Euclidean and non-Euclidean algebra

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1. Program

In this talk we begin by formulating Postulates 2, 5, and 1 of Euclid's *Elements* algebraically.

The result is a purely spatial axiomatization of the variety $\mathbf{Aff}_{\mathbb{Q}}$ of vector spaces over the rationals.

Our axiomatization is equivalent to the customary numerical one based on linear combinations whose coefficients sum to unity.

We then extend this to $\mathbf{Aff}_{\mathbb{Q}[\mathbf{i}]}$, complex rationals, via a suitably axiomatized binary operation $x \circ y$ giving the point y goes to when xy is rotated counterclockwise 90° about x.

Lastly we axiomatize respectively elliptical and hyperbolic space by modifying a detail of our algebraic formulation of Postulate 5.

2. Historical background

It is plausible that the Pyramid at Djoser (2700 BC) was laid out with the help of cartesian coordinates. Laying out its intricate system of inner chambers would have been considerably harder if shaped like the Circular Pyramids of Western Mexico.

In any event it surely did not appeal to anything like Book I of Euclid's *Elements*.

It was over two millennia before the Greeks noticed that the side of the pyramid's base (or any square) was incommensurate with its diagonal.

Max Dehn (1926) has speculated that this incommensurability led Euclid to formulate Books I-IV in purely spatial terms free of any concept of number.

3. Euclid's postulates

Book I gives 23 definitions (e.g. "A straight line is a line which lies evenly with the points on itself") together with the following five postulates.

1. To draw a straight line between any two points.
2. To produce a finite straight line continuously [to any finite multiple of its length].
3. To draw a circle with specified center and radius.
4. That all right angles are equal.
5. That two lines inclined inwards meet.

Postulates 1-3 are constructions while 4-5 are assertions.

4. Algebraic formulation

Our algebraic version of Euclid's postulates starts with the second, which we modify as, "to produce a finite straight line as far again."

If x and y are the endpoints of the given line, we write xy for the point reached by so producing it and abbreviate $(xy)z$ to xyz.

We axiomatize Postulates 2 and 5 as follows.

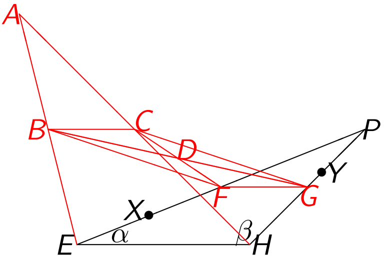
$$xx = x \qquad (1)$$

$$(xy)y = x \qquad (2)$$

$$(wx)(yz) = (wy)(xz) \qquad (3)$$

Call these **G1, G2, G3**.

5. Expression of Euclid's 5th Postulate



Euclid's 5th: *EX* & *HY*, when inclined inwards, meet when *produced*.

Euclid's criterion for "inclined": $\alpha + \beta < 180^o$.

Our criterion: existence of a *witness triangle* ΔAEH with parallelogram *BCGF* (centroid *D*) s.t *B*, *C* at midpoints of *AE*, *AH*.

Our version of the 5th: *EF* and *HG*, when obtained by extending the four sides of the skew quadrilateral *ABDC*, meet when *extended*.

$$\underset{E \quad \to \quad F \quad = \quad H \quad \to \quad G}{A \to B \to (C \to D) = A \to C \to (B \to D)} \qquad \textbf{(G3)}$$

G3 $\quad xy(zw) = xz(yw) \mid xywz = xzwy \mid xywzywz = x \mid x102102 = x$

6. Consequences

Rename $wx(yz) = wy(xz)$ to $xy(zw) = xz(yw)$.
Then set $w = z$ to obtain $xy(zz) = xz(yz)$. Use $zz = z$ to obtain

$$xyz = xz(yz)$$

Call this **G2.5**. It is weaker than **G3**, and asserts that the image of a geodesic under inversion (reflection) in a point (here z) is a geodesic. (And by **G2** inversion in a point is an involution.)
Applications: (i) Non-Euclidean algebra (later).
(ii) Parenthesis elimation. Substitute xz for x in **G2.5** and simplify to yield

$$x(yz) = xzyz.$$

Use this and the convention $(xy)z = xyz$ to recursively eliminate all parentheses from any term (does not need **G3**).

