A ten-slide survey on Chu

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0. Outline

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1. Six prefatory notes

History: Chu spaces were first defined in a V-enriched setting by Michael Barr's master's student Peter (Po-Hsiang) Chu and appeared as a long appendix in Barr's 1979 book $*$*-autonomous categories*.

Rediscovery: Game semantics for linear logic [Lafont&Streicher,LICS'91].

Our use: As a model of concurrency [Gupta & Pratt, STOC'93].

Nature: Chu spaces generalize topological spaces by allowing open sets to (i) be fuzzy (K-valued membership) and (ii) express other structures besides those allowed by $\bigcup$ and $\cap$.

Unification: With presheaves: $\rightarrow$ *typed* Chu spaces (sorts & properties).

Warning: Philosophical framework: functor-free foundations (experimental, should annoy category theorists).
Hides the Yoneda embedding under the bonnet.

2. Sorted categories: exploiting associativity (CT preview)

A **sorted category** $(\mathcal{C}, \Sigma)$ consists of a locally small category $\mathcal{C}$ and a (small) set $\Sigma \subseteq ob(\mathcal{C})$ of objects of $\mathcal{C}$ comprising the **sorts** $s, t, \ldots$.

Theorem

(i) Every sorted category extends canonically to a presheaf category.
(ii) All presheaf categories arise in this way.

Proof.

$$Y : \mathcal{S} \overset{Y'}{\hookrightarrow} \mathcal{C} \overset{Z}{\twoheadrightarrow} \hat{\mathcal{C}} \hookrightarrow \mathrm{Psh}(\mathcal{S}) = \mathbf{Set}^{\mathcal{S}^{op}} \quad \text{where}$$

- $\mathcal{S}$ is the full subcategory of $\mathcal{C}$ s.t. $ob(\mathcal{S}) = \Sigma$ ("subYoneda");
- Z quotients homsets of $\mathcal{C}$ by extensionality; and
- $\hat{\mathcal{C}}$ is a subcategory of $\mathrm{Psh}(\mathcal{S})$ (not necessarily full).

□

Extremal cases: (i) $\mathcal{C} = \mathcal{S}$ $(= \hat{\mathcal{C}})$. (ii) $\mathcal{C} = \hat{\mathcal{C}} = \mathrm{Psh}(\mathcal{S})$.

2. Sorted categories: exploiting associativity

A **sorted category** $(\mathcal{C}, \Sigma)$ consists of a locally small category $\mathcal{C}$ and a (small) set $\Sigma \subseteq ob(\mathcal{C})$ of objects of $\mathcal{C}$ comprising the **sorts** $s, t, \ldots$.

Each object α of $\mathcal{C}$ has an **extension** A as a heterogeneous algebra with

- for each sort s, a carrier $A_s = \{a : s \to \alpha\}$; and
- for each $f : t \to s$ in $\mathcal{S}$, an operation $f_A : A_s \to A_t : a \mapsto af$

Each morphism $h : \alpha \to \beta$ has extensions $\hat{h}_s : A_s \to B_s : a \mapsto ha$.

Theorem

Every extension $\hat{h} : A \to B$ is a homomorphism.

Proof.

$$t \xrightarrow{f} s \xrightarrow{a} A \xrightarrow{h} B$$

expresses both $h(f_A(a))$ and $f_B(h(a))$, equal by associativity. □

3. Examples

Set. A **set** is an object of a sorted category $(\mathcal{C}, \{1\})$ such that $\{1\}$ is **rigid** $(\mathcal{C}(1,1)| = 1)$.

$\hat{\mathcal{C}}$ quotients $\mathcal{C}$ by extensionality: if $f, g : \alpha \to \beta$ have the same extension (i.e. same right actions on all A_s) then $f = g$. *Remarks:*

1. $\hat{\mathcal{S}} = \mathcal{S}$. (Trivial)
2. $\hat{\mathcal{C}}$ is a subcategory of **Set**.

Grph: A graph is an object of a sorted category $(\mathcal{C}, \{V, E\})$ such that the objects V and E are rigid and the only morphisms between them are $s, t : V \to E$. *Remarks:*

1. $\hat{\mathcal{S}} = \mathcal{S}$, because the right actions of s and t on the identity at E are themselves and they are distinct.
2. $\hat{\mathcal{C}}$ is a subcategory of **Grph**.

4. Bityped categories: sorts + properties, points + states

A **bityped category** $(\mathcal{C}, \Sigma, \Pi)$ consists of a locally small category $\mathcal{C}$ and disjoint sets $\Sigma, \Pi \subseteq ob(\mathcal{C})$ of objects of $\mathcal{C}$ comprising respectively the sorts $s, t, \ldots$, and the **properties** $p, q, \ldots$.

