The Class CAT of Locally Small Categories as a Functor-Free Framework for Foundations and Philosophy

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1. Outline

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2. Motivations for foundations and philosophy
3. The class $\mathcal{CAT}$ as an alternative to the theory ZFC
4. The category **Set**: uniqueness in CAT (up to equivalence)
5. Graphs: Irreflexive and reflexive
6. Presheaves and toposes
7. Chu(Set, K) as a universal framework
8. Qualia
9. Typed Chu spaces
10. Evolution of human consciousness

2. Motivations for foundations and philosophy

- Q? Foundations: What rests on what? A: Nonwellfounded cycles ok
- Q? What is ZFC? Mathematics, metamathematics, term rewriting?
- Q? Can algebraic structures be constructed more economically?
  A: $\mathcal{CAT}$ to the rescue.
- Q? Is point-set topology fundamentally different from algebra?
  A: Each mirrors the other. (Descartes would have liked that.)
- Q? Of what possible use are categories without functors?
  A: Treating $\mathcal{CAT}$ as merely a class will get us a long way.
- Q? Are properties intrinsically intensional?
  A: We propose an extensional notion of "property".
- Q? Is "red" more a noun or an adjective?
  A: (C.I.Lewis) "Red" as a quale is equally noun and adjective.
- Q? How long must evolution of human consciousness take?
  A: $\mathcal{CAT}$ could accelerate natural selection.
- Q? Is the distinction between sort and property a fundamental feature of human consciousness? A: Open question.

3. The class CAT as an alternative to the theory ZFC

ZFC starts from the binary relation $\in$ of **set membership**.
$\mathcal{CAT}$ starts from the "binary" operation $\circ$ of **function composition**.

Sizes: *Small* : $\emptyset, \{\emptyset\}, \omega, \epsilon_0, \mathbb{Z}, \mathbb{Q}, c, \aleph_{\epsilon_{47}}$. *Medium*: **Set**, **Grp**. *Large*: $\mathcal{CAT}$.

$\mathcal{CAT}$ is the class of medium-sized locally small reflexive graphs, each equipped with an associative multiplication defined on length-2 paths. Each self-loop 1_x at x acts as a two-sided **identity** for multiplication.

Object = vertex, **morphism** = edge, **composition** = multiplication.
Isomorphism: An $f : x \to y$ s.t. $\exists g : y \to x.[gf = 1_x \wedge fg = 1_y]$.

Examples. **Set**; **Grp**; monoids $(|ob(\mathcal{A})| = 1)$; posets $(|\mathcal{A}(x, y)| \leq 1)$.

Subcategory $A \subseteq B$, **Isomorphic** cats $A \sim B$: by analogy with groups.
Equivalence: $A \equiv B$ when A, B have isomorphic subcategories $A' \sim B'$, every object $a \in A$ is isomorphic to an object $a' \in A'$, and ditto for B.

4. The category **Set**: uniqueness in CAT (up to $\equiv$)

A **set-like category** $(A, \mathbf{1})$ is a category $A \in \mathcal{C}AT$ with a distinguished object $\mathbf{1}$ such that the full subcategory of A whose only object is $\mathbf{1}$ has $1_{\mathbf{1}}$ as its only morphism, and for any two morphisms $f, g : X \to Y$ in A, if for all morphisms $x : \mathbf{1} \to X$, $fx = gx$, then $f = g$.

The **carrier** of object X is the homset $\mathcal{A}(\mathbf{1}, X)$.

A set-like category $(\mathcal{A}, \mathbf{1})$ is **full** when it is its only set-like carrier-preserving extension (**CPE**) by morphisms. A full set-like category $(\mathcal{A}, \mathbf{1})$ is **complete** when it is equivalent to its every full set-like CPE.

Theorem

There exists a complete full set-like category. Call "it" **Set***.*

Proof.

Make each homset in $\mathcal{C}AT$ a distinct carrier in **Set**. □

5. Graphs: Irreflexive and reflexive

A **graph-like category** $(\mathcal{A}, V, E)$ is a category $\mathcal{A}$ with distinguished objects V, E whose only non-identity morphisms are $s, t : V \to E$. Each object G has vertices and edges drawn from $\mathcal{A}(V, G)$ and $\mathcal{A}(E, G)$. Morphisms are graph homomorphisms by associativity of composition.

Define **full** and **complete** by analogy with set-like categories.

Theorem

There exists a complete full graph-like category. Call "it" **Grph**.

Proof.

Make each pair of homsets in $\mathcal{C}AT$ a pair $(\mathbf{Grph}(V, P), \mathbf{Grph}(E, P))$ and make each such pair P a distinct graph G by composing the edges in $\mathbf{Grph}(E, G)$ with s, t in all distinct ways. □

Introduce functions $i : V \to E$, $si, ti : E \to E$ in all possible ways, call this category **RGrph**. All full rgraph-like extensions are equivalent.

