Orthocurrence as both Interaction and Observation

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1 Introduction

Thesis: Interaction and observation:
dual views of the same phenomenon.

Example. Two people A, B passing in the street
form an Allen interval algebra.

May be viewed:
Symmetrically as their interaction $A \otimes B$
OR
Asymmetrically as $A \multimap B$, namely B from A's viewpoint.
Assymetric because $B \multimap A$ may be very different.

Both observers and observed may be complex entities
distributed in space and/or time.

Interaction is associated with physics
e.g. Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$ of two electrons.

Observation is associated with psychology.
Formalize $A \multimap B$ as a function space.
The outcome of an observation is a function of A
giving the point (value) of B seen from each point $a \in A$.

Examples

Let 1 denote a single point with an unobstructed view.
This is a local (nondistributed) observer.
Then $1 \multimap B \cong B$.
That is, B viewed from a single unobstructed point is seen simply as B.
One observation of B yields one point (value) of B.

Let $2 = 1 + 1$ denote two unobstructed viewpoints.
This is a distributed observer.
Then $2 \multimap B \cong B^2 = B \times B$
(note: not $B + B$, but $(B^2)^\perp = B^\perp + B^\perp$).

Obstruction Example 1
A pair of stations a, b along a train line
with the property:
Any train passing station b has already passed station a.

This rules out the case:
Station a reports no train
Station b reports a train

Obstruction Example 2

A branching line with a station on each branch
with the property:
A train can reach only one station.

This rules out the case of both stations reporting a train.

Paradox:
If interaction is symmetric and observation is asymmetric,
how can they be two views of the same thing?

Resolution:
Observation is really symmetric.
Although $B \multimap A$ may be utterly unlike $A \multimap B$,
we do have $A \multimap B \cong B^{\perp} \multimap A^{\perp}$.

Formalization:
In terms of matrices or Chu spaces or *couples*.

A couple A is an $|A| \times |A^{\perp}|$ matrix.
Matrix entries drawn from a set Σ, the *alphabet*.

$A^{\perp}$ is the transpose or *dual* of A.

$|A|$ is the set of rows $a, b, \ldots$.

$|A^{\perp}|$ is the set of columns $x, y, \ldots$.

Write $a.x$ (or $r(a, x)$) for the entry at row a column x.

Represent Example 1 as the couple $\begin{matrix} a \\ b \end{matrix}\boxed{\begin{matrix} 011 \\ 001 \end{matrix}}$

The rows are the two stations.
Columns are states, understood as possible observations.

Missing column: $\boxed{\begin{matrix} 0 \\ 1 \end{matrix}}$

Represent Example 2 as the couple $\begin{matrix} a \\ b \end{matrix}\boxed{\begin{matrix} 010 \\ 001 \end{matrix}}$

Missing column: $\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}$

The matrix view is a triviality
that yields the following profound insight.

$A \multimap B$ consists of the states of $A \otimes B^{\perp}$,
i.e. of A interacting with *the states of* B.

This is isomorphic to the states of $B \otimes A^{\perp}$,
i.e. of B interacting with the states of A.

The Chu calculator

2 Axiomatization

($\perp$-less alternative to proceedings)

$$\begin{aligned}(A \otimes B) \otimes C &\cong A \otimes (B \otimes C)\\ A \otimes B &\cong B \otimes A\\ A^{\perp\perp} &= A\end{aligned}$$

Definition

$A \multimap B = (A \otimes B^{\perp})^{\perp}$

Rationale

A function from A to B is a state of $A \otimes B^{\perp}$ because
(i) f is a *function*, in that f can be represented as all $|A|$ instances of $f(a)$ each represented by a word of length $|B^{\perp}|$.

(ii) f is *continuous*, in that f can be represented as all $|B^{\perp}|$ instances of $f'(x)$ ($x \in |B^{\perp}|$) each represented by a word of length $|A|$.

Consequences

$$
\begin{aligned}
(A \multimap B)^{\perp} &= A \otimes B^{\perp} \\
A \otimes B &= (A \multimap B^{\perp})^{\perp} \\
A \multimap B &\cong B^{\perp} \multimap A^{\perp} \\
(A \otimes B) \multimap C &\cong A \multimap (B \multimap C)
\end{aligned}
$$

Concreteness

These axioms are satisfied by many $*$-autonomous categories.
Very abstract, so make concrete with $|A|$:

$$
\begin{array}{rcl}
|A \otimes B| & = & |A| \times |B| \\
|A \multimap B| & \subseteq & (|A| \to |B|) \times (|B^{\perp}| \to |A^{\perp}|)
\end{array}
$$

So a morphism from A to B is
a pair $(f : |A| \to |B|, f' : |B^{\perp}| \to |A^{\perp}|)$

Crosswords

We have $(A \otimes B)^\perp \cong A \multimap B^\perp \cong B \multimap A^\perp$
A state of $A \otimes B$ fills in rectangle $|A| \times |B|$.
Row a of that rectangle is a state $f(a)$ of B;
Column b of the rectangle is a state $f'(b)$ of A.

Achieve this with the crossword solution requirement.
With $|A^\perp|$ as the across dictionary
and $|B^\perp|$ as the down dictionary,
a state of $A \otimes B$ is any crossword solution.

Formally, $f(a)(b) = f'(b)(a)$.
(More precisely, $b._B f(a) = a._A f'(b)$)

In the case of $A \multimap B$ this becomes the requirement for
morphism $(f : |A| \to |B|, f' : |B^\perp| \to |A^\perp|)$ from A to B
that $f(a).x = a.f'(x)$.
This is *adjointness*.

Properties

A *property* of a couple is any augmentation of the couple with additional states.
A couple is its own strongest property.
Inclusion of state sets is implication of properties.
Intersection of state sets is conjunction of properties.
The weakest property of couple A is $true_{|A|}$,
the discrete couple on $|A|$.

Homomorphisms

Given a property Π of A,
a function $f : |A| \to |B|$ takes Π to the set
$\{x : B \to \Sigma | x \circ f : A \to \Sigma \ \in \ \Pi\}$

A function $f : |A| \to |B|$ is a *homomorphism* of couples A, B when it maps properties of A to properties of B.

Theorem f is a homomorphism iff f is continuous.

Theorem Every small category C embeds in $\mathbf{Chu}_{|C|}$.

Represent each object c of C as the following couple.
Points of c are all maps to c
States of c are all maps from c
Matrix entries are the composites.

Theorem Every category of structures and their homomorphisms of total arity n embeds in couples over 2^n.

Represent structure (S, R) as the couple:
$|A| = S$
and states $x \in (2^n)^S$ $(= (2^S)^n)$
such that for every tuple $(a_1, \ldots, a_n) \in R$
there exists i in $1..n$ with $a_i \in x_i$.

Theorem (Full Completeness)
(Devarajan-Hughes-Plotkin-Pratt)

The binary logical transformations of functors obtained as composites of $\otimes$ and $\multimap$ in $\mathbf{Chu}_2$ are in bijection with the proof nets of Girard's Multiplicative Linear Logic.