$\mathsf{Chu}(\mathsf{Set}, K)$ without sets

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Abstract. We define the case $V = \mathsf{Set}$ of $\mathsf{Chu}(V, K)$ without reference to Set. Somewhat analogously to toposes and abelian categories as 1-categorical abstractions of Set and Ab, we propose rigid couple categories, RCOUP, as a uniform 2-categorical abstraction of the Chu_K categories. Theorem 1 obtains the Chu_K's as the 2-final couple categories of the connected components of RCOUP. Theorem 2 obtains RCOUP as closure of the discrete 2-category of Chu_κ's (κ the canonical K of that cardinality) under equivalences, subobjects, and 2-cell duplication.

Extended Abstract

1. Chu Categories

Chu space over a set K: A triple (A, r, X) with sets A and X and a function $r : A \times X \to K$.

Chu_K: Category of all Chu spaces over K with morphisms all pairs $(f{:}A \to B,\ g{:}Y \to X)$ from (A, r, X) to (B, s, Y) s.t. $s(f(a), y) = r(a, g(y))$, $a \in A, y \in Y$.

Goal: Define Chu_K directly without reference to sets.

2. Couple Categories

Couple category: Bipointed category $(C, \mathcal{G}, \mathcal{K})$.

Couple functor: Functor between couple categories preserving $\mathcal{G}$ and $\mathcal{K}$ and fully faithful on all $C(\mathcal{G}, \mathcal{A})$ and $C(\mathcal{A}, \mathcal{K})$ (i.e. $\forall a, a' : \mathcal{G} \to \mathcal{A}, \forall x, x' : \mathcal{A} \to \mathcal{K} \ldots$).

COUP: The 2-category of couple categories.

$\mathsf{COUP}_{\iota\kappa\lambda}$: $|C(\mathcal{G}, \mathcal{G})| = \iota$, $|C(\mathcal{G}, \mathcal{K})| = \kappa$, $|C(\mathcal{K}, \mathcal{K})| = \lambda$.

COUP is disconnected: $\mathsf{COUP} = \sum_{\iota\kappa\lambda} \mathsf{COUP}_{\iota\kappa\lambda}$, because couple functors preserve ι, κ, λ.

$Y_{\mathcal{GK}} \stackrel{\text{def}}{=} \lambda\mathcal{A}.(C(\mathcal{G}, \mathcal{A}) \times C^{\text{op}}(\mathcal{K}, \mathcal{A})) : C \to \mathsf{Set} \times \mathsf{Set}^{\text{op}}$.

Duplication, $\ker(Y_{\mathcal{GK}})$. Call $f, g : \mathcal{A} \to \mathcal{B}$ ***duplicates*** when $fa = ga$ for all $a : \mathcal{G} \to \mathcal{A}$ and $yf = yg$ for all $y : \mathcal{B} \to \mathcal{K}$. Extend to 2-cells of COUP.

Lemma 1 (F quasifaithful). *If $F(f) = F(g)$ for F a couple functor in* COUP*, then f, g are duplicates.*

Chu forgetful: $U_K : \mathsf{Chu}_K \to \mathsf{Set} \times \mathsf{Set}^{\text{op}}$, given by $U_K((A, r, X)) = (A, X)$, $U_K((f, g)) = (f, g)$. $\text{Im}(U_K)$ is a couple category, $\mathcal{G} = (1, K), \mathcal{K} = (K, 1)$.

RCOUP (Rigid COUP): $|C(\mathcal{G}, \mathcal{G})| = |C(\mathcal{K}, \mathcal{K})| = 1$.

(*Nonrigid example.* $(\mathbf{LCAb}, \mathbb{Z}, \odot)$: $\iota = \lambda = \omega$, $\kappa = 2^\omega$.)

RCOUP is disconnected: $\mathsf{RCOUP} = \sum_\kappa \mathsf{RCOUP}_\kappa$.

Lemma 2. *In* RCOUP*, $Y_{\mathcal{GK}}$ factors through U_K.*

Let $H_{\mathcal{GK}} : C \to \mathsf{Chu}_K$ solve $Y_{\mathcal{GK}} = U_K H_{\mathcal{GK}}$, one such for each of the $|K|!$ bijections between $C(\mathcal{G}, \mathcal{K})$ and K. Any choice of $H_{\mathcal{GK}}$ associates Chu spaces to couples and identifies all and only duplicates.

Theorem 1. *The categories Chu_K are the 2-final categories of the connected components of* RCOUP.

$\mathsf{CHU} \stackrel{\text{def}}{=} \sum_\kappa \{\mathsf{Chu}_\kappa\}$ over all cardinals κ (fixpoints $|\kappa| = \kappa$), viewing $\{\mathsf{Chu}_\kappa\}$ and hence CHU as discrete 2-categories of categories.

$ESD(\mathcal{C})$: Closure of 2-category $\mathcal{C}$ under equivalences, subobjects (back along monic couple functors), and 2-cell duplication.

Theorem 2. $\mathsf{RCOUP} = ESD(\mathsf{CHU})$.

3. Couples

Couple: an object of a couple category.

Discrete (codiscrete) couple: A couple $\mathcal{A}$ in C such that $H_{\mathcal{GK}} : C \to \mathsf{Chu}_{C(\mathcal{G},\mathcal{K})}$ is fully faithful on $C(\mathcal{A}, \mathcal{B})$ ($C(\mathcal{B}, \mathcal{A})$) for all couples $\mathcal{B}$ in C.

Point of view. Chu spaces are couples (a first order notion) that are both discrete and codiscrete (fullness here is second order if Chu_K is defined via 2-finality).

4. Individuals, Predicates, Values

Let $\mathcal{A}$ be a couple in C. Write $|\mathcal{A}|$ for its carrier $C(\mathcal{G}, \mathcal{A})$ and $\mathcal{K}^{\mathcal{A}}$ for its cocarrier $C(\mathcal{A}, \mathcal{K})$.

Individual of $\mathcal{A}$: $a : \mathcal{G} \to \mathcal{A}$, an element of $|\mathcal{A}|$.

Predicate on $\mathcal{A}$: $x : \mathcal{A} \to \mathcal{K}$, an element of $\mathcal{K}^{\mathcal{A}}$.

Value in C: $v : \mathcal{G} \to \mathcal{K}$, an element of $|\mathcal{K}| = \mathcal{K}^{\mathcal{G}}$.

Remarks. (i) Values are both individuals and predicates, a more uniform schizophrenia than the sporadically schizophrenic objects of Stone duality.

(ii) $Y_{\mathcal{GK}}$ faithful removes D from Theorem 2.

(iii) Chu_K is $*$-autonomous; is this so for $\iota = \lambda \neq 1$?

(iv) Does the V-enriched case of Theorem 1 yield $\mathsf{Chu}(V, k)$? (Take rigidity as $C(\mathcal{G}, \mathcal{G}) = C(\mathcal{K}, \mathcal{K}) = I$.)

(v) If functors are second order, connected components and 2-finality would be third order. If this is reduced to second order by abstracting functors to 1-cells, what do Chu spaces so defined abstract to?

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