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Comonoids in chu: a large cartesian closed sibling of topological spaces

Vaughan R. Pratt[1]

Computer Science Dept.
Stanford University
Stanford CA 94305, USA
http://boole.stanford.edu/pratt.html

Abstract

$\mathbf{Com}_K$ may be defined as the (cartesian closed) category of comonoids in $\mathbf{chu}_K$, or equivalently as dictionaries D for which any crossword over D has its main diagonal in D. $\mathbf{Com}_2$ resembles **Top**, ordinary topological spaces. Common to both are the Alexandroff posets and the Scott DCPOs, while the topological space $\mathbb{R}$ and the dual DCPO $\{-\infty < \ldots < -2 < -1 < 0\}$ jointly witness the incomparability of $\mathbf{Com}_2$ and **Top**. Such comonoids support a notion of bitopology admitting limits simultaneously for convergence and divergence. We raise the questions of whether a comonoid in $\mathbf{chu}_2$ can be fully specified in terms of its specialization order and omitted cuts, and which cuts are optional. These questions have been actively pursued for four weeks as of this writing on the theory-edge mailing list in response to Puzzle 1.5 starting with http://groups.yahoo.com/group/theory-edge/messages/6957.

1 Introduction

The comonoids of this paper can be described in different ways for different audiences. The sort of audience that likes to get directly to the definition should skip forthwith to the next section. Although the present audience as the attendees of a conference on coalgebraic methods can be assumed to be relatively sophisticated in the methods of category theory and comonoids, the notion of "ordinary comonoid" that we assume here is sufficiently elementary as to be accessible to a much wider mathematical audience. In the interests of conveying an intuitive feel not only for these comonoids but for coalgebraic methodology in general, we have taken the liberty of going into pedagogically more detail than customary for a research paper. Another factor disqualifying this as a straight research paper is that the propositions herein are all either known (some less well than others) or obvious: the genuine novelties are to be found not in the answers but the questions, which we hope the reader will find both challenging and interesting.

[1] Email: pratt@cs.stanford.edu

A fan of the New York Times daily crossword would understand a comonoid as a dictionary of equal-length words with the properties that any square crossword whose every horizontal and vertical word appears in the dictionary must turn out to have its main diagonal also in the dictionary, and every word of the requisite length having all letters the same must be in the dictionary.

A point-set topologist might prefer to see it as a variant on the notion of topological space. We leave untouched the definition of continuous function as one for which the inverse image of every open set is open. But instead of requiring the set of open sets to be closed under arbitrary union and finite intersection, they can be any selection of subsets that includes the empty set and the whole comonoid, and that makes continuity joint: if $f(a,b)$ is continuous *separately* in each of a and b then it is continuous *jointly* in a and b, meaning that $f(a,a)$ is continuous in a. The closed sets continue to be the complements of the open sets.

Unlike topological spaces, comonoids enjoy the same duality principle as posets, lattices, and categories. This is because the closed sets of a comonoid are the open sets of another comonoid on the same set of points.

The participants in this coalgebra workshop would recognize it most readily as a comonoid $(\mathcal{A},\delta,\varepsilon)$ in **chu**, the monoidal category of (bi)extensional Chu spaces [1,4,3,8], where $\mathcal{A}$ is such a Chu space and $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\varepsilon : A \to I$ are Chu morphisms satisfying the coassociativity and two counit equations. Compare this with the notion of "monoid in a monoidal category $(C, \otimes, I)$" as a triple $(\mathcal{A},\mu,\eta)$ where $\mathcal{A}$ is an object of C and $\mu : A \otimes A \to A$, $\eta : I \to A$ are morphisms of C satisfying the associativity and two unit equations, the principal difference being the reversal of the arrows.

Now the notion of "monoid in C" is customarily contracted to just "monoid" when C is **Set**. If we follow the same convention for comonoids however, we find that a comonoid is nothing more than an ordinary set obscurely described. More precisely, every set A admits a unique comonoid structure (A,δ,ε) where $\delta : A \to A \times A$ is the diagonal map $\delta(a) = (a,a)$ and $\varepsilon : A \to \{*\}$ is the constant map $\varepsilon(a) = *$, and every function between two sets is a homomorphism between the respective unique comonoid structures on those sets. That is, the category of comonoids in **Set** is equivalent (in fact isomorphic) to **Set** itself.

