The Stone Gamut: A Coordinatization of Mathematics

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Abstract

We give a uniform representation of the objects of mathematical practice as Chu spaces, forming a concrete self-dual bicomplete closed category and hence a constructive model of linear logic. This representation distributes mathematics over a two-dimensional space we call the Stone gamut. The Stone gamut is coordinatized horizontally by coherence, ranging from −1 for sets to 1 for complete atomic Boolean algebras (CABA's), and vertically by complexity of language. Complexity 0 contains only sets, CABA's, and the inconsistent empty set. Complexity 1 admits noninteracting set-CABA pairs. The entire Stone duality menagerie of partial distributive lattices enters at complexity 2. Groups, rings, fields, graphs, and categories have all entered by level 16, and every category of relational structures and their homomorphisms eventually appears. The key is the identification of continuous functions and homomorphisms, which puts Stone-Pontrjagin duality on a uniform basis by merging algebra and topology into a simple common framework.

1 Mathematics from matrices

We organize much of mathematics into a single category **Chu** of Chu spaces, or games as Lafont and Streicher have called them [LS91]. A Chu space is just a matrix that we shall denote ⫤, but unlike the matrices of linear algebra, which serve as representations of linear transformations, Chu spaces serve as representations of the objects of mathematics, and their essence resides in how they transform.

This organization permits a general two-dimensional classification of mathematical objects that we call the *Stone gamut*[1], distributed horizontally by shape and vertically by diversity of entries. Along the horizontal axis we find Boolean algebras at the *coherent* or "tall" end, sets at the *discrete* or "flat" end, and all other objects in between, with finite-dimensional vector spaces and complete semilattices in the middle as square matrices. In the vertical direction we find lattice structures near the bottom, binary relations higher up, groups higher still, and so on.

Chu spaces were first described in enriched generality by M. Barr to his student P.-H. Chu, whose master's thesis on $\mathbf{Chu}(V,k)$, the V-enriched category produced by what since came to be called the Chu construction, became the appendix to Barr's monograph on *-autonomous categories [Bar79]. The latter subject generated no interest at the time but a decade later was recognized by Seely [See89] as furnishing Girard's linear logic [Gir87] with a natural constructive semantics. Barr then proposed the Chu construction as a means of producing constructive models of linear logic [Bar91].

The subsequent history of Chu spaces has been one of successive weakenings of the enriching category V. Barr and Chu took V to be any symmetric monoidal closed category with pullbacks, de Paiva [dP89] and Brown and Gurr [BG90] restricted to order enrichment, and finally Lafont and Streicher banished enrichment altogether by taking $V = \mathbf{Set}$ [LS91] and calling the resulting objects games after von Neumann and Morgenstern.

Chu spaces are indeed games, and moreover of the asynchronous kind, ideally suiting them as a model of concurrent behavior. However the term "Chu construction" predates it, and Barr has proposed to us in conversation the more concrete "Chu space," which has the advantage over the general term "game" of requiring no additional disambiguating qualification.

*This work was supported by ONR under grant number N00014-92-J-1974.

[1] "Spectrum," the obvious candidate for this appliction, already has a standard meaning in Stone duality, namely the representation of the dual space of a lattice by its prime ideals. "A gamut of games" is in the "exaltation of larks" tradition.

We recommend **Set** as the default enriching category for Chu spaces, and suggest "V-enriched Chu space" for the case of general V. We take $\mathbf{Chu}_K$ to be an abbreviation for $\mathbf{Chu}(\mathbf{Set}, K)$, and **Chu** to be an abbreviation for $\mathbf{Chu}_{\mathbb{N}}$. It would seem a *very* interesting question what uncountable K offers, with regard to both algebraic and logical strength.

Our own rationale for the choice $V = \mathbf{Set}$ is the surprising size of **Chu**, which not only meets the model theoretic requirements of first order logic by providing relational structures but transcends them by including topological spaces. Such universality together with the fact that $\mathbf{Chu}_K$ is closed, i.e. enriched in itself, would appear to render V-enrichment redundant for most practical applications, and prompts the speculation that **Chu** is "the largest concrete category" in a suitable sense. Of possible relevance are Pavlović's characterization of $\mathbf{Chu}(V, k)$ as the cofreely generated self-dual closed category on V [Pav93] and Rosebrugh and Wood's appealing one-axiom characterization of **Set** as the unique choice of V (any category) for which the Yoneda embedding $Y_V : V \to \mathbf{Set}^{V^{\mathrm{op}}}$ has a string of four left adjoints [RW94].

Another rationale for choosing $V = \mathbf{Set}$ is its accessibility as a universally understood category, with sets and functions nowadays being taught early on. Whereas category theory is unavoidable for the general V-enriched case, the ordinary case can be treated quite comprehensively from the conventional set-theoretic perspective of universal algebra. One day everyone will be equally fluent in category theory and set theory, but in the meantime any subject that can be explained purely set-theoretically makes itself accessible to a far wider audience.

A third rationale is the completeness of **Set**, which has all limits and colimits as well as splitting idempotents and a regular generator (1) and regular cogenerator (1+1). **Chu** inherits all these (but not cartesian closedness), making it an algebraically well-endowed category when taken in combination with its *-autonomous or self-dual symmetric monoidal closed structure and its concreteness. These structural properties correspond to the operations[2] of linear logic as defined and axiomatized by J.-Y. Girard [Gir87], namely $A^\perp$ (dual), $A \otimes B$ (the symmetric monoid, with tensor unit $\top$ and right adjoint $A \multimap B$ making it closed), $A \times B$ (the case of binary products, with unit 1, dualized by $A^\perp$ to coproduct $A+B$ with unit 0), and $!A$ (underlying set of A, dual to $?A$ as the extension of A to a **CABA** whose ultrafilters (atoms) are the states of A).[3] While category theory remains the best motivation for this particular organization, as well as the language of this metadiscussion, all these operations are definable in traditional set theoretic terms.

Chu spaces first entered our investigations of the nature of concurrent computation in June of 1992 when we found with our student V. Gupta [Gup93, GP93, Gup94] that they captured exactly the notion of partial distributive lattice that we had been trying to pin down [Pra92d] as an extension of our earlier notion of event space [Pra92c, Pra92a, Pra92b]. More recently [Pra95] we have arrived at the view of $=\!\!|$ and its converse $\models$ as the dual relations of *causality* and *consciousness*. The entry $a =\!\!| x$ expresses the causal link from physical event a to mental state x whereby the *actual* time of a (relative to x) stamps itself on x. The converse link $x \models a$ points backwards in time and exhibits x's *record* of the time of a. Thus the same entry is ambiguously interpretable as physical time and mental information (of that time). This simple ambiguity forms the basis for one direction of Descartes' *causal interaction* of body and mind, a mathematically more attractive seat of causal interaction than Descartes' proposal of the pineal gland [Pra95].

Each Chu space has its own clock, whose possible values are its events, and its own history, whose possible values are its states. Events constitute the points or instants of the space's idea of time, which is structured by the data in its rows. States form the possible records of its view of the past, similarly structured by its columns. Events occur at definite times but need not convey precise information. States contain precise information but need not be localized in time. The dynamic interpretation of Chu spaces has more recently been considerably extended by van Glabbeek and Plotkin under the rubric of configuration systems [VGP95].