Definition $\mathcal{K} : \mathcal{S} \Rightarrow \mathcal{P}$ is the full bipartite subcategory (bimodule, profunctor, distributor) of $\mathcal{C}$ whose objects are in $\Sigma \cup \Pi$. $\mathcal{K}_s^p = \mathcal{C}(s, p)$.

Each object α of $\mathcal{C}$ has an **extension** $(A, ;, X)$ as a **typed Chu space over** $\mathcal{K}$ with

- $\forall$ sorts s, a carrier $A_s = \{a : s \to \alpha\}$;
- $\forall$ $f : t \to s$ in $\mathcal{S}$, an operation $f_A : A_s \to A_t : a \mapsto t \xrightarrow{f} s \xrightarrow{a} \alpha$;
- $\forall$ $a \in A_s$ and $x \in X_p$, a **quale** $a; x = s \xrightarrow{a} \alpha \xrightarrow{x} p$;
- $\forall$ properties p, a cocarrier $X_p = \{x : \alpha \to p\}$; and
- $\forall$ $g : p \to q$ in Π, an operation $g_X : X_p \to X_q : x \mapsto \alpha \xrightarrow{x} p \xrightarrow{g} q$

$$Y : \mathcal{K} \overset{Y'}{\hookrightarrow} \mathcal{C} \overset{Z}{\twoheadrightarrow} \hat{\mathcal{C}} \hookrightarrow \mathbf{TChu}_{\mathcal{K}}$$

4. Bityped categories: sorts + properties, points + states

Each morphism $h : \alpha \to \beta$ of $\mathcal{C}$ has an extension pair (h, h') consisting of

a left extension: for each s a function $h_s : A_s \to B_s : a \mapsto ha$; and

a right extension: for each p a function $h'_p : Y_p \to X_p : y \mapsto yh$.

Theorem

Every morphism $h : \alpha \to \beta$ of $(\mathcal{C}, \Sigma, \Pi)$ is an adjoint (co)homomorphism.

Proof.

$$t \xrightarrow{f} s \xrightarrow{a} \alpha \xrightarrow{h} \beta \xrightarrow{y} p \xrightarrow{g} q$$

(cf. red items on previous slide)

Homomorphism $h(af) = (ha)f == h(f_A(a)) = f_B(h(a))$ (points)

Adjoint $y(ha) = (yh)a == h(a); y = a; h'(y)$ (biactions)

Cohomomorphism $(gy)h = g(yh) == h'(g_B(y)) = g_A(h'(y))$ (states) □

5. Chu spaces

Slogan: Chu spaces:typed Chu spaces::sets:presheaves

A **Chu space** over K is an object α of a sorted category $(\mathcal{C}, \{1, \perp\})$ such that 1 and $\perp$ are rigid and $\mathcal{C}(1, \perp) = K$.

A Chu space has a carrier $A = \{a : 1 \to \alpha\}$, a cocarrier $X = \{x : \alpha \to \perp\}$, and a K-valued $A \times X$ binary relation ; (a matrix) satisfying $a; x = 1 \stackrel{a}{\to} \alpha \stackrel{x}{\to} \perp$. $\alpha^{\perp} = (X, \breve{;}, A)$.

Associate to α the set $\{\lambda a.(a; x) | x \in X\} \subseteq K^A$ (the columns of α). Call this the **character** of α, denoted $\chi(\alpha)$.

The power set 2^{K^A} is a complete atomic Boolean algebra (CABA). A **property** $\phi \subseteq K^A$ of α is a superset of $\chi(\alpha)$; that is, the properties of α form the upset $\uparrow \chi(\alpha)$ in 2^{K^A}. This is the CABA obtained from 2^{K^A} by omitting those functions omitted by $\chi(A)$.

The properties of α constitute the formulas of a propositional logic by interpreting inclusion as implication, intersection as conjunction, and so on for all Boolean connectives.

6. Classes of Properties

Chu spaces can be limited to various classes by imposing closure properties on their states. For example when $K = 2$, making states sets, we may require closure under arbitrary union and arbitrary intersection. Chu spaces of this kind represent preordered sets.

But no matter what restrictions we place on Chu spaces, their homomorphisms always preserve properties, as follows.

Theorem

Given a morphism $h : \alpha \to \beta$, every property of α is a property of the image $h(\alpha)$ in β.

Proof.

