditional forms, namely five with a general conclusion and ten with a particular conclusion The remaining nine have "existential import": three are conditioned on the middle term being inhabited and the remaining six on the subject S being inhabited. He nominated Barbara (AAA-1), Darii (AII-1), Darapti (AAI-3), and Barbari (AAI-1) as suitable representatives of these four classes, as well as acknowledging Hilbert and Ackermann as an earlier source of this grouping.

Timothy Smiley in "What is a syllogism?" [9] and John Corcoran in "Aristotle's Natural Deduction System" [2] both raised the further objection to Łukasiewicz that a syllogism is not meant to be understood as a sentence but rather as a form of argument involving multiple sentences. Corcoran pointed out that Łukasiewicz's sentences were fragments of second order logic, to be judged as true or false. Smiley and Corcoran maintained that syllogisms should be judged according to how soundly they reason, for which the prevailing terminology among "syllogisters" is "valid" or "invalid" (my own preference would be for "sound" or "unsound" since "valid" is understood more commonly as a property of sentences meaning "true under all interpretations of the symbols in the language").

My own first exposure to logic was in high school in 1961 when our swim team's coach lent me a thin booklet on the assertoric syllogisms of Aristotle. As a college freshman in 1962 I took a course in philosophy whose logic section used Copi and neglected syllogisms. In 1968 I wrote a computer program to parse and reduce to disjunctive normal form the syllogisms Lewis Carroll had published as a series in a newspaper, possible because every sentence was general (no particulars).[5] In 2006 my interest in syllogisms was revived by Wikipedia's article on the subject. In 2015 it occurred to me to look for a structural reason for the fact that the 24 syllogisms were organized as four figures each with six syllogisms, and I subsequently drew up such a $6 * 4$ table for a paper offering a connection between Aristotle, Boole, and categories [6].

More recently (the point of this paper), I realized that a better factoring of 24 was $(5 + 3) * (1 + 2)$, which expands to $5 + 3 + 10 + 6$. These are the number of syllogisms equivalent to each of Reichenbach's essential syllogisms *Barbara*, *Darapti*, *Darii*, and *Barbari*. Furthermore 5 of the 10 syllogisms in the class *Darii* are derivable from the 5 in *Barbara* by application of contraposition to the major premise, and the remaining 5 by the same operation on the minor premise. The analogous relation holds between the 3 syllogisms of *Darapti* and the 6 syllogisms of *Barbari*.

The next section presents the concept of Aristotle's syllogistic from a point of view intended to motivate our contribution.

6 The Aristotelian syllogisms

From here on our goal is not to find any fault with either the 24 valid syllogisms or with any prior treatments, but simply to prove that substitution and symmetry suffice to reduce their number to four, and to expose an interesting structure created by the rule of contraposition, visualized with the graph of Figure 1 in §2.

Even with no background in zoology, hopefully you would accept the following line of reasoning.

> *If no llamas are fish,*
> *and all sharks are fish,*
> *then no sharks are llamas.*

Likewise with no background in the print world, you might just as happily accept

> *If no newspapers are books,*
> *and all monographs are books,*
> *then no monographs are newspapers.*

Now "no sharks are llamas" is not *logically* equivalent to "no monographs are newspapers", not even slightly since one speaks of animals and the other of printed matter. However these clearly inequivalent conclusions are arrived at from their respective premises by *deductively* equivalent lines of reasoning.

Aristotle is the first on record in the western world to study deductive equivalence of arguments. Had the Greek words for "llama" and "newspaper" been recognizable as categories to Aristotle he would have called each of these two arguments a **valid syllogism** and identified them as *substitution instances* of their common syllogistic form "no P are M, all S are M, therefore no S are P".

This **form** can be succinctly written as PeM,SaM $\vdash$ SeP. Even more succinctly it can be written in two parts as EAE-2.

6.1 Moods

The first part of EAE-2, namely EAE, is called the **mood**. It picks out the three connectives or **copulas**, one per sentence, between the two categories or **terms** of the sentence. Besides A and E there are also I and O.

The two positive connectives are A and I, being respectively the general and the particular positive forms. XaY expresses "all X are Y" or $\forall i.X(i) \rightarrow Y(i)$, while XiY expresses "some X are Y" or $\exists i.X(i) \wedge Y(i)$.