The $=\!\!|$ relation is the only primitive link needed for this theory of behavior. The causing of events and the extraction of information from states are lesser relations which we have shown [Pra95, p.118] to be straightforwardly derivable from $=\!\!|$ using the binary relation operations of right and left residuation. The right residual $=\!\!|\backslash=\!\!|$ is such that $x(=\!\!|\backslash=\!\!|)y$ holds just when $a =\!\!| x$ implies $a =\!\!| y$ for all events a, giving the natural inclusion order on columns, which preorders

[2]Our notation is that of Barr [Bar91] and Seely [See89].

[3]Besides ? there is another useful monad $\Sigma : \mathbf{Chu} \to \mathbf{Chu}$ whose effect on $\mathcal{A} = (A, =\!\!|, X)$ is to replace X by the empty set. $\Sigma\mathcal{A}$ is usable as an alphabet, since every function $f : A \to B$ is a Chu transform $f : \mathcal{A} \to \Sigma\mathcal{B}$. Unlike ? however this monad cannot be restricted to the subcategory of **Chu** consisting of T_0 extensional spaces since T_0 evidently fails for $\Sigma\mathcal{A}$ when $|A| = 2$.

states by permitted transitions[4]. The left residual $\dashv/\dashv$, which yields the *converse* of the natural inclusion order on rows, preorders events by necessary temporal precedence. Residuation can be defined purely equationally [Lam58, Joh82, Pra90], yielding an attractive logic of these derived relations.

The present paper turns from these dynamic considerations to the potential applications of Chu spaces to the foundations of mathematics. We first became aware of this possibility when Stone duality intruded into our concurrency work as an annoying and obscure complication that we would have much preferred to do without but were obliged to acknowledge as an inevitable part of the time-information duality of infinite concurrent processes. As the situation with Stone duality became clearer we started to realize that Chu spaces realized a far greater diversity of objects than just the vector spaces, topological spaces, and coherent spaces reported by Lafont and Streicher [LS91]. In particular all the "partial distributive lattice" categories (e.g. locales, semilattices, and algebraic lattices, but not abelian groups or nondistributive lattices) treated by Johnstone [Joh82] arise at $K = 2$.

Discussions with M. Barr following an impromptu talk on Chu spaces at MFPS'93 led us to the theorem [Pra93] that every category C of (possibly multisorted) relational structures of total arity k and their homomorphisms was realizable in the category of Chu spaces over 2^k, in the sense that there exists a full and faithful functor $F : C \to \mathbf{Chu}_{2^k}$ (the notion of *representation*) which commutes with the underlying set functors (the stronger notion of *realization* [PT80]). In particular, every elementary class (class of structures definable in first order logic) and its homomorphisms, e.g. groups, rings, forms such a category. Topological spaces are not relational structures, yet are realizable in $\mathbf{Chu}_2$, while topological groups are realizable in $\mathbf{Chu}_{16}$. In general structures can be freely conjoined using **Chu**'s tensor product, taking the diagonal of the tensor when the carrier is common (treated elsewhere).

Chu spaces and category theory may each be regarded as a simplification of the organization of mathematics imposed by universal algebra. An algebra, more generally a relational structure, is a set (possibly divided into sorts, either explicitly in the signature or implicitly by unary relations) which transforms by homomorphisms. Algebras having the same signature are organized into categories, leading to a three-level hierarchy: individuals such as points and numbers, structures such as topological spaces and groups, and categories such as **Top** and **Grp**.

Category theory simplifies the universal algebra hierarchy by demoting the lowest level, individuals, to the inessential role of derived concept. Mathematics is then organized into categories and categories into geometrical cells. Ordinary categories admit objects and morphisms as cells of dimension 0 and 1 respectively. A morphism has a two-object boundary consisting of a *source* and a *target*, and morphisms compose associatively.

Category theory recovers the individual as a derived notion for any given category C via the homfunctor $\mathrm{hom} : C^{\mathrm{op}} \times C \to \mathbf{Set}$. One selects a suitable object $\top$ of C, e.g. a final object or the tensor unit, and defines the elements of each object x of C to be the morphisms from $\top$ to x. This makes $\mathrm{hom}(\top, -) : C \to \mathbf{Set}$ the *underlying set functor* for C, a *representable* functor represented by $\top$, which when faithful makes C concrete.

Chu spaces leave untouched the concreteness of relational structures, and instead remove the boundaries between the categories that universal algebra and category theory are in agreement on. In this organization, "universal algebra" connotes not a subject but an object: a Chu space *is* a universal algebra, and mathematics becomes a universal concrete category **Chu** of Chu spaces.

The traditional notion of category as classification is then recovered as a derived notion by identifying most of the familiar categories of mathematics as full concrete subcategories of **Chu**. Included among these are all categories of relational structures and their homomorphisms standardly defined: posets, groups, graphs, etc., along with topological spaces but not to our knowledge hypergraphs [DW80].

This paper explores the gamut of Stone duality at $K = 2$. We show that "continuous function" is equivalent to "homomorphism" for a very general notion of the latter. We recast the proof of our universal realization theorem in terms of a view of Chu spaces over 2^k as $A \times k \times X$ Chu spaces over 2 in which k is stationary, transforming neither forwards like A nor backwards like X. And we briefly examine the notion of concreteness. Further material on Chu spaces may be found on World Wide Web at http://boole.stanford.edu.

2 Basic Notions

2.1 Chu Spaces

Definition 1 For any set K of *values*, a Chu space $\mathcal{A} = (A, \dashv, X)$ *over* K consists of a set A of *points* (el-

[4]Unlike the transitions of ordinary automata, this kind may be associated with multiple asynchronous events. With more than one event its time becomes ill-defined, analogously to the atemporal collapse of the quantum-mechanical wavefunction.

ements, events, locations, variables) constituting the *carrier* of $\mathcal{A}$, a set X of *states* (opens, functionals, freedoms) constituting the *cocarrier*, and a function $\mathrel{=\!\!|} : A \times X \to K$ serving to organize $\mathcal{A}$ as an $A \times X$ matrix of values drawn from K. ∎

We denote the entry at (a, x) as either $\mathrel{=\!\!|}(a, x)$ or $a \mathrel{=\!\!|} x$, and refer to it as the value at the point (location, variable) a in state x. As mere indices, events and states are intensional. We use tilde uniformly to indicate their extensional denotation. The function $\widetilde{\mathrel{=\!\!|}} : A \to K^X$ is defined as satisfying $\widetilde{\mathrel{=\!\!|}}(a)(x) = \mathrel{=\!\!|}(a, x)$; dually $\widetilde{\models} : X \to K^A$ satisfies $\widetilde{\models}(x)(a) = \models(x, a)$. We define *row* a to be $\widetilde{\mathrel{=\!\!|}}(a)$, written $\widetilde{a}$, and likewise *column* x as $\widetilde{x} = \widetilde{\models}(x)$. We denote the set $\{\widetilde{a} \mid a \in A\}$ of rows of $\mathcal{A}$ as $\widetilde{A}$ and likewise the set of columns of $\mathcal{A}$ as $\widetilde{X}$.

A Chu space therefore manifests itself ambiguously as a multiset A of rows from K^X and a multiset X of columns from K^A, constituting respectively the physical (concrete, conjunctive, yang) and mental (coconcrete, disjunctive, yin) views of a Chu space. We may regard rows and columns as characteristic functions of subsets of respectively X and A, where K is taken to consist of the degrees of membership, with $K = 2 = \{0, 1\}$ giving the ordinary notion of membership, 1 for in and 0 for out.