It suffices to show $\chi(h(\alpha)) \subseteq \chi(\alpha)$. Each $y : \beta \to \perp$ induces a function on points ha in B whose value at each such ha is yha. But this can be interpreted as a function on A indexed by yh, which is in $\chi(\alpha)$. □

7. Some full subcats of Chu

Many *concrete* categories are definable as full subcategories of $\mathbf{Chu}_K$ for suitable K. The earliest examples were found by Lafont and Streicher [LICS'91], namely **Top** (with $K = 2$) and $\mathbf{Vct}_K$ for K any field. For the former the states must be closed under meet and arbitrary join. For the latter the states are all linear functionals on the represented space.

Finite dimensional vector spaces over GF(2) can be represented as Chu spaces over 2 whose rows and columns are both closed under XOR.

Preordered sets have states closed under $\bigcup$ and $\bigcap$.

Meet-semilattices are representable with $K = 2$ by closing the rows under $\wedge$ and the columns under $\bigcap$ and infinitary joins. Rows and columns interfere in a way that prevents their closure under respectively finite meets and finite joins, as a generalization of the evident impossibility of having constant rows and different constant columns.

For more examples google for Coimbra Chu spaces.

8. Orthocurrence

Orthocurrence $\alpha \otimes \beta$ is what I was calling tensor product of pomsets in the early 1980s before I noticed certain resemblances with linear logic.

Bilinearity for Chu spaces. Think of a subset $\mathcal{X} \subseteq K^A$ as a dictionary of $|\mathcal{X}|$ words of length $|A|$ over the alphabet K. Define $\mathcal{X} \parr \mathcal{X}'$ to be the set of all crosswords $A \times A' \to K$ such that the across words are all from $\mathcal{X}'$ and the down words are all from $\mathcal{X}$.

Definition $\alpha \otimes \beta = (A \times B, ;, \chi(\alpha) \parr \chi(\beta))$.

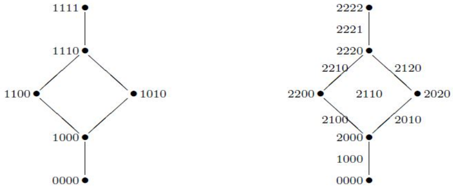
$(a, b); x$ is the letter at position (a,b) of crossword x.

Linear logic defines the operation $\multimap$ of linear implication as $\alpha \multimap \beta = (\alpha \otimes \beta^{\perp})^{\perp}$. The points of $\alpha \multimap \beta$ constitute the Chu morphisms $h : \alpha \to \beta$ while its states are pairs (a, y). The value of $h; (a, y)$ is $h(a); y$ in β.

9. Interval algebras

Let $A = \{a, a'\}, B = \{b, b'\}$. Write 10 for the state in K^A assigning 1 to a and 0 to a'. Omitting only this state defines $\chi(\alpha)$ for the Chu space α representing the two-element chain $\{a < a'\}$, and likewise for β.

An Allen-style interval algebra consists of the possible configurations or states of two intervals sliding past each other. The following show these configurations with $K = 2$ and 3 respectively.



These are obtained as the states of $\alpha \otimes \beta$ for each K. For $K = 2$ the two qualia are "before" and "after". $K = 3$ adds "at". The tensor product then yields the 13 configurations of the Allen interval algebra.

9. Interval algebras

Anger and Rodriguez added "near" and obtained 29 states by hand. Motivated by relativity they further added "enter" and "exit" to express entering and leaving the light cone. Again by hand, they obtained 82 states.

Using the Chu caculator at chu.stanford.edu we confirmed both counts by evaluating the tensor products of the corresponding Chu spaces. Given the difficulty of enumerating so many configurations by hand, as well as the possibility of bugs in the calculator, we found this agreement impressive.

More on these interval algebras in section 4.3 of the paper "Higher dimensional automata revisited" (google the title).

10. Beyond Chu

Dropping the requirement that Σ and Π be rigid singletons takes us beyond Chu spaces.

Removing $\perp$ from Π reduces bityped categories to sorted categories since objects can no longer have states and there are no other conditions specific to bityped categories that do not apply to sorted categories.

Example Red-blue graphs with weighted edges.

Replacing $\{1\}$ by $\{V, E\}$ and $\{\perp\}$ by $\{C, W\}$ allows four homsets from Σ to Π consisting of the possible colors and weights of vertices and edges. We aren't interested in colors of edges or weights of vertices and therefore take $\mathcal{K}_E^C = \mathcal{K}_V^W = \{\perp\}$ (one quale in each homset, denoting "undefined"). To color each vertex red or blue, take $\mathcal{K}_V^C = \{\bullet, \bullet\}$. To assign weights to edges, take $\mathcal{K}_E^W$ to be one of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, etc. as desired.