The two negative connectives are E and O. XeY denotes the substitution instance or **contrary** of XaY obtained by substituting $\overline{\mathrm{Y}}$ for Y. Similarly XoY expresses the contrary of XiY resulting from the same substitution, $\overline{\mathrm{Y}}$ for Y.

There being three connectives in a syllogism, each having four possibilities, it follows that there are $4^3 = 64$ possible moods.

Aristotle's **Square of Opposition** arranges these four connectives to form the square $\begin{matrix}\mathrm{A} & \mathrm{E}\\ \mathrm{I} & \mathrm{O}\end{matrix}$. Horizontal movement in the square corresponds to taking the contrary, i.e. negating just the right hand term. Diagonal movement contradicts, i.e. negating the whole sentence.

Those two directions of movement are invertible. There is also a downward movement, from A to I and from E to O, that Aristotle refers to as subalternation and views as a form of weakening. But besides not being invertible, it also raises the following problem of *existential import.*

> *If all black swans are black,*
> *and all black swans are swans,*
> ** then some swans are black.*

(The * indicates a dubious conclusion.) The problem is that both premises are always true, albeit vacuously so in the case when no black swans exist.

Yet Aristotle recognized this line of reasoning as what we now call *Darapti* in the third figure (the middle term on the left in both premises), namely as strengthening the minor premise in *Datisi*, which in this instance would be that *some* black swans are swans. However it is not really stronger because this supposed strengthening drops any mention of existence. Thus whereas *Datisi* is sound, *Darapti* is only sound if its conclusion is true to begin with.

But if we assume the conclusion as a premise, one might reasonably ask what is the point of judging such a circular argument as sound? Aristotle's justification of syllogisms is that they serve to add their conclusions to our knowledge. Since circular arguments don't do that, should Aristotle have rejected *Darapti*?

The general position today seems to be not to question the soundness of a circular syllogism but rather the need to *properly* augment our knowledge. Useless though the number zero might have seemed in the past, mathematicians eventually embraced it for the sake of completeness. Circular arguments have a similar status, which Reichenbach expressed for syllogisms like *Darapti* as a tacit third premise MiM, some black swans are black swans. (And for syllogisms like *Barbari*, SiS.)

Were it not for the problem of existential import, there would be only two kinds of syllogisms, general and particular. Addressing this problem doubles that number to the four treated by Reichenbach.

6.2 Figures

A syllogistic form contains two copies of each of S, P, and M. Ignoring the mood, their distribution determines the **figure** of the form.

The conclusion is always of the form SP.

Each premise is one of the forms M* or *M where * is a placeholder for one of S or P. There being two premises, there are four possibilities, each leaving two places denoted by the two *'s, one for S and one for P. On the face of it, it would seem that these can be placed in either order, making a total of 8 possibilities.

There being 64 moods, there are therefore $64 * 8 = 512$ possible forms.

But order of premises is immaterial from the standpoint of deductive equivalence. Aristotle exploited this by normalizing the order to put the premise containing P first, called the **major premise**. This reduces the number of premise configurations from 8 to 4, namely those shown in Table 2.

1	2	3	4
MP SM	PM SM	MP MS	PM MS
SP	SP	SP	SP

Table 2. The four figures.

The first three of these are Aristotle's three figures, numbered accordingly from 1 to 3; the fourth was added by his student Theophrastus. The major premise is MP or PM according to whether the figure number is odd or even respectively. The minor premise is SM or MS according to whether the figure is one of the first two or the last two respectively.[3]

The upshot is therefore $64 \times 4 = 256$ well-formed syllogistic forms, each determined by its mood and figure. Our example is an **instance** of the form EAE-2. All substitution instances of a given form are deemed equally valid.

7 Obversion and conversion

A syllogism with an odd-numbered figure has P on the right of both the major premise and the conclusion, while those in the Second Figure have M on the right of both premises. The **obverse** of such a syllogism is the result of taking the contrary of the two sentences with a common right side, thereby changing the mood but not the figure.