A Chu space is T_0 when $\widetilde{\mathrel{=\!\!|}}$ is injective (no repeated rows), *extensional* when $\widetilde{\models}$ is injective (no repeated columns). The Chu space obtained by identifying all pairs x, y of column indices for which $\widetilde{x} = \widetilde{y}$, and likewise for row indices, is called the *skeleton* of that Chu space. A *normal* Chu space is one satisfying $\widetilde{x} = x$ for all $x \in X$, and can be written as simply (A, X), $\mathrel{=\!\!|}$ being inferrable as application of functions $x : A \to K$ to elements $a \in A$. A normal space is automatically extensional, indeed $X = \widetilde{X}$, but need not be T_0.

To every Chu space $\mathcal{A} = (A, \mathrel{=\!\!|}, X)$ is associated its *dual* space $\mathcal{A}^\perp = (X, \models, A)$, where $\models : X \times A \to K$ is the transpose or converse of $\mathrel{=\!\!|}$ defined by $x \models a \;=\; a \mathrel{=\!\!|} x$. Since $\mathcal{A}^{\perp\perp} = \mathcal{A}$, the dual space contains the same information as $\mathcal{A}$ but in transposed form, and is therefore just an alternative view of $\mathcal{A}$, which we think of as $\mathcal{A}$ seen "in the mirror." Duality interchanges "T_0" and "extensional." We define a *conormal* space to be the dual of a normal space, automatically T_0 but not necessarily extensional.

2.2 Chu Transforms

Given two Chu spaces $\mathcal{A}$ and $\mathcal{A}'$, a *Chu transform* from $\mathcal{A}$ to $\mathcal{A}'$ consists of functions $f : A \to A'$ and $g : X' \to X$ satisfying the *adjointness condition* $f(a) \mathrel{=\!\!|}' x' \;=\; a \mathrel{=\!\!|} g(x')$ for all $a \in A$ and $x' \in X'$.

Chu transforms compose as $(f', g')(f, g) = (f'f, gg')$; this composition evidently yields a Chu transform, is associative, and has $(1_A, 1_X)$ as its identity at each Chu space $(A, \mathrel{=\!\!|}, X)$. Chu spaces over K thereby form a category which we denote $\mathbf{Chu}_K$. An isomorphism of Chu spaces is an isomorphism in this category; equivalently a Chu transform consisting of a pair of bijections. Two Chu spaces are *equivalent* when they have isomorphic skeletons, by analogy with categorical equivalence.

The adjointness condition may be seen to express exactly the notion of continuity in point set topology via λ-abstraction on a (leaving x' universally quantified over X') followed by η-reduction, thus.

$$
\begin{array}{rcll}
\lambda a. f(a) \mathrel{=\!\!|}' x' & = & \lambda a. a \mathrel{=\!\!|} g(x') & (\lambda\text{-abstract}) \\
\lambda a. \widetilde{x'}(f(a)) & = & \lambda a. \widetilde{g(x')}(a) & (\text{columnize}) \\
\widetilde{x'} \circ f & = & \widetilde{g(x')} & (\eta\text{-reduce}) \\
\exists \widetilde{x} \in \widetilde{X}[\ \widetilde{x'} \circ f & = & \widetilde{x}\ ] & (\text{de-Skolemize}) \\
\exists \widetilde{x} \in \widetilde{X}[\ \{a \mid f(a) \in \widetilde{x'}\} & = & \widetilde{x}\ ] & (\text{set-ify}) \\
f^{-1}(\widetilde{x'}) & \in & \widetilde{X} & (\text{defn. of } f^{-1})
\end{array}
$$

The last two steps identify $\widetilde{x'} \circ f$ as the characteristic function of a subset of A (assuming K-valued membership) with the set $\{a \mid f(a) \in \widetilde{x'}\}$, making it recognizable as the expansion of the usual definition of the inverse image function f^{-1} applied to $\widetilde{x'}$. But this is exactly the continuity condition of point set topology with columns treated as open sets.

Duality. Transposition of the objects of $\mathbf{Chu}_K$ extends in the obvious way to its morphisms, sending (f, g) to (g, f). This makes transposition a functor from $\mathbf{Chu}_K$ to $\mathbf{Chu}_K{}^{\mathrm{op}}$. Transposition is of course an involution, $\mathcal{A}^{\perp\perp} = \mathcal{A}$, and hence an isomorphism of $\mathbf{Chu}_K$ and $\mathbf{Chu}_K{}^{\mathrm{op}}$. This makes transposition a *duality* in the sense of a contravariant equivalence of categories from $\mathbf{Chu}_K$ to itself. Hence $\mathbf{Chu}_K$ *is a self-dual category.*

Carrier and cocarrier. Whereas an algebra or relational structure has only the one underlying set or *carrier*, a Chu space has both a carrier A and a *cocarrier* X. We define the underlying-set functor $U_K : \mathbf{Chu}_K \to \mathbf{Set}$ as $U_K(A, \mathrel{=\!\!|}, X) = A$, $U_K(f, g) = f$, and the underlying-antiset functor $V_K : \mathbf{Chu}_K \to \mathbf{Set}^{\mathrm{op}}$ as $V_K(A, \mathrel{=\!\!|}, X) = X$, $V_K(f, g) = g$.

2.3 The Stone Gamut

The Stone gamut of the title classifies Chu spaces as follows.

When A and X are finite, we assign to $\mathcal{A}$ a real number in the interval $[-1, 1]$, the *discreteness* of $\mathcal{A}$, denoted $\delta\mathcal{A}$, whose sign is that of $|\widetilde{X}| - |\widetilde{A}|$. The *coherence* of $\mathcal{A}$ is defined as $\delta(\mathcal{A}^\perp)$. The successive shapes ▭, □, ▯ display the passage from discrete to coherent in terms of the form factor of the Chu space after identifying all repeated rows and columns (which keeps $|A|$ and $|X|$ within an exponential of each other). Sets have discreteness 1 and coherence -1, which we associate with particulate matter or dust. At the other extreme are Boolean algebras with discreteness -1 and coherence 1, which we associate with logic, mind, or organization[5], for the reasons given in the next two sections. In the middle, at zero discreteness and coherence, we have the "healthy mind in a healthy body" objects such as finite-dimensional vector spaces (Hilbert spaces drop the dimension restriction) and complete semilattices. The counterpart in physics of our discrete-coherent gamut is the particle-wave gamut.

We demand the following properties of discreteness. (i) It should depend only on the number of distinct rows and columns of $\mathcal{A}$. (ii) It should satisfy $-1 \leq \delta\mathcal{A} \leq 1$. (iii) It should satisfy $\delta(\mathcal{A}^\perp) = -\delta\mathcal{A}$, i.e. coherence is simply the negation of discreteness. The following candidate measure of discreteness meets these criteria. For $K \leq 1$, $\delta\mathcal{A} = |\widetilde{X}| - |\widetilde{A}|$, while for $K \geq 2$, $\delta\mathcal{A} = \frac{P-Q}{P+Q}$ where $P = |K^X - \widetilde{A}|$ is the number of "missing" points and dually $Q = |K^A - \widetilde{X}|$ is the number of missing states. For $K \geq 2$, if $P+Q=0$ then $|K^{K^A}| = |A|$ but this is impossible whence so is division by 0. This measure has the odd property that the only discreteness possible in $[-\frac{1}{3}, \frac{1}{3}]$ is 0, namely for square Chu spaces; we pose the problem of finding a more uniformly distributed measure meeting the above criteria.