7. Equivalent formulations of the 5th postulate

Flat version of Postulate 5:

$$
\begin{aligned}
wxyz &= wxy(zee) && (4)\\
&= wx(ze)(ye) && (5)\\
&= wz(xe)(ye) && (6)\\
&= wzy(xee) && (7)\\
&= wzyx && (8)
\end{aligned}
$$

$$wxyz = wzyx$$

Recover $wx(yz) = wy(xz)$ from this and $xyz = xz(yz)$.
Use **G2** three times to convert this to

$$wxyzxyz = w$$

Application: Euclidean case of non-Euclidean algebra.

8. Midpoints

Define the ternary relation $M(x, y, z)$ as $xz = y$. Call M an **operation** when for all x, y exactly one z, denoted $x \oplus y$, satisfies $M(x, y, z)$. This condition can be expressed as follows.

$$x \oplus y \;=\; z \;\text{ iff }\; xz \;=\; y$$

This equivalence can be split into two equations as follows.

$$x \oplus (xz) \;=\; z \qquad\qquad x(x \oplus y) \;=\; y$$

Theorem 1 *$x \oplus y$ is defined iff $|\{z \mid xz = y\}| = 1$.*
For example $N \oplus S$ (N and S poles) is undefined on the globe.
Theorem 2 *If $h(A \to B) = h(A) \to h(B)$ for all $A, B \in S$ (i.e. the category* **Gsp***) then $h(A \oplus B) = h(A) \oplus h(B)$ when $A \oplus B$ is defined.*

9. Centroids

Defining $A \xrightarrow{n} B$ as on Slide 4, generalize midpoint $A_1 \oplus A_2$ to centroid $A_1 \oplus \ldots \oplus A_n$ as a partial n-ary operation via

$$A_1 \oplus \ldots \oplus A_n = B \text{ iff } (A_1 \oplus \ldots \oplus A_{n-1}) \xrightarrow{n} B \;=\; A_n, \quad n \geq 3$$

$$\text{Split as} \quad A_1 \oplus \ldots \oplus A_{n-1} \oplus ((A_1 \oplus \ldots \oplus A_{n-1}) \xrightarrow{n} B) = B$$

$$(A_1 \oplus \ldots \oplus A_{n-1}) \xrightarrow{n} (A_1 \oplus \ldots \oplus A_n) \;=\; A_n$$

Theorem 3 *For $n \geq 3$, $A_1 \oplus \ldots \oplus A_n$ is defined iff $A_1 \oplus \ldots \oplus A_{n-1}$ is defined and $|\{B \mid (A_1 \oplus \ldots \oplus A_{n-1}) \xrightarrow{n} B = A_n\}| = 1$.*

Theorem 4 *The subvariety (!) of* **Gsp** *consisting of the flat centroidal ($\oplus$ total) spaces is equivalent to the category* $\mathbf{Aff}_{\mathbb{Q}}$ *of affine spaces over the rationals.* (*pace* Löwenheim-Skolem)

Extends to $\mathbf{Vct}_{\mathbb{Q}}$ by adjoining a constant **O** as the origin. Further expansions in the same vein permit $\mathbb{Q}$ to be extended to $\mathbb{Q}[\mathbf{i}]$ (complex rationals), $\mathbb{R}$, and $\mathbb{C}$ (complex numbers).