It is customary to regard obversion as an inference rule in its own right. In this paper we view it as a substitution instance in which not-P is substituted for P, effected by taking the contrary. Since substitution is the means by which Aristotle reduced a potential infinity of possible syllogistic forms to a small finite number, it is surely fair to continue to use substitution to further reduce the number.

The **converse** of a syllogism with a mood containing E or I is the result of exchanging the terms of the sentence with that copula. (If the sentence is the conclusion then S and P need to be renamed as each other, and hence the premises need to be exchanged.) Like obversion, conversion is customarily viewed as an inference rule, but since symmetry is tacitly used to reduce the 512 forms to 256 with the convention that the major premise comes first, again it would seem fair to treat a further possible reduction via a commutativity as equally justifiable without having to call it a separate inference rule but merely another application of symmetry.

That is, we are proposing to use *only* the two principles sufficient to reduce the number to 256, to further reduce that number to four.

[3]This pattern will be familiar to those acquainted with counting in binary.

There are traditionally 24 valid syllogisms when including the conditionally valid ones. Five of them have a general conclusion, all of them unconditionally valid; call these the general syllogisms.

Theorem 2. *When obversion and conversion are used to identify syllogisms, there remains only one syllogism with a general conclusion.*

Proof. Aristotle's two axioms are AAA-1 and EAE-1, *Barbara* and *Celarent*. From the point of view adopted here, these are substitution instances of each other and hence should not be counted as two distinct forms. That is, EAE-1 follows from AAA-1 by obversion. Applying converse, then obverse, then converse yields respectively EAE-2, AEE-2, and AEE-4.

In this way we have identified all five syllogisms having a general conclusion, by alternating obversion with conversion. This alternation is shown in Figure 1, connecting the syllogisms numbered 1-5. □

Having accounted for one class of syllogisms in Table 1, we have three more to account for. Now there are 10 unconditionally valid syllogisms with a particular conclusion, call these the particular syllogisms, the second class. There are 9 conditionally valid syllogisms, 3 of which require the middle term to be inhabited, call these the ∃M syllogisms, the third class. The remaining 6 require the subject of the conclusion to be inhabited; call these the ∃S syllogisms, the fourth class.

Theorem 3. *Alternating obversion and conversion suffice to reduce the valid syllogisms to just four: a general syllogism, a particular syllogism, a ∃M syllogism, and a ∃S syllogism.*

Proof. Figure 1 without the directed (vertical) edges shows how this works for each of the four groups. The diagonal edges and the edge between 19 and 22 complete the connections. In the case of syllogisms 8, 15 and 21, conversion is applied to the conclusion, entailing switching the premises, which leaves the Second and Third Figures unchanged (their premises have M in the same column) but interchanges the first and fourth figures.

□

Reichenbach's four essential syllogisms *Barbara* (AAA-1), *Darii* (AII-1), *Darapti* (AAI-3), and *Barbari* (AAI-1) [8] are suitable representatives of each of these four classes.

8 Contraposition

There are exactly twice as many syllogisms on the right of Table 1 as on the left. This raises the question of whether there might be some underlying structural reason for this.

We answer this with the observation that contraposition creates a 2-to-1 surjection from the right side onto the left.

Equivalently, there is a bijection between the first and second columns, and a second bijection between the first and third columns. These two bijections are created from the two premises in each row of the first column: the major premise yields the second column while the minor premise yields the third.

The basic operation of contraposition is to negate (take the contradictory of) both the selected premise and the conclusion, and exchange them.

This inevitably breaks the naming rules for the three pairs of terms. The terms in the resulting conclusion are renamed to S and P, and the remaining term (which appears in both premises) is renamed to M. The major premise of the result is whichever premise contains P, and the two premises are arranged accordingly to put the major premise first.

To illustrate, for the third line in our table we have EAE-2 in the first column, namely PeM,SaM $\vdash$ SeP. Negating the major premise and the conclusion turns this into PiM,SaM $\vdash$ SiP. Exchanging them yields SiP,SaM $\vdash$ PiM. Renaming P to S and M to P (whence S becomes M) turns this into MiS,MaP $\vdash$ SiP. Lastly we move the new major premise to the first position giving MaP,MiS $\vdash$ SiP, or AII-3.

