Notes on Event structures and Chu

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1 Introduction

We consider the relation between event structures and $\mathbf{Chu}(\mathbf{Set}, 2)$, particularly the full subcategory **Sys** of set systems.

2 Event Structures and Configurations

Definition 1 *1 An event structure is a structure* $(E, \vdash)$ *where* $\vdash \subset \mathcal{P}(E) \times \mathcal{P}(E)$.

2 A configuration of the event structure is a subset y *of* E *such that if* $u \subset y$ *and* $u \vdash v$ *then* $v \cap y$ *is inhabited. We write* $\mathcal{C}(E)$ *for the set of configurations of* E.

There are two interpretations of event structures. In one, E is the set of *events* and $\vdash$ is called the *enabling* relation. Alternatively, E is the set of *propositions*, $\vdash$ is the *consequence* relation, and $\mathcal{C}(E)$ is the set of *models*.

Some classes of event Structures By imposing conditions on the allowed consequences, $u \vdash v$, one can pick out many interesting classes of event structure. We just consider a few illustrative ones:

Finite Horn u is finite and v is a singleton.

Information u is finite and v is a singleton, or empty.

Finitary u and v are finite.

Geometric u is finite

Note the implications:

$$\textbf{Finite Horn} \supset \textbf{Information} \supset \textbf{Finitary} \supset \textbf{Geometric}$$

We can make a category **Events** of event structures and turn $\mathcal{C}$ into a functor from **Events** to **Sys**. A *morphism* from $(E, \vdash)$ to $(E', \vdash^{'})$ is a function $f : E \to E'$ such that if $u \vdash v$, then $fu \vdash^{'} fv$. With the evident composition and identity, this defines the category **Events**.

Then $\mathcal{C} : \mathbf{Events} \to \mathbf{Sys}$ is defined by putting:

$$\mathcal{C}(E, \vdash) = (E, \mathcal{C}(E))$$

on objects, and

$$\mathcal{C}(f) = (f, f^{-1})$$

on morphisms. One has that $\mathcal{C}$ is full on objects, but much more is true. For $\mathcal{C}$ has a right adjoint $\mathcal{E}$ where

$$\mathcal{E}(E, Y) = (E, \{(u, v) \mid \forall y \in Y.(u \subset y) \supset (v \cap y) \neq \emptyset\})$$

and

$$\mathcal{E}(f, g) = f$$

$\mathcal{E}$ is full and faithful (on morphisms); indeed **Sys** is a reflective subcategory of **Events**, modulo $\mathcal{E}$, with the reflection being $\mathcal{C}$

(Vaughan how does the attribution go here; as I recall, you already knew the full on objects part if not phrased in this way ?)

With each class of event structures, given by a condition P one has a full subcategory, $\mathbf{Events}_P$ of **Events** (with the whole category corresponding to the universally true condition).

Definition 2 *A binary condition on sets is preserved by function application iff whenever $f : E \to E'$ for sets E and E' and u, v are subsets of E such that (u, v) satisfies the condition, then also (fu, fv) satisfies the condition.*

Let P and Q be conditions preserved by function application and suppose that P implies Q. Then $\mathbf{Events}_P$ is a coreflective subcategory of $\mathbf{Events}_Q$; the coreflection removes all the consequences that do not satisfy P.

3 Completeness

Let P be a condition preserved by function application, and write $\mathcal{E}_P$ for the composition of $\mathcal{E}$ and the corestriction to $\mathbf{Events}_P$. This is right adjoint to $\mathcal{C}_P$, the composition of the inclusion and $\mathcal{C}$. In less technical language, we are considering event structures, or consequence relations satisfying P; $\mathcal{C}_P$ yields the set of models; and $\mathcal{E}_P$ yields the consequence relations in P that hold of a given set of interpretations.