Any such one-dimensional notion of discreteness can provide only a crude measure of the interconnectedness of an object. As one adds states to a Chu space $\mathcal{A} = (A, X)$, with X going from 0 to K^X, discreteness increases monotonically as does the number of morphisms from $\mathcal{A}$ to any other fixed Chu space $\mathcal{A}'$, since adding states to the source only makes it easier to satisfy continuity. Furthermore the number of morphisms to the conormal space $\perp = (K, 1)$ tracks discreteness monotonically no matter how X changes, since those morphisms are in bijection with X and discreteness is monotonic in $|X|$ for fixed A. However this does not generalize: replacing $\perp$ by some other Chu space $\mathcal{A}'$ may break this mononotonic link between discreteness and number of morphisms to $\mathcal{A}'$.

We also associate to $\mathcal{A}$ (of any cardinality) its *complexity* K, which by abuse of notation we identify with $|K|$. This quantity measures the precision to which the state of the space can be measured at each location. Posets are realizable as Chu spaces of complexity 2, binary relations 4, and groups 8 (not known to be tight). Vector spaces over an infinite field appear to require infinite complexity; the cardinality of the field suffices [LS91], but we conjecture that the Löwenheim-Skolem theorem suitably adapted should make countable complexity sufficient for realization by Chu spaces up to elementary equivalence. More generally we conjecture that $\mathbf{Chu}_{\mathbb{N}}$ suffices for all of elementary (first-order-definable) mathematics, while extending it with topology.

3 The Stone Gamut

3.1 Sets and Antisets

We begin our exploration of the Stone gamut with its endpoints, respectively sets and antisets. For any K, our standard realization of the set A in $\mathbf{Chu}_K$ will be as the normal Chu space $\mathcal{A} = (A, K^A)$. Since every function from A to K exists as a column of $\mathcal{A}$, every $f : \mathcal{A} \to \mathcal{A}'$ from $\mathcal{A}'$ to an arbitrary Chu space $\mathcal{A}'$ has some $g : X' \to X$ making f continuous. Furthermore all Chu transforms from $\mathcal{A}$ arise in this way, making this representation of A by (A, K^A) a realization.

When $\mathcal{A}$ is realized as (A, J^A) for J any fixed subset of K, even the empty subset, then this argument remains sound for those targets $\mathcal{A}'$ of f whose entries are restricted to J. The conormal Chu space (J^X, X) therefore realizes the antiset X for any fixed $J \subseteq K$.

Antisets permit the following elegant definition of $\mathbf{Chu}_2$. Let **Rel** denote the category of sets and their binary relations. Then the arrow category of **Rel**, namely the functor category $\mathbf{Rel}^2$ (where 2 is the category with two objects and one nonidentity arrow between them) has as objects all binary relations and as morphisms all commuting squares of binary relations. Then $\mathbf{Chu}_2$ can be seen to be the (nonfull) subcategory of $\mathbf{Rel}^2$ obtained by restricting the morphisms $f : (A, \models, X) \to (A', \models', X')$ to those for which the relation from A to A' is a function and that from X to X' is an antifunction.

$\mathbf{Set}^{\rm op}$ may be understood as either a concrete or coconcrete category. Its concrete representation is standardly given by the contravariant power set functor, which faithfully embeds $\mathbf{Set}^{\rm op}$ in **Set**, sending the antiset X (construed as an object of **Set**) to its power set 2^X and each antifunction $f^\smile : X \to Y$ (converse of $f : Y \to X$) to a distinct function $2^f : 2^X \to 2^Y$ defined by $2^f(g) = g \circ f$ where $g : X \to 2$ (i.e. $g \in 2^X$) is (the characteristic function of) some subset of X and the indicated composition belongs to **Set**. Power sets being complete atomic Boolean algebras (CABA's),

[5] But not with inconsistency, for which $|X| = 0$, only possible for $|\widetilde{A}| = 1$ corresponding to the inconsistent Boolean algebra satisfying 0=1.

and 2^f, also known as the inverse image function f^{-1}, being a complete Boolean algebra homomorphism, the above contravariant power set functor factors through the forgetful functor $U : \mathbf{CABA} \to \mathbf{Set}$ to yield an equivalence $\mathbf{Set}^{\mathrm{op}} \cong \mathbf{CABA}$, the well-known duality of sets and complete atomic Boolean algebras.

As for the coconcrete representation of $\mathbf{Set}^{\mathrm{op}}$, this may be understood categorically by simply taking $\mathbf{Set}^{\mathrm{op}}$ as the basis for coconcreteness: a coconcrete category is just one with a faithful functor to $\mathbf{Set}^{\mathrm{op}}$. $\mathbf{Set}^{\mathrm{op}}$ is then its own coconcrete representation, via its identity functor.

From the set theoretic viewpoint however, **Set** is customarily obtained as a (nonfull) subcategory of the category **Rel** of sets and binary relations, having as its objects those of **Rel**, and having for its morphisms $R : A \to B$ those binary relations $R \subseteq A \times B$ for which $I_A \subseteq R;R^{\smile}$ and $R^{\smile};R \subseteq I_B$. Mindful of Bill Thurston's almost allergic reaction at last year's Universal Algebra and Category Theory conference at Berkeley to the very idea of $\mathbf{Set}^{\mathrm{op}}$ as **Set** with its morphisms reversed, we obtain $\mathbf{Set}^{\mathrm{op}}$ directly from **Rel** as those relations R satisfying $I_A \supseteq R;R^{\smile}$ and $R^{\smile};R \supseteq I_B$. This has the effect of defining the morphisms of $\mathbf{Set}^{\mathrm{op}}$ to be those binary relations $f^{\smile} : X \to Y$ expressible as the converse of a function $f : Y \to X$, which we call an antifunction. We shall call a set that transforms by antifunctions an *antiset*. Those uncomfortable with $\mathbf{Set}^{\mathrm{op}}$ but comfortable with **Rel** should find the category of antisets and antifunctions reassuringly familiar.

3.2 Functions and Antifunctions

We now consider the structural significance of functions and antifunctions. We shall argue that functions identify and adjoin while antifunctions copy and delete, and infer that states should transform via antifunctions rather than functions. This is because homomorphisms are supposed to respect structure and therefore should not adjoin new states since that would create new degrees of freedom for the transformed space.

Writing $\ker f$ for the quotient of X induced by the (equivalence relation) kernel of f, and $\mathrm{im} f$ for the subset of Y that f lands in, $f : X \to Y$ factors uniquely as $X \to \ker f \to \mathrm{im} f \to Y$, consisting (going from X to Y) of three operations: *identify*, *rename*, *adjoin*. That is, we first identify x and y just when $f(x) = f(y)$. Then we rename (bijectively) each equivalence class $[x]$ to $f(x)$, yielding the elements of $\mathrm{im} f$. Finally we adjoin $Y - \mathrm{im} f$ to $\mathrm{im} f$ to yield Y.

By duality, antifunctions $f^{\smile} : X \to Y$ factor uniquely, again from X to Y, as *delete*, *rename*, and *copy*, being the converses of *adjoin*, *rename*, and *identify* in that order.

We neglect renaming as less significant structurally than the other four operations. Identification and adjoining are natural for mathematical transformations because they make sense *when the elements are anonymous*, or unlabeled as combinatorialists put it, the point of view from which one can see only two groups of order four. However a computer user who is editing a file or transforming an array thinks in terms of points with specific labels on them or values stored in them. These are properly understood as states, for which identification raises the question of what to do with conflicting labels, while adjoining raises the question of what to label the new elements. These questions do not arise with *copy* and *delete* as every computer user knows instinctively.

There is a simple analysis here. Fix a domain K of values. Associated with each set A of locations is a set K^A of functions $\mathit{valof}_A : A \to K$ constituting the possible states of the system. *Copy* is an antifunction $f^{\smile} : A \to A'$ ostensibly acting on locations to expand A to A' by duplicating some of A's locations, achieved with a surjective function $f : A' \to A$ associating to each new location a the old location $f(a)$ of which a is a copy. Similarly *delete* is the converse of an injective function whose converse deletes locations not in the image of f.