10. Geodesic spaces and non-Euclidean algebra

Weaken Postulate 5 to right distributivity,

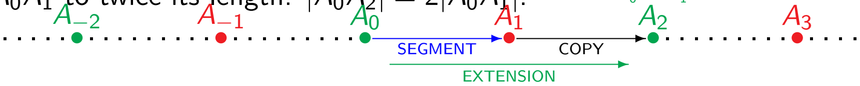
$$abc = ac(bc).$$

Thinking of ba, a, b, ab, etc. as points evenly spaced along a geodesic γ, right distributivity expresses a symmetry of γ about an arbitrary point c, namely that the inversion γc in $c = \ldots, bac, ac, bc, abc, \ldots$ is itself a geodesic, namely $\ldots, bc(ac), ac, bc, ac(bc), \ldots$.
These algebras have sometimes been identified with quandles as used to algebraicize knot theory. This is wrong because the quandle operations interpreted in **Grp** are $b^{-1}ab$ and bab^{-1}, which collapse in **Ab** to $ab = a$, whereas the above is $ba^{-1}b$ which is very useful in **Ab**.

A geodesic space or **geode** is an algebraic structure with a binary operation $x \to y$, or xy, of **extension** (with xyz for $(xy)z$) satisfying

G0 $xx = x$ **G1** $xyy = x$ **G2** $xyz = xz(yz)$

Geometrically, segment A_0A_1 is *extended to* $A_2 = A_0 \to A_1$ by producing A_0A_1 to twice its length: $|A_0A_2| = 2|A_0A_1|$.



Symmetric spaces: Affine, hyperbolic, elliptic, etc.
Groups: Interpret $x \to y$ as $yx^{-1}y$ (abelian groups: $2y - x$)
Number systems: Integers, rationals, reals, complex numbers, etc.
Combinatorial structures: sets, dice, etc.

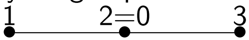
13. The category **Gsp**

A **discrete geodesic** $\gamma(A_0, A_1)$ is a subspace generated by A_0, A_1.
A **geodesic** in S is a directed union of discrete geodesics in S.

Examples: $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{E}$ (§11). Not $\mathbb{R}$ (not fully represented).

Geodesics properly generalize cyclic groups.

Example: $\mathbb{E} = \mathbb{Z}_4/\{0 = 2\}$.



S is **torsion-free** when every finite geodesic in S is a point.

The **connected components** of $\gamma(A_0, A_1)$ are $\ldots, A_{-2}, A_0, A_2, \ldots$ and $\ldots A_{-1}, A_1, A_3, \ldots$. These become one component just when $A_0 = A_{2n+1}$ for some n, as with $\mathbb{Z}_3$, $\mathbb{Z}_5$, etc.

Geode homomorphism: a map $h : S \to T$ s.t. $h(xy) = h(x)h(y)$.
Denote by **Gsp** the category of geodes and their homomorphisms.

14. Sets

Theorem 5. *For any space S, the following are equivalent.*

(i) $\gamma(A,B) = \{A,B\}$ *for all* $A, B \in S$ (cf. $\gamma(N,S)$, N&S poles).

(ii) *The connected components of S are its points.*

(iii) $xy = x$ *for all* $x, y \in S$.

A **set** is a geode S with any (hence all) of those properties.

Define $U_{\mathbf{SetGsp}} : \mathbf{Set} \to \mathbf{Gsp}$ as $U_{\mathbf{SetGsp}}(X) = (X, \pi_1^2)$, i.e. $xy \stackrel{\text{def}}{=} x$.
Left adjoint $F_{\mathbf{GspSet}}(S)$ = the set of connected components of S.
Cf. $\mathcal{D} : \mathbf{Set} \to \mathbf{Top}$ where $\mathcal{D}(X) = (X, 2^X)$, a discrete space.
These embed **Set** fully in **Top** (**Pos**, **Grph**, **Cat**, etc.) and **Gsp**.

In **Top** etc. the embedding $\mathcal{D}$ preserves colimits.

In **Gsp** the (reflective) embedding $U_{\mathbf{SetGsp}}$ preserves limits!