Since **chu** has an appealingly elementary definition, while the comonoids therein are more elementary yet in that they can be defined without reference to **chu**, we propose **chu** as a natural choice of C in "comonoid in C" as the meaning of "ordinary comonoid."

This notion finds various applications. In Girard's linear logic, such comonoids constitute the most general model of terms of the form $!A$ when A is modeled by Chu spaces.

In domain theory, they provide a particularly large cartesian closed category that fully embeds DCPOs [2], dual DCPOs, and even biDCPOs, as well as Alexandroff (i.e. maximally discrete) posets and other objects in between these extremes. In this role they provide an elementary and useful alternative to Scott's cartesian closed category EQU whose objects are T_0 spaces with total equivalence relations and whose morphisms are equivariant continuous maps. (Comonoids do not however form a topos, the finite comonoids being just the finite posets.)

In analysis they permit an extension of the notion of limit of a converging series to that of limit of a diverging series, in such a way that both types of limits can coexist in the one structure. (In this extension the latter kind would be the true limits and the former would more naturally then be called the colimits.) In the continuum the latter kind of limit could only be an open set, but one could imagine more general structures having more interesting limits of this kind, such as obfuscation strategies intended to hide what starts out in plain view.

In category theory, comonoids serve to make the point that, for a suitably closed category C, cartesian closedness as a global property of C can be expressed instead as a local property of individual objects of C, namely that each be a comonoid.

In fuzzy set theory, comonoids are just as much at home with fuzzy open sets as with "sharp" or two-valued-membership ones. The onset of fuzziness does not impair the cartesian closedness of a closed category of comonoids, whose structure is independent of the choice of alphabet.

The main focus of this paper will be on properties of comonoids, principally in the dyadic case.

We conclude the paper with the following open problems. Is every dyadic comonoid $\mathcal{A}$ just some weakening of the Alexandroff topology on the specialization order of $\mathcal{A}$? If so, which such weakenings are comonoids? If not, what applications exist for the counterexamples?

A test case for the first question is when the specialization order is discrete, i.e. the comonoid is T1 in the sense that no two points are comparable in terms of their containing sets of open sets. In particular, is every T1 comonoid discrete?

Comonoids are sufficiently accessible as to appeal to a wide mathematical audience, witness the considerable interest they have generated starting with `http://groups.yahoo.com/gr` on the theory-edge mailing list moderated by V.Z. Nuri.

2 Elementary Definitions

In this paper, "comonoid" will mean "comonoid in $\mathbf{chu_2}$." The methodical approach to defining "comonoid" would therefore be first to define the more general notion of comonoid in a monoidal category C, and then specialize C to the category $\mathbf{chu_2}$, that is, the category of dyadic biextensional Chu spaces, or Boolean matrices with no repeated rows or columns. Fortunately there is an entirely elementary definition having the further virtue of brevity. We postpone for the moment the connection with the official notion of comonoid in a monoidal category.

A ***comonoid*** $\mathcal{A} = (A, X)$ is a set A together with a set X of subsets of A with the following two properties.

(i) X contains A and the empty set.

(ii) Let C be any $A \times A$ matrix of 0's and 1's such that for all a in A, X contains both $\{b|C_{ab} = 1\}$ and $\{b|C_{ba} = 1\}$. Then X also contains $\{b|C_{bb} = 1\}$.

If we take A to be the set of positions for letters in a word, then X can be viewed as a dictionary of words over the alphabet $\{0, 1\}$, all of the same length, namely $|A|$. Any such word represents the subset of A consisting of those positions at which a 1 appears.

Condition (i) requires that both constant words, $00\dots0$ and $11\dots1$, appear in the dictionary.

Condition (ii) can be understood intuitively by regarding C as a filled-in square crossword with no black squares. The premise of the condition is that every row and every column of C must appear in the dictionary, the standard crossword condition. The condition itself then says that the main diagonal of any such crossword must also appear in the dictionary.

Comonoids closely resemble topological spaces. The latter is defined by replacing condition (ii) by the requirement that X be closed under arbitrary union and finite intersection. (Condition (i) is traditionally retained explicitly for topological spaces, presumably to avoid the distraction of defining empty union and intersection; for comonoids the corresponding alternative is even more distracting.) With this analogy in mind, call the elements of A points and the elements of X open sets.

A morphism $f : (A, X) \to (A', X')$ of comonoids is a function $f : A \to A'$ such that for all $Y \in X'$, the inverse image $f^{-1}(Y)$ is in X. This is exactly the definition of continuous function for topological spaces.