The action of *copy* and *delete* on locations backwards in time translates to functions mapping states forward in time. The latter send each valof_A to a specific $\mathit{valof}_{A'}$, namely $\mathit{valof}_A \circ f$ where $f : A' \to A$ is the converse of the antifunction $f^{\smile}$.

Now the pattern we observe here, namely $f(g) = g \circ f$, is familiar as a basic source of contravariance, namely the $^{\mathrm{op}}$ in **Set**'s homfunctor $\mathrm{Hom} : \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \to \mathbf{Set}$, with the representable functor $\mathrm{Hom}(-, K)$ (represented by K) sending $\mathbf{Set}^{\mathrm{op}}$ contravariantly into **Set**, a faithful representation provided $|K| \geq 2$,

We propose the viewpoint that *identify* and *adjoin* are the fundamental mathematical or denotational transformations, and *copy* and *delete* are the fundamental computational or operational transformations.

3.3 $\mathbf{Set}^{\mathrm{op}}$and CABA

There is another community besides the computer scientists that migrates naturally to the $\mathbf{Set}^{\mathrm{op}}$ end of the Stone gamut, namely the logicians. A much stronger alliance seems to be growing up between computer science and logic than between computer science and algebra. On the face of it this seems like a social phenomenon. But the $\mathbf{Set}^{\mathrm{op}}$ perspective suggests a more foundational origin for this alliance, argued as follows.

Speaking concretely, **Set** and $\mathbf{Set}^{\mathrm{op}}$ are at opposite extremes with regard to their theory, as follows. **Set** has the empty signature and the empty theory (measured by axioms at least, the equational tautologies $x = x$ are always present). We have already pointed out the equivalence of $\mathbf{Set}^{\mathrm{op}}$ and **CABA**. Now the free CABA generated by the set X is exactly 2^{2^X} (atomicity saves this situation in comparison to the lack of infinite free complete Boolean algebras), whence we can understand the Boolean operations of any arity X including infinite as *all* functions from 2^X to 2. (This generalizes to CABA's the well-known representation of the n-ary Boolean operations as the functions from 2^n to 2, e.g. the 16 binary Boolean operations and 256 ternary ones, with finiteness of n being a distinguishing feature of Boolean algebras in contrast to complete Boolean algebras.) The "all" makes this signature the maximum possible in this sense. Furthermore the equational theory is the maximum possible in the sense that adding *any* new equation in this language makes the theory collapse to the inconsistent theory $x = y$.

So $\mathbf{Set}^{\mathrm{op}}$ is logical not merely because it is propositional logic but because it has a maximal equational theory, dual to **Set** being the minimal (empty but for $x = x$) equational theory, both with regard to signature and equations. That is, it is as logical as it can be, at least with respect to $K = 2$. Our realization theorem for k-ary relational structures [Pra93] as Chu spaces over $K = 2^k$ shows that bringing in larger K lets us extend propositional logic to the rest of mathematics. Interestingly, any $|K| \geq 2$ serves to represent the same (up to isomorphism) embedding of $\mathbf{Set}^{\mathrm{op}}$ in **Set**, since the cube K^n behaves structurally like 2^n in that each of the n dimensions collapses entirely as a unit (an identification, dual to deleting a dimension) whether of length 2 or K. Only for $K = 1$ does this collapsing become degenerate. Hence the impact of larger K is not felt at the "goal-posts" at the end (where one index set A or X of the Chu space is K^X or K^A of the other) but only in the interior proper of the Stone gamut.

4 Continuous = Homomorphism

At the initiative largely of D. Scott, "continuous function" has been used informally for a number of years as synonymous with "homomorphism" (as a function preserving the appropriate sup's and inf's) for certain narrow classes of lattices. In this section we turn this informal synonym into a theorem. The next section uses this theorem to embed the many categories of basic Stone duality in $\mathbf{Chu_2}$. The final section uses it to subsume that part of concrete mathematics dealing with relational structures such as groups, fields, etc.

To give meaning to the term "homomorphism" in the following theorem, we define the following notions. A *possible property* ϕ *of* a general Chu space $\mathcal{A} = (A, \models, X)$ over K is a subset $\phi \subseteq K^A$. A *property* of $\mathcal{A}$ is a possible property of $\mathcal{A}$ satisfying $\widetilde{X} \subseteq \phi$, i.e. lying in the interval $[\widetilde{X}, K^A]$. The properties of $\mathcal{A}$ then form the power set $2^{K^A - \widetilde{X}}$, this notion of property being independent of the choice of K. As a power set the properties of $\mathcal{A}$ form a CABA with inclusion being implication, intersection conjunction, and union disjunction (including infinite and empty conjunctions and disjunctions).

For $K = 2$, properties as elements of 2^{2^A} are just Boolean operations of arity $|A|$. Each Boolean operation is associated with the property consisting of its satisfying assignments. For example $a \to b$ is satisfied by three subsets of $\{a, b\}$ (each subset defining the assignment of true to the members of the subset and false to the rest of A) but not by $\{a\}$.

An *axiomatization* is a set of properties, and denotes its conjunction (intersection), a property and also the Chu space axiomatized by that property (as its strongest property). One use of axiomatizations is to define a class of Chu spaces axiomatizable by axioms *of a particular form*; for example posets are those Chu spaces axiomatizable by atomic implications $a \to b$.

We now define substitution. Fix two Chu spaces $\mathcal{A} = (A, X)$ and $\mathcal{A}' = (A', X')$. Given a possible property ϕ of $\mathcal{A}$, the function $f : A \to A'$ induces a *substitution* of $f(a)$ for a in ϕ, understood informally for the moment, to yield a possible property of $\mathcal{A}'$ that we shall notate as $\phi \circ f$. While substitution works very much like composition, a viewpoint our notation encourages, the sorts do not match up here. Instead we define $\phi \circ f$ formally to be the set of those functions $g : A' \to K$ for which $g \circ f \in \phi$, this set being a possible property of $\mathcal{A}'$. This definition justifies the notation $\phi \circ f$, which can be seen to indeed be substitution.

In this way each function $f : A \to A'$ induces a corresponding substitution $- \circ f : 2^{K^A} \to 2^{K^{A'}}$ sending possible properties of $\mathcal{A}$ to possible properties of $\mathcal{A}'$. We define a *homomorphism* of Chu spaces to be a function $f : A \to A'$ between Chu spaces whose induced substitution sends properties of $\mathcal{A}$ to properties of $\mathcal{A}'$.

Theorem 2 *(Continuous = Homomorphism) Chu* *transforms are exactly homomorphisms.*

Proof: $f : A \to A'$ is a Chu transform iff for every state $g : A' \to K$ of X', $g \circ f : A \to K$ is a state of X. But the latter is just a restatement of the condition of being a homomorphism. ∎

That is, the essence of continuity is the preservation of structure.

If we think of states as permissible degrees of freedom, a function is continuous just when it does not create new degrees of freedom, in the sense that for every degree of freedom $g : A' \to K$ of the target, $g \circ f$ must be a degree of freedom of the source. Chu transforms may copy and delete degrees of freedom (the essential antifunctions) but they may not adjoin new ones (functional behavior).

5 The Stone categories

In this section we exhibit a great many categories associated with Stone duality and treated by Johnstone [Joh82], and embed them all as full concrete duality-preserving subcategories of $\mathbf{Chu}_2$. A key benefit of this perspective is that the notoriously difficult notion of Stone duality reduces simply to matrix transposition.