In **Set**, $1 + 1 = 2$ and $2^{\aleph_0} = \beth_1$ (discrete continuum).
In **Top**, $1 + 1 = 2$ but $2^{\aleph_0}$ = Cantor space, not discrete.
In **Gsp**, $2^{\aleph_0} = \beth_1$, discrete (!), but $1 + 1 = \mathbb{Z}$, a homogeneous (no origin) geodesic with two connected components.

15. Normal form terms and free spaces

A **normal form** geodesic algebra term over a set X of variables is one with no parentheses or stuttering, namely a finite nonempty word $x_1 x_2 \ldots x_n$ over alphabet X with no consecutive repetitions.

Theorem 6. All terms are reducible to normal form using G0-G2. (G2 removes parentheses while G1 and G0 remove repetitions.)

Theorem 7. The normal form terms over X form a geode.

Denote this space by $F(X)$, the **free space** on X consisting of the "X-ary" operations. $F(\{\}) = \mathbf{0}$ (initial), $F(\{0\}) = \mathbf{1}$ (final).

$F(\{0,1\}) = \mathbf{1} + \mathbf{1}$ has two connected components $\mathbf{0}\alpha$ and $\mathbf{1}\alpha$.

It is an infinite discrete geodesic $\gamma(0,1) = \{0 \xrightarrow{n} 1\} =$

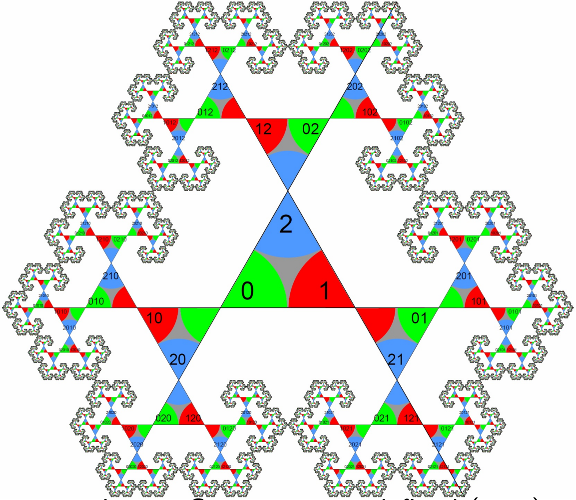
$$\mathbb{Z} \quad = \quad \ldots, 1010, 010, 10, 0, 1, 01, 101, 0101, \ldots$$

Call this *geodesimal notation*, tally notation with sign and parity bits.

Geodesimal operations: $x \xrightarrow{3} y = yxy$, $x \xrightarrow{-3} y = yxyx$, etc.

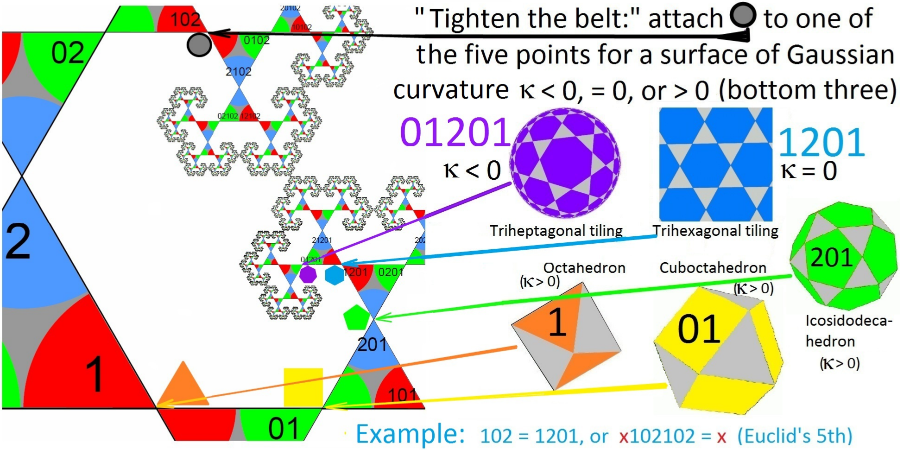
16. The free space 1+1+1.