Now a completeness question arises: for what consequence relations $(E, \vdash)$ in $\mathbf{Events}_P$ do we have that the relations holding in the set of all models of $(E, \vdash)$ are exactly those holding in $(E, \vdash)$? That is when do we have:

$$\mathcal{E}_P(\mathcal{C}_P(E, \vdash)) = (E, \vdash)$$

When this holds, we say that $(E, \vdash)$ is *complete*.

Certain closure conditions on an event structure $(E, \vdash)$ are at hand here:

Reflexivity $u \vdash v$, if $u \cap v \neq \emptyset$

Weakening If $u \vdash v$ then $u \cup w \vdash v \cup w$, for any $w \subseteq E$.

Cut If $u \vdash v, e$ and $u', e \vdash v'$ then $u, u' \vdash v, v'$ (where, as often, we write comma for union and confuse a singleton with its unique element).

Definition 3 *A finitary relation is a Scott consequence relation (resp. Tarski consequence relation) iff it is satisfies Reflexivity, Weakening and Cut, restricted to finitary consequences (resp. finite Horn consequence relations).*

Theorem 1 *1 A finitary relation is complete iff it is a Scott consequence relation*

2 A finite Horn relation is complete iff it is a Tarski consequence relation.

3 An information system is complete iff satisfies Reflexivity, Weakening and Cut, restricted to Information consequences

This notion of a Scott consequence relation and the first part of the theorem are due to Scott; see Proposition 1.3 in [Sco74]and also [Gab81]. Scott attributes the result to Lindenbaum, as the proof is the same as the original proof of Lindenbaum's Theorem. Nothing similar seems to be available for geometric or general consequence relations. Complete information systems are, essentially, the same things as Information Systems in the sense of Scott; see [Sco82, LW81]. (This was also observed in [DG81].)

4 Characterisation

One can seek to characterise the range of a functor C_P. An example is:

Proposition 1 *(X,Y) is in the range of $\mathcal{C}_{\mathbf{Finitary}}$ (up to isomorphism) iff Y is closed in the product topology on 2^X.*

More traditionally, one can seek to characterise the set of models $\mathcal{C}(E)$ of a class of consequence relations $(E,\vdash)$ in terms of some structure obtained from the set system $(E,\mathcal{C}(E))$ associated to the consequence relation. For example, one can associate a topology to $\mathcal{C}(E)$ taking as subbasis the U_e $(e \in E)$ where:

$$U_e = \{x \in \mathcal{C}(E) \mid e \in x\}$$

If $(E,\vdash)$ is geometric, then this topology is sober.

Theorem 2 *A topological space X can be obtained in this way from a finitary consequence relation, up to isomorphism, iff it is a compact, totally order-disconnected space*

This is due to Droste and Göbel, see [DG81]

Again one may, more simply, consider the subset partial order on $\mathcal{C}(E)$.

Theorem 3 *A partial order can be obtained in this way from an information system iff it is a Scott domain.*

This is, again, due to Scott; see [Sco82, LW81].

5 Infinitary Consequence

Completeness considers the P-consequences of a consequence relation $(E,\vdash)$. One can also consider the general, infinitary consequences, that is: $\mathcal{E}(\mathcal{C}_P(E,\vdash))$

Theorem 4 *Let $(E,\vdash)$ be a finitary (resp. geometric) consequence relation. Then $u \vdash v$ is a consequence of $(E,\vdash)$ iff it is a weakening of a finitary (resp. geometric) consequence of $(E,\vdash)$.*

The finitary part of this theorem is proved in its contrapositive form by Scott; see Proposition 1.4 in [Sco74]

6 Recursion

The category **Events** has finite products and sums, and a tensor. Via $\mathcal{C}$ these correspond, up to natural isomorphism, to the corresponding functors on **Sys**. Turning **Events** into a large cpo, one can solve recursive event structure equations by normal least fixed-point arguments, and hence the corresponding recursive set system equations. The idea of solving event structure equations and then transferring the solutions to another category was pioneered by Scott in [Sco82] and laid out in detail by Larsen and Winskel in in [LW81]; there the other category was that of Scott domains.

References

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