A recurring theme is that the objects of any given category can be characterized either logically or algebraically ("model theoretically"). Posets provide the prototypical example: logically they are those Chu spaces over 2 axiomatizable with atomic propositions $a \to b$, while algebraically they are those Chu spaces over 2 whose columns are closed under arbitrary union and intersection. This is a distant sibling to Birkhoff's HSP theorem relating classes defined by equations to classes closed under homomorphisms, subalgebras, and direct products, thinking of each state x as a model and X as the class of all models (satisfying assignments) of the Chu space. Van Glabbeek and Plotkin give a number of such correspondences in the table in section 3 [VGP95] pertinent to concurrent computation; we complement their table with a similar list of further such correspondences.

Sets. A set is a Chu space axiomatizable with no axioms; equivalently, an extensional T_0 Chu space whose columns form a complete atomic Boolean algebra or CABA, that is, are closed under complement and arbitrary union. That is, sets are dual to CABA's, argued later. The normal Chu space representing the set A is $(A, 2^A)$. Every function $f : A \to B$ between sets $(A, 2^A)$ and $(B, 2^B)$ is a Chu transform because $(A, 2^A)$ has all possible columns whence we can always find g making (f, g) a Chu transform. A better way to see this however is to use the Continuous = Homomorphism theorem, that the Chu transforms from $\mathcal{A}$ to $\mathcal{B}$ are those functions $f : A \to B$ that preserve the axioms of $\mathcal{A}$, which must be all functions when $\mathcal{A}$ has the empty set of axioms.

Pointed Sets. A pointed set is a Chu space with the one axiom $a = 0$ (or any other constant from K), this element being the "point." Equivalently it is the result of adjoining a constant row to the Chu realization of a set. (Thus a constant is quite literally a constant row.) For $K = \mathbf{2}$ bipointed sets are also possible, axiomatized as $a = 0$, $b = 1$; in general up to K points are possible. Chu transforms between pointed sets preserve the point: $f(0) = 0$.

Preorders. A preorder is a Chu space axiomatized by "atomic implications," namely propositions of the form $a \to b$ where a and b are variables (points of A). A partial order is a T_0 preorder. Preserving properties then means preserving axioms $a \to b$, i.e. $f(a) \to f(b)$ must hold in the target. Hence Chu transforms between preorders are exactly monotone functions.

Theorem 3 *An extensional Chu space realizes a preorder if and only if its columns form a complete lattice under arbitrary (including empty and infinite) union and intersection.*

Proof: (Only if) Let Γ be a set of atomic implications defining the given preorder. Suppose that the intersection of some set Z of assignments each satisfying all implications of Γ fails to satisfy some $a \to b$ in Γ. Then it must assign 1 to a and 0 to b. But in that case every assignment in Z must assign 1 to a, whence every such assignment must also assign 1 to b, so the intersection cannot have assigned 0 to b after all. Dually, if the union of Z assigns 1 to a and 0 to b, it must assign 0 to b in every assignment of Z and hence can assign 1 to a in no assignment of Z, whence the union cannot have assigned 1 to a after all. So the satisfying assignments of any set of atomic implications is closed under arbitrary union and disjunction.

(If) Assume the columns of $\mathcal{A}$ under union and intersection form a complete lattice.[6] It suffices to show that the set Γ of atomic implications holding in $\mathcal{A}$ axiomatizes $\mathcal{A}$, i.e. that $\mathcal{A}$ contains all satisfying assignments of Γ. Let $x \subseteq A$ be any such assignment. For each $a \in A$ form the intersection of all columns of $\mathcal{A}$ containing a, itself a column of $\mathcal{A}$ containing a, call it y_a. Now form the union $\bigcup_{a \in x} y_a$ to yield a column z of $\mathcal{A}$, which must be a superset of column x.

[6] It is worth mentioning that this is a stronger assumption than that the columns of $\mathcal{A}$ partially ordered by inclusion form a complete lattice, since the meets and joins thereof then need not coincide with intersection and union.

Now suppose $b \in y - x$. Then there exists $a \in x$ such that $b \in y_a$, whence b is in every column of $\mathcal{A}$ containing a, whence $a \to b$ is in Γ. But x contains a and not b, contradicting the assumption that x satisfies Γ. Hence $b \in y - x$ cannot exist, i.e. $y = x$. However y was constructed from columns of $\mathcal{A}$ by arbitrary union and intersection and therefore is itself a column of $\mathcal{A}$, whence so is x. ∎

This corresponds to the result that posets are dual to profinite distributive lattices [Joh82, p.249], with columns playing the role of ultrafilters (lattice homomorphisms maps to the 2-element lattice), less the important fact, taken as an hypothesis of the theorem here, that the posets have enough points to make them extensional.) A normal T_0 Chu space whose rows are closed under arbitrary union and intersection realizes a profinite distributive lattice, which for our purposes suffices for a definition of this notion; consult Johnstone (op.cit.) for an alternative definition.[7] That this is a categorical duality follows immediately from the self-duality of **Chu**.

This result makes it easy to demonstrate the duality of sets and CABA's we promised earlier. One direction is clear: sets contain all possible columns and hence form a CABA by set theory. For the other direction, a CABA is a profinite distributive lattice, whence the theory of its dual is axiomatizable by atomic implications. When $a \leq b$ in a poset for distinct a, b there must exist a satisfying assignment making $a = 0$ and $b = 1$; the complementary assignment, which exists in a CABA, then contradicts $a \leq b$, showing that the theory of the dual of a CABA cannot contain any atomic implications $a \to b$ for distinct a, b, and hence is axiomatizable with the empty set of axioms.

To complete the argument that this is a realization we need the Chu transforms between posets realized in this way as Chu spaces to be exactly the monotone functions. Now monotonicity is the condition that if $a \leq b$ holds in (is a property of) the source then $f(a) \leq f(b)$ holds in the target. Since the only axioms are atomic implications, monotonicity is equivalent to being axiom-preserving, equivalently property-preserving, hence a Chu transform, and we are done.

The following fact about posets will prove useful when we come to locales. The notion of column-maximality is that no distinct column can be added while holding A fixed and preserving the specified row properties, and characterizes the free object on A with those properties. Row-maximality is the dual notion.

Theorem 4 *A T_0 extensional Chu space $\mathcal{A}$ whose columns are closed under arbitrary union and intersection is automatically row-maximal with respect to those properties.*

Proof: By the previous theorem and the T_0 property, $\mathcal{A}$ realizes a poset P. Hence adding a new row to form $\mathcal{A}'$ without disturbing those properties corresponds to adding an element a to P to yield a new poset P'. But the principal order filter in P' generated by a, and the result of deleting a from that filter, are both order filters of P' and hence must be distinct columns of $\mathcal{A}'$. However deleting row a makes them identical columns of $\mathcal{A}$, contradicting extensionality. ∎

Topological spaces. A topological space is an extensional Chu space whose columns are closed under arbitrary union and *finite* (including empty) intersection. We showed previously that Chu transforms can be defined exactly as for continuous functions between topological spaces, whence for Chu spaces that *are* topological spaces the Chu transforms between are exactly their usual continuous functions, mentioned in passing by Lafont and Streicher [LS91].

Locales. A *spatial* (i.e. extensional) *locale* [Isb72, Joh82] is a *row-maximal* T_0 topological space, one to which no point can be added without defeating the requisite closure properties of the columns. Dropping "spatial" means dropping the usual assumption of extensionality for topological spaces. Thus a *locale* is a row-maximal T_0 Chu space whose columns are closed under arbitrary union and finite intersection. In the non-spatial case we do not attempt to understand the opens of a locale as sets of points, instead regarding the points of locales as sets of opens, locales being T_0 instead of extensional.