3 connected components $0\alpha, 1\alpha, 2\alpha$



All points out to ∞ shown. Curvature κ undefined $(-\infty)$.

Triangles congruent by defn. but $\angle$, $\angle$, and $\angle$ incomparable.



All spaces (including $1+1+1$ itself) homogeneous.

Not shown: Sets ($xy = x$, §3), Dice ($xyxy = x$, §11).

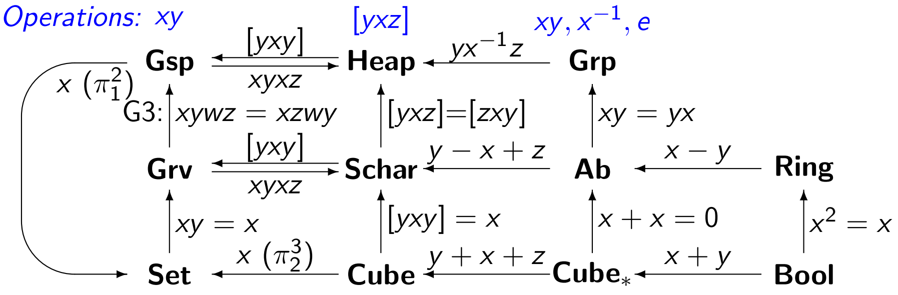
18. Dice and subdirect irreducibles of **Grv**

The **edge** $\mathbb{E} = \mathbb{E}_3 = \{1, 0 = 2, 3\}$ is the unique geodesic with an odd number of points and two connected components.

- $\mathbb{E}_3 = \mathbb{Z}_4/\{0 = 2\}$
- $\mathbb{E}_6 = \mathbb{Z}_8/\{0 = 4, 2 = 6\}$
- $\mathbb{E}_{12} = \mathbb{Z}_{16}/\{0 = 8, 2 = 10, 4 = 12, 6 = 14\}$, etc.

Ab and **Grv** have the same SI's (subdirect irreducibles), namely $\mathbb{Z}_{p^n}$, $n \leq \infty$, as groves, except for $p = 2$ when $\mathbb{Z}_{4.2^n}$ is replaced by $\mathbb{E}_{3.2^n}$ in **Grv**. ($\mathbb{Z}_{p^\infty}$ is the Prüfer p-group = the direct limit of the inclusion $\mathbb{Z}_{p^0} \subseteq \mathbb{Z}_{p^1} \subseteq \mathbb{Z}_{p^2} \subseteq \ldots$.) Key fact: $\mathbb{Z}_4$ is a subdirect product of $\mathbb{E}$'s.
$\mathbb{E} \in \mathcal{V}$ iff $\mathbb{Z}_4 \in \mathcal{V}$ for all varieties $\mathcal{V} \subseteq$ **Gsp**.
A **die** is a subspace of $\mathbb{E}^n$, $n \leq \infty$. Equivalently, a model of
$xx = xyy = x$, $xyxy{=}x$.
Dice $= HSP(\mathbb{Z}_4) = SP(\mathbb{E}) \subset$ **Grv**.

20. The geodesic neighborhood



Every path in this commutative diagram denotes a forgetful functor, hence one with a left adjoint. Vertical arrows *forget* the indicated *equation*, horizontal arrows *interpret* the blue *operation* above as the arrow's label. E.g. the left adjoint of the functor $U_{\mathbf{AbGrp}} : \mathbf{Ab} \rightarrow \mathbf{Grp}$ is abelianization, the arrow to **Schar** from **Ab** interprets **Schar**'s [yxz] as $y - x + z$ in **Ab**, the left adjoint of the functor $U_{\mathbf{SetGsp}} : \mathbf{Set} \rightarrow \mathbf{Gsp}$ gives the set $F_{\mathbf{GspSet}}(S)$ of connected components of S, and so on.