The ***specialization order*** of a structure (A, X) is the preordering $\leq$ of A defined by $a \leq b$ just when every open set containing a also contains b; equivalently, when $cl\{a\} \subseteq cl\{b\}$ where $cl\{a\}$ denotes the closure of (the least closed set containing) the singleton $\{a\}$.

The ***Alexandroff topology*** on a preordered set $(A, \leq)$ has for its open sets the order filters of the preordered set, equivalently the monotone functions from $(A, \leq)$ to the chain $0 < 1$. This topology is readily seen to be closed under arbitrary union and arbitrary intersection. Slightly less obviously, every topology closed under arbitrary intersection is the Alexandroff topology of the specialization order of that topology. The Alexandroff topology on $0 < 1$ itself is called the ***Sierpinski topology***.

Proposition 2.1 *The Alexandroff topology on a poset is a comonoid.*

Proof. For each point a let m^a denote the intersection of all open sets containing a, let $f : \mathcal{A}^2 \to \mathbf{2}$ be separately continuous in each argument, and let $B = \{a | f(a, a) = 1\}$. For joint continuity of f it suffices to show that B is open.

So let $C = \bigcup_{a \in B} m^a$, a union of intersections of open sets and hence itself open. Clearly $B \subseteq C$, so to show B is open it suffices to show $C \subseteq B$.

So suppose $c \in C$. Then for some $a \in B$, $c \in m^a$. Hence every open set containing a must also contain c. Hence for any continuous $g : \mathcal{A} \to \mathbf{2}$, $g(a) \leq g(c)$. So $f(a, a) \leq f(a, c) \leq f(c, c)$. Since $a \in B$, $f(a, a) = 1$, so $f(c, c) = 1$, whence $c \in B$. □

A ***directed set*** D of a poset $(A, \leq)$ is a nonempty subset $D \subseteq A$ such that every pair of elements of D has a common upper bound in D. We shall treat ***directed upwards*** as synonymous with directed, and define a ***downwards directed set*** to be a subset of $(A, \leq)$ which is directed (upwards) in the order dual $(A, \geq)$ of $(A, \leq)$ (the result of turning the latter upside down).

A ***directed-complete partial order*** (DCPO) is a poset whose every directed set has a supremum (least upper bound). A ***dual DCPO*** is a poset whose every downward directed set has an infimum (greatest lower bound). A ***biDCPO*** is a DCPO that is also a dual DCPO. It is immediate that the image of a directed set under a monotone function is directed, and likewise for downward directed sets.

A morphism $f : (A, \leq) \to (B, \leq)$ of DCPOs is a monotone function that preserves the supremum of every directed set. That is, if D is directed with supremum a, then its (necessarily directed) image $f(D)$ has supremum $f(a)$.

The ***Scott topology*** on a DCPO has for its open sets the DCPO morphisms to the two-element chain $\mathbf{2} = \{0 < 1\}$ (as a DCPO). Equivalently, every continuous $f : \mathcal{A} \to \mathbf{2}$ is monotonic, and if it vanishes everywhere on some directed set D then it continues to vanish at $\bigvee D$ (no unexpected jumps in the limit, necessarily upwards by monotonicity). The Scott and Alexandroff topologies coincide on finite posets.

Proposition 2.2 *A DCPO with the Scott topology is a comonoid.*

Proof. Let D be directed, let $f : \mathcal{A}^2 \to \mathbf{2}$ be a DCPO morphism in each of its arguments separately, and let $f(a, a) = 0$ for all $a \in D$. Given $a, b \in D$ there exists $c \in D$ with $a \leq c$ and $b \leq c$. But $f(c, c) = 0$ so $f(a, b) = 0$. Hence $f(\bigvee D, b) = 0$ (directed sup over a), whence $f(\bigvee D, \bigvee D) = 0$ (directed sup over b). $\square$

Proposition 2.3 *The continuum $\mathbb{R}$ as standardly topologized is not a comonoid.*

Proof. Let $f : \mathbb{R}^2 \to \mathbf{2}$ satisfy $f(x, x) = 0$ for $x \neq 0$, and let $f(x, y) = 1$ elsewhere. Then $f(x, y)$ is continuous separately in x and y since the empty space (when the fixed variable is zero) and singletons (otherwise) are closed in the topology of the continuum. However $f(x, x)$ cannot be continuous because singletons (in this case the origin) are not open. $\square$