A *frame* (op.cit.) is the dual of a locale, that is, a column-maximal extensional Chu space whose rows are closed under arbitrary union and finite intersection. Column-maximality ensures that these closure properties are the *only* properties enjoyed by a frame, which the columns witness by being all frame homomorphisms to the conormal 2×1 Chu space $\perp$. It follows that Chu transforms between frames are exactly frame homomorphisms.

Illustrative examples are provided by the posets $P = (\mathbb{N}, \leq)$ and $Q = (\mathbb{N}, \geq)$, realized as the respective Chu spaces $(\mathbb{N}, \geq, \mathbb{N} \cup \{\infty\})$, $(\mathbb{N}, <, \mathbb{N} \cup \{\infty\})$. Adjoining a new row to Q destroys closure of states

[7] Here is a more conventionally abstract but simple and novel definition of "profinite distributive lattice." A distributive lattice is *profinite* when it is complete and its maximal chains are *nowhere dense*, that is, every proper interval (pair $a < b$ and all elements between) includes a *gap*, meaning a proper interval containing only its two endpoints.

under arbitrary union and finite intersection (the latter forces 1 in column ∞), making Q row-maximal and so a locale. However adjoining to P the point "$\infty - 1$" (a row of all 1's except in column ∞) preserves the localic closure requirements so P is not a locale. But $\infty - 1$ is all we can so add, making $\mathbb{N} \cup \{\infty - 1\}$ a locale (but not a poset).

Semilattices. The semilattice $(A, \vee)$ is axiomatized by all equivalences $a \vee b \equiv c$ holding in $(A, \vee)$, one for each pair a, b in A. For f to preserve these axioms is to have $f(a) \vee f(b) \equiv f(c)$ hold in the target. But this is just the condition for f to be a semilattice homomorphism, giving us a realization in **Chu** of the category of semilattices.

Equivalently a semilattice is an extensional Chu space whose rows are closed under binary union and which is column-maximal subject to the other conditions. Column-maximality merely ensures that all columns satisfying the axioms are put in.

The dual of a semilattice is an algebraic lattice [Joh82, p.252].

Complete semilattices. The complete semilattice $(A, \bigvee)$ has all joins, including the empty join and infinite joins. It is axiomatized as for a semilattice, but the left hand sides of its equivalences may be either infinite joins or 0. The dual of a complete semilattice is itself a complete semilattice; thus the subcategory of **Chu** consisting of these complete semilattices is a self-dual subcategory (the same duality).

Distributive Lattices. The idea for the semilattice $(A, \vee)$ is extended to the lattice $(A, \vee, \wedge)$ by adding to the semilattice equations for $\vee$ all equations $a \wedge b = c$ holding in $(A, \vee, \wedge)$ for each a, b in A. Distributivity being a Boolean tautology, it follows that all lattices so represented are distributive. The second (equivalent) formulation of semilattices also extends to distributive lattices along the same lines.

Boolean algebras. A Boolean algebra is a complemented distributive lattice, hence as a Chu space it suffices to add the requirement that the set of columns be closed under complement. Equivalently a Boolean algebra is an extensional Chu space whose rows form a Boolean algebra under pointwise Boolean combinations (complement and binary union suffice) and which is column-maximal subject to these conditions.

The dual of a Boolean algebra can be obtained as always by transposition. What we get however need not have its set of columns closed under arbitrary union, in which case this dual will not be a topological space. But M. Stone's theorem [Sto36] is that the dual of a Boolean algebra is a totally disconnected compact Hausdorff space. This creates a nice little puzzle: how can the dual of a Boolean algebra simultaneously be a topological space and an object which does not obviously behave like a topological space.

There is a straightforward explanation, which at the same time yields an elementary proof of Stone's theorem stated as a categorical duality.

Theorem 5 *The Chu transforms between a pair of transposed Chu realizations $\mathcal{A}$, $\mathcal{B}$ of two Boolean algebras are the same as the Chu transforms between the results $\mathcal{A}'$, $\mathcal{B}'$ of closing the column sets of $\mathcal{A}$ and $\mathcal{B}$ under arbitrary union, namely the topologies generated by the columns as a basis of clopen sets.*

(These spaces being extensional, we may understand Chu transforms between them to be just functions $f : A \to A'$ for which there exists $g : X' \to X$ such that (f, g) satisfies the adjointness condition. By not giving g explicitly, the Chu transform remains unaffected by changes to g when we perform the above closure.)

Proof: We first close the source, then the target, and observe that neither adjustment changes the set of Chu transforms. We omit the standard argument that closing produces no new clopens[8].

Closing the columns of the source under arbitrary union can only add to the possible Chu transforms, since this makes it easier to find a counterpart for a target column in the source. Let $f : A \to B$ be a function that was not a Chu transform but became one after closing the source. Now the target is still a transposed Boolean algebra so its columns are closed under complement, whence so is the set of their compositions with f. But there are no new clopens in the source, hence no new source column can be responsible for making f a Chu transform, so f must have been a Chu transform before closing the source.

Closing the columns of the target under arbitary union can only delete Chu transforms, since we now have new target columns to find counterparts for. But since the new target columns are arbitrary unions of old ones, and all Boolean combinations of columns commute with composition with f (a simple way of seeing that f^{-1} is a CABA homomorphism), the necessary source columns will also be arbitrary unions of old ones, which exist because we previously so closed the source columns. Hence Chu transforms between transposed Boolean algebras are the same thing as Chu transforms, and hence continuous functions, between the topological spaces they generate. ∎

[8] The trick is to kill any prospective new clopen by exhibiting a suitable witness row formed as an ultrafilter completed by an application of Kuratowksi's lemma from a certain disjoint ideal-filter pair.

This answers our question of why the Chu dual of a Boolean algebra is not a topological space when the Stone dual is. The answer is that both transform identically, whence either will do as the dual. Stone's theorem takes the duality one step further by characterizing the generated topological spaces as all and only the totally separated compact spaces, but this is irrelevant to our question.

There are two points worth noting here. First, the hard part of the proof was getting from the Chu dual to the Stone dual, which established a concrete isomorphism of categories. Thus the meat of Stone's theorem resides in a *covariant* correspondence of categories; the contravariant part responsible for its being a duality was established by the completely trivial process of matrix transposition. In this sense duality is a red herring for Stone's theorem.

Second, if the goal was only to identify a canonical concrete category of structures representably dual to Boolean algebras and hence suitable for representing them, the Chu dual does just as well as the Stone dual and with much less fuss, indeed it is an automatic consequence of general Chu duality. The harder passage to topological spaces was necessary only because these were the nearest mathematical objects available then. The identification and understanding of the much larger category of Chu spaces legitimizes stopping at the Chu dual.

Vector spaces over $GF(2)$. An unexpected entry in this long list of full concrete subcategories of $\mathbf{Chu}_2$ is that of vector spaces over $GF(2)$. These are extensional Chu spaces containing the constantly zero row, with rows closed under binary exclusive-or, and column-maximal subject to these conditions.

6 Universality

The categories $\mathbf{Str}_k$ of k-ary relational structures and their homomorphisms where k is any ordinal are universal categories for mathematics to the extent that they realize many familiar categories: binary relations and graphs at $k = 2$, groups and Boolean algebras at $k = 3$, rings, fields, and categories at $k = 4$, etc. We previously showed [Pra93, p.153-4] that the self-dual category $\mathbf{Chu}_{2^k}$ of Chu spaces over the power set of $k = \{0, 1, \ldots, k-1\}$ realizes $\mathbf{Str}_k$. The following reorganization of that proof, in particular the notion of state as a tuple permitted not to hold, ties the argument much better to Continuity = Homomorphism.

