Chu realizes all small concrete categories

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January 9, 2005

Abstract

The category **Chu** is *concretely* universal for much of concrete mathematics; in particular it concretely represents or *realizes* all categories of relational structures and their homomorphisms, as well as all topological such. This note extends these results to all small concrete categories, equivalently all small subcategories of **Set**. The category C is realized in **Chu**(**Set**, K) where K is the disjoint union of the underlying sets of objects of C. Each object is realized as the normal Chu space (A, X) where X consists of all functions from A in C astricted to K.

1 Motivation

Call a category D *universal* when it fully embeds every small category C. V. Trnková proved the existence of universal categories [Trn66]. Hedrlín and Lambek then showed that the category **Sgrp** of semigroups was universal [HL69].

Pultr and Trnková [PT80] call such a full embedding $F : C \to D$ a *representation*. We understand each object c in C to be represented by the object $F(c)$ in D, and each morphism $f : c \to c'$ by the morphism $F(f) : F(c) \to F(c')$. A full embedding is a fully faithful functor, meaning that F associates the morphisms of the homset $C(c, c')$ bijectively to those of $D(F(c), F(c'))$. The importance of fullness is that the representing object $F(c)$ transforms in exactly the "same" ways as the represented object c.

So why have semigroups not replaced relational structures as the foundations of mathematics? While one can imagine various nontechnical reasons, a likely technical reason is that the elements of a structure are at least as important to us as its possible transformations, yet their is no universal way of deducing the former from the latter. Consider for example the subsemigroups of the semigroup of natural numbers under addition, which with their usual semigroup homomorphisms form a small category. Hedrlín and Lambek represent such semigroups using uncountable semigroups, even when the represented semigroups have only two or three elements.

Pultr and Trnková [PT80] address this concern with the notion of a *realization* as a concrete representation of a concrete category, namely a representation that commutes with the underlying-set functors. Reflection reveals the notion of realization to be literally more concrete than that of representation. If we think of a concrete category as one having sets for objects and functions for morphisms, then a realization realizes each set as itself and each function as itself.

Normally one represents something with something else, so our representation of sets and functions with themselves might seem redundant. However this notion of representation needs to be understood in context. First, fullness of the realization means that the homset of all functions between the representatives of two objects in the representing category is identically the same homset as that between the objects themselves in the represented category; thus the representation "knows" which functions are permitted (e.g. all and only the group homomorphisms from G to G'). Second, many categories $C, C', \ldots$ may fully embed in different regions of the same category D, allowing the latter to serve as a universal category with respect to its full subcategories.

The purpose of this note is to show that the category of Chu spaces realizes every small concrete category.

This result improves on the Hedrlín-Lambek representation in two ways. First, it is a realization, thereby addressing the objection raised above. From this we see that every small category of sets and functions standardly composed (i.e. every small but not necessarily full subcategory of **Set**) is a *full* subcategory of the concrete category **Chu**. Second, the embedding is much easier; the role of universal category simply comes more naturally to **Chu**, which is better equipped for the job.

Although Chu spaces are much less well known than semigroups, we regard the former as more important and fundamental. As a concrete, complete, and self-dual symmetric monoidal closed category, **Chu** is a natural model of linear logic. We first encountered Chu spaces ourselves while searching for a suitably general and natural model of true concurrency [GP93]. We have shown that **Chu** realizes many large concrete categories of mathematics, in particular that of all relational structures and their homomorphisms [Pra93, Pra95], which in turn realizes **Grp**, $\mathbf{Vct}_k$, and most other familiar concrete mathematical categories. It also realizes the usual categories that can be formed from these by adding topology, such as topological groups, topological Boolean algebras, topological vector spaces, etc.

2 Result

We first settle on terminology.

Following Pultr and Trnková [PT80], a *representation* of a category C by a category D is a fully faithful functor $F : C \to D$. A *realization* of a concrete category (C, U_C) by a concrete category (D, U_D) is a representation of C by D which commutes with the underlying-set functors.

A *normal Chu space* (A, X) *over* a set K consists of a set A, the underlying set or carrier, and a set[1] X of functions $g : A \to K$, the *states*. A Chu morphism between normal Chu spaces (A, X), (A', X') is a function $f : A \to A'$ such that for every $g : A' \to K$ in X', $gf \in X$. More details about Chu spaces may be found elsewhere [Pra95].

The essential idea of our representation is to represent the morphisms of C as their underlying functions with their codomains expanded or astricted to a single common codomain K. Composing with an inclusion on the right is called a *restriction*, on the left an *astriction*. That is, let $B \subseteq B'$. Composing $f : B' \to C$ with this inclusion restricts f to $f' : B \to C$ (the domain of f is restricted or shrunk to B). Composing this inclusion with $f : A \to B$ astricts f to $f' : A \to B'$ (the codomain of f is astricted or expanded to B').

Theorem 1 *For every small concrete category C there exists K such that* **Chu**(**Set**, K) *realizes C.*

Proof: Let $U_C : C \to$ **Set** be the underlying set functor of C. Take the set K to be the disjoint union of the underlying sets of the objects of C, a set because C is small. Associated to each object of C is an inclusion from that object's underlying set to K. Realize object c of C as the normal Chu space (A, X) where $A = U_C(c)$ and X consists of the astrictions to K of the underlying functions of the morphisms from c.

It suffices to show that for objects c, c' in C having associated underlying sets A, A', the set $C(c, c')$ of morphisms $f : c \to c'$ is mapped bijectively by U_C to the set of Chu transforms between the above realizations of c and c'. In the following we take (A, X) and (A', X') to be the above-defined realizations of c and c' respectively, and take $F = U_C(f) : A \to A'$ as the function realizing the morphism $f : c \to c'$.

Let $f : c \to c'$ be any morphism of C, realized as $F : A \to A'$. Every $G : A' \to K$ in X' is the astriction to K of some function A' to A'' realizing some morphism $g : c' \to c''$. Then the realization of $gf : c \to c''$, astricted to K, appears in X, whence $F : A \to A'$ is a Chu transform.

Now let $F : A \to A'$ be any Chu transform from (A, X) to (A', X'). The inclusion $i_{A'}$ of A' in K is in X', whence composing it with F must yield a function in X. But this composition is simply the astriction to K of $F : A \to A'$. To belong to X, F must realize some morphism from c to c'. ∎

References

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[1] Without "normal" this becomes a multiset. More precisely, X is taken to be any set whatsoever and the Chu space itself is then determined by a function from X to K^A, equivalently from $A \times X$ to K or from A to K^X.

November 1993.

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