6. Presheaves and toposes

Any category $\mathcal{A}$ and set $J \subseteq ob(\mathcal{A})$ of objects thereof determines the full subcategory $\mathcal{J} \subseteq \mathcal{A}$ such that $ob(\mathcal{J}) = J$. Call such a category a $\mathcal{J}$-like category. J indexes the carriers $\mathcal{A}(j, A)$ of each object A while the morphisms $f : k \to j$ of $\mathcal{J}$ index the unary operations $af : \mathcal{A}(j, A) \to \mathcal{A}(k, A)$ defined by the *right* action of f on the elements a of $\mathcal{A}(j, A)$.

This makes each object A of such a pairing $(\mathcal{A}, J)$ a heterogeneous algebra, and each morphism $h : A \to B$ a homomorphism of algebras wrt those operations by its *left* action on $a \in \mathcal{A}(j, A)$.

$$k \xrightarrow{f} j \xrightarrow{a} A \xrightarrow{h} B$$

Presheaves are simply generalized graphs with $\{V, E\}$ generalized to J and operation symbols as the morphisms of $\mathcal{T} = \mathcal{J}^{op}$.

Every category $Psh(\mathcal{J})$ of presheaves on $\mathcal{J}$ is a topos.

7. $\mathbf{Chu}(\mathbf{Set}, K)$ as a universal framework

A **Chu space** (A, r, X) over a set K consists of sets A and X and a function $r : A \times X \to K$, i.e. matrix. A **Chu transform** $(h, h') : (A, r, x) \to (B, s, Y)$ is a pair $h : A \to B$, $h' : Y \to X$ of functions satisfying $s(h(a), y) = r(a, h'(y))$ for all $a \in A, y \in Y$.

As for **Set** etc., let $J = \{\mathbf{1}\}$ in $\mathcal{A}$, let $L = \{\bot\}$ be a second rigid object in $\mathcal{A}$, and let $\mathcal{A}(\mathbf{1}, \bot) = K$. For any object ArX of $\mathcal{A}$, take $A = \mathcal{A}(\mathbf{1}, ArX)$ to be the carrier of ArX as for **Set**, take $X = \mathcal{A}(ArX, \bot)$ to be the **cocarrier** of ArX, and $\forall a \in A, x \in X$ take $r(a, x) = xa$.

Every morphism $h : ArX \to BsY$ of $\mathcal{A}$ acts on $a \in A, y \in Y$ thus.

$$\mathbf{1} \xrightarrow{a} ArX \xrightarrow{h} BsY \xrightarrow{y} \bot$$

Now $y(ha) = (yh)a$, that is, $s(h(a), y) = r(a, h'(y))$ where h, h' denote respectively the left and right actions of h, making h a Chu transform.

Chu spaces are of interest because they can represent a wide range of mathematical objects, algebraic, topological, and both.

8. Qualia

The new object $\perp$ that **Chu** brings to **Set** can be understood as a Chu space in its own right having as its carrier the set K. But it can also be understood as furnishing $\mathbf{1}$ with states, a notion that is nonexistent in **Set**. Each element of the set $K = \mathcal{A}(\mathbf{1}, \perp)$ thus has the ambiguous quality of being simultaneously a covariant point and a contravariant state.

If we view points $a \in A$ as concrete entities and states $x \in X$ as mental states, this ambiguity of the elements of K may provide a consistent interpretation of C.I. Lewis's notion of qualia, as having simultaneously a perceptual or psychological quality yet also being a real thing given the need for perceived objects to be real.

9. Typed Chu spaces

Unify the base $\mathcal{J}$ of presheaves and the qualia $K : \mathbf{1} \to \perp$ in Chu spaces as follows.

$$t \xrightarrow{f} s \xrightarrow{a} A \xrightarrow{h} B \xrightarrow{x} p \xrightarrow{\varphi} q$$

- s, t Sorts in base $\mathcal{J}$.
- $f : t \to s$ The opposite of an operation symbol in $\mathcal{T} = \mathcal{J}^{op}$.
- A, B universes ("communes").
- $a : s \to A$ Element of sort s in universe A
- $h : A \to B$ Morphism in $\mathcal{A}$ transforming elements forwards and states backwards.
- $x : B \to p$ state or predicate for property p in universe B.
- p, q properties in dual base $\mathcal{L}$.
- $\varphi : p \to q$ Predicate transformer in $\mathcal{L}$ acting on p-predicates.
- $K_s^p = \mathcal{A}(s, p)$, qualia $k : s \to p$ of sort s for propery p. $\mathcal{K} : \mathcal{L} \not\rightarrow \mathcal{J}$.

10. Evolution of human consciousness

Human consciousness seems able to distinguish sorts and properties. Perhaps other animals have similar abilities, but since we cannot as yet communicate sufficiently well with them we can only speak about human consciousness.

Proposal: Thought emerges from categories as graphs with composable edges constructed at random, with randomly chosen distinguished objects $j, k, \ldots$ acting as sorts and $p, q, \ldots$ acting as properties. We then organize our comprehension of a scene as a universe possessing elements or points and states or predicates.

Natural selection then acts to favor those structures that are most helpful to survival.

While one could design a great many other ways of accomplishing the same thing, this particular approach is sufficiently simple in organization that it could well be discovered early in human evolution.

THANK YOU