Repeating this process for the minor premise, we obtain PeM,SoM $\vdash$ SiP. The exchange then yields PeM,SiP $\vdash$ SoM. The requisite renaming just exchanges M and P to give MeP,SiM $\vdash$ SoP, namely EIO-1.

Lemma 4. *Independently of mood, for syllogisms in figures 1 to 3 contraposition changes the figure according to the following transition diagrams, one for each premise.*

$$\text{Major: } 2 \Rightarrow 3 \leftrightarrows 1$$

$$\text{Minor: } 3 \Rightarrow 2 \leftrightarrows 1$$

The double right arrow between 2 and 3 signifies the need to switch the premises at the end. Syllogisms in figure 4 remain there, and always need to switch the premises at the end.

Proof. The contradictory of a sentence leaves the terms unchanged, while exchanging the two contradicted sentences, and exchanging the premises if needed, permutes the three terms on the left independently of those on the right. It follows that the requisite change to the figure is independent of the mood of the syllogism. The actual transitions can then be verified by inspection. Note that the Fourth Figure has the unique property that all three of S,M,P each appear both on the left and the right (there are no duplicates on either side) and contraposition cannot change that property. □

Theorem 5. *In each row of Table 1, the syllogisms in the second and third column are obtained from the one in the first column by applying contraposition to respectively its major and minor premise.*

Proof. The transition tables in the lemma make it easy to verify the Figures in the second and third columns. To verify the Moods, first check that the result conclusion is the contradictory of the selected premise. The contradictory of the initial conclusion is moved to the selected premise or the unselected one according to whether the transition is a single or double arrow respectively (always the unselected one in the case of the Fourth Figure). The unselected premise then moves unchanged to the remaining position of the result. □

Figure 1 summarizes Theorems 2 and 4 with a visualization of the connections between the 24 syllogisms, using the numbering from Table 1.

The thick horizontal lines denote the nine obversions. The thin lines denote the 14 conversions: nine horizontal, four diagonal any one of which suffices to connect the second and third columns of Table 1, and one connecting 19 and 22 (the only one available), for a total of 23 unoriented lines. The oriented vertical lines (those with arrowheads) denote the 16 contrapositions, pointing up or down according to whether they act on the major or minor premise respectively of the syllogisms in column 1.. All lines are invertible, with the proviso that the oriented lines include the orientation information (up or down) when being inverted (it is necessary to know which premise led to that syllogism).

9 Proof of Theorem 1

At this point we have everything needed to prove Theorem 1 save what it means for Figure 1 to constitute a proof system.

Proof systems customarily have axioms as the initial theorems from which the remaining theorems are derived. Since all the edges of Figure 1 constitute equivalences, we could arbitrarily take any one syllogism in a connected component as the axiom from which the rest of that component is derived. However since this choice is completely arbitrary, making such a choice adds nothing of signifiance. Hence we may as well stop with just Figure 1 itself as the proof system and dispense with choosing a representative of each component as an axiom.

There are actually two proof systems in Figure 1. Omitting the directed edges gives the system that reduces the 24 assertoric syllogisms to four, while including them gives the system that reduces the 24 to just two syllogisms.

This is all that needs to be said to justify calling Figure 1 a proof system.

10 Conclusion

We took the position that the principle of substitution used to reduce the potential infinity of valid syllogistic forms to a finite number was of the same nature as the rule of obversion, and inferred that obversion should therefore be allowed to further reduce that number with the same justification. We also argued that the principle justifying the convention of putting the major premise first was of the same nature as the principle justifying the rule of conversion, so that too should be allowed to further reduce the number.

We showed that this allowed the 24 assertoric syllogisms, including the conditionally valid ones, to be further reduced to just four: one with a general conclusion, one with a particular conclusion, one contingent on its middle term being inhabited, and one contingent on its subject being inhabited. This matches up exactly to Reichenbach's four essential syllogisms.[8].

This classification accounts for respectively 5, 10, 3, and 6 syllogisms identifiable using obversion and conversion. We then pointed out a one-to-two correspondence between the first column of Table 1 and the other two columns created by the rule of contraposition, thereby further reducing the four essential syllogisms to two, namely the unconditional syllogism and the conditional one.

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