A k-ary *relational structure* (A, ρ) consists of a set A, the *carrier*, and a k-ary relation $\rho \subseteq A^k$. A *homomorphism* $f : (A, \rho) \to (B, \sigma)$ of such structures is a function $f : A \to B$ such that $t \in \rho$ implies $f \circ t \in \sigma$. These structures and homomorphisms, with the usual composition, form the category $\mathbf{Str}_k$.

A *k-ruple* (for "relational tuple") *over* A is a binary relation R from k to A, which we shall regard interchangeably as a set $R \subseteq k \times A$ and a function $r : A \to 2^k$ as appropriate, related by $r(a) = \{i \mid (i, a) \in R\}$, $R = \{(i, a) \mid i \in r(a)\}$, and also as $t : k \to A$ for the special case of ruples that are tuples (functions). k-ruples are ordered by inclusion when viewed as subsets of $k \times A$.

The composition $r \circ t$ of a ruple r with a tuple t assumes the representations $r : A \to 2^k$, $t : k \to A$. As $r \circ t : k \to 2^k$, this composite is itself a binary relation on k; $I \le r \circ t$ ($r \circ t$ is reflexive) just when r extends t.

Define $\hat{\rho}$ as the set $\{r : A \to 2^k \mid \exists t \in \rho \,.\, I \le r \circ t\}$ of extensions of tuples of ρ, and $\overline{\hat{\rho}} = (2^k)^A - \hat{\rho}$ as its complement. We realize (A, ρ) in $\mathbf{Chu}_{2^k}$ as $(A, \overline{\hat{\rho}})$; in words, the normal Chu space with point set A whose states are those $r : A \to 2^k$ extending no tuple of ρ.

Theorem 6 *A function $f : A \to B$ is a Chu transform $f : (A, \overline{\hat{\rho}}) \to (B, \overline{\hat{\sigma}})$ if and only if it is a homomorphism $f : (A, \rho) \to (B, \sigma)$.*

Proof: It suffices to show $\forall r{:}B \to 2^k [r \circ f \in \hat{\rho} \Rightarrow r \in \hat{\sigma}]$ iff $\forall t{:}k \to A[t \in \rho \Rightarrow f \circ t \in \sigma]$. The following argument is organized around the associativity of $r \circ f \circ t$.

(If) Given $r : B \to 2^k$, if $r \circ f \in \hat{\rho}$ then there must exist $t \in \rho$ such that $I \le r \circ f \circ t$. But then $f \circ t \in \sigma$ (f is a homomorphism), whence $r \in \hat{\sigma}$.

(Only if) Given $t \in \rho$, take $r : B \to 2^k$ to be the minimal (trivial) extension of $f \circ t$ to a ruple, i.e. $I \le r \circ f \circ t$ with equality just when f is injective. Then $r \circ f \in \hat{\rho}$ (definition of $\hat{\rho}$), whence $r \in \hat{\sigma}$ (f is a Chu transform). But then there must exist some $t' : k \to B$ such that $t' \in \sigma$ and $I \le r \circ t'$. Hence r extends t'. But r is (trivially extends) a tuple, whence $t' = r = f \circ t$, whence $f \circ t \in \sigma$. ∎

A ruple may be understood as an ambiguous tuple t, namely a conjunction of propositions (j, a) asserting $t_j = a$, with the ambiguity arising through either over- or under-specification at any given coordinate. States are taken to be those ruples not required to stand in the relation ρ, which disqualifies any state asserting enough to name a tuple of ρ (it may assert more); this makes a much better connection with the Continuity = Homomorphism theorem than our previous form of the argument [Pra93]. Our proof does not go through when the "paradoxical" states are omitted, those over-specifying t_j as simultaneously having distinct values a and a'. This parallels the Schrödinger cat paradox in which a cat can be both dead and alive in the same state. Quantum mechanics precludes the omission of such states for apparently similar reasons.

Another view of $\dashv''$ is obtained by "currying" $\dashv : A \times X \to 2^k$ to $\dashv'' : A \times k \times X \to 2$ and regarding a Chu transform $f : A \to B$, $g : Y \to X$ as having a third function, the identity function 1_k on k (as the k-element set $\{0, 1, \ldots, k-1\}$). The adjointness condition is then stated as $\dashv'''(f(a), j, y) = \dashv''(a, j, g(y))$ for all $a \in A$, $j < k$, and $y \in Y$, turning 2^k-valued logic into ordinary 2-valued logic by taking k to be a set of variables stationary in time and information, i.e. transforming neither forwards nor backwards.

7 Concreteness

We conclude with some remarks about concreteness, a key benefit of the realization theorem of the previous section. When concreteness is ignored, any small category fully embeds in the category of semigroups [HL69], one way of making it concrete. However this embedding will in general represent objects otherwise representable with two or three elements by a semigroup as large as the embedded category. Such a concretization is incomprehensible and impractical.

Not every category is concrete on the face of it. There seem to be too many binary relations from A to B to make **Rel** concrete. But then $\mathrm{Hom}(1, A)$ as the "underlying set of the set A" turns out to be the contravariant power set functor 2^A, a monad on **Set** having **Rel** as its Kleisli category, concretized by the monad as the free complete semilattices, a very pleasant, insightful, and generally useful concretization of a self-dual category. The corresponding Eilenberg-Moore category is **CSLat**, all complete semilattices. If anything our intuition tends to underestimate the concreteness of mathematics.

Linear logic is the "structured programming" of proof theory, giving logical form to the informal manipulation of the internal homfunctor. But mathematics does not live by the internal homfunctor alone, and many situations are better understood via the external homfunctor $\mathrm{Hom} : C^{\mathrm{op}} \times C : \mathbf{Set}$ carrying logic into the familiar cartesian closed world of sets, where one can reason about elements and functions unencumbered by signature and coherence. Linear logic caters to this passage with the exponential $!A$, interpreted concretely for $\mathbf{Chu}_K$ as the comonad sending $(A, \dashv, X)$ to (A, K^A), whose coKleisli category is **Set** [LS91]. Domain theory proposes other cartesian closed categories offering universal solvability of recursive equations, but the familiarity of sets is a lot to give up for universal recursion, and mathematical practice would seem more faithfully reflected by the understanding of $!A$ as an ordinary set.

The essence of sets resides in their discreteness (lack of coherence) and size, combined in the notion of cardinal where one counts points. Concern with size also appears in linear logic, where one counts the use of premises; whereas each occurrence of the formula A *must* be counted while its elements *cannot* be counted, the exponential reverses this: it is permitted not to count occurrences of $!A$, and it is possible to count the elements of $!A$. This conservation principle for counting mimics the conservation of meets and joins in Chu spaces over 2, omitted here, see section 5.1 of ftp://boole.stanford.edu/pub/DVI/bud.dvi.Z on event-state interference, where Theorem 12 states that *A proper meet in $\mathcal{A}$ precludes some proper join in $\mathcal{A}^\perp$*, an immediate corollary of which is *If $\mathcal{A}$ has all meets then $\mathcal{A}^\perp$ has no proper joins.* The tradeoff is accurate enough that, to within axiomatic details such as whether induction is a feature of counting and numbers, the logic of $!A$ is exactly the same in both the proof theoretic and semantic worlds.

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