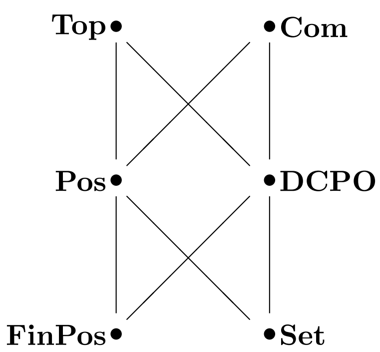
(This choice of f constitutes a crossword whose rows and columns are open sets but whose main diagonal is not. Viewed as the characteristic function of a subset of the plane, this subset is not open with respect to the usual product topology on the plane, but only because of its misbehavior along the diagonal $y = x$ in the neighborhood of the origin.)

By symmetry of the definition of comonoid, given a DCPO with the Scott topology, we may call its order dual a dual DCPO. This is again a comonoid, but it need not be a topological space. Morphisms of dual DCPOs preserve the downward directed infs, while morphisms of biDCPOs preserve both directions of these directed bounds.

Proposition 2.4 *The poset $\{-\infty < \ldots < -2 < -1 < 0\}$ with the Alexandroff topology less the cut between $-\infty$ and the integers is a comonoid but not a topological space.*

Proof. This structure can be obtained by taking the Scott topology on $\{0 < 1 < 2 < \ldots < \infty\}$ and replacing its open sets by its closed sets; equivalently, by complementing the bits of the words in X. This is a comonoid by symmetry of the definition of comonoid with respect to 0 and 1. However the union of the finite cuts (cuts between $-n - 1$ and $-n$) is the infinite cut separating $-\infty$ from the finite integers, which the Scott topology omits, contradicting the requirement that a topology be closed under arbitrary union. $\square$

The above propositions establish the following hierarchy.



The orderings in this Hasse diagram of categories denote full embeddings. The category **Pos** of posets certainly has the categories **FinPos** of finite posets and **Set** of all sets as full subcategories. Finite posets are DCPOs because a subset is directed just when it has a greatest element, which is therefore its sup. Sets are DCPOs because the only directed subsets are singletons. Posets are made topological spaces with the Alexandroff topology, while DCPOs are made topological spaces with the Scott topology. Posets and DCPOs are both comonoids as proved above.

The immediately preceding propositions establish the incomparability of **Top** and **Com**. The incomparability of **Pos** and **DCPO** is witnessed by the chain $\{0 < 1 < 2 < \ldots < \infty\}$: with the Alexandroff topology it is a poset and not a DCPO, while the Scott topology makes it a DCPO and not a poset. Lastly, **FinPos** and **Set** are obviously incomparable.

These incomparabilities immediately establish the strictness of all embeddings, in that no embedding in the diagram is between equivalent categories.

We close this section with a useful property of comonoids.

Proposition 2.5 *Comonoids are closed under finite union and intersection.*

Proof. Let $x, y \in X$ be two open sets of $\mathcal{A} = (A, X)$, and let z be the subset of $\mathcal{A} \times \mathcal{A}$ consisting of those (a, b) for which either $a \in x$ or $b \in y$. This is an open set of $\mathcal{A} \times \mathcal{A}$ because rows are open sets of X whether or not $b \in y$ (if not, the empty set is still an open set of $\mathcal{A}$), and dually for columns. The diagonal being $x \cup y$, it follows that $x \cup y$ must be an open set of $\mathcal{A}$. The same argument with conjunction in place of disjunction shows that $x \cap y$ must also be an open set. The case of empty union and intersection are covered explicitly by the definition of comonoid. $\Box$

3 Comonoids in C

Up to this point we have defined an elementary notion of ordinary comonoid, in much the same elementary style as one would define an ordinary monoid, using only sets, functions, and products in the definition itself without recourse to categories. The difference however is that whereas ordinary monoids are, categorically speaking, monoids in the category **Set**, what we are here calling ordinary comonoids are comonoids in the category **chu** of biextensional Chu spaces and their continuous functions.

Most working mathematicians today understand algebras concretely: an algebra is a set A and a family of operations on A of various arities. Any teacher of algebra introducing their class to the notion of a monoid (A, μ, η) would be considered derelict in their pedagogical duty if the first few examples were not all taken from the category **Set** of sets and their functions. A monoid in **Set** consists of a set A and operations μ, η of respective arities 2 and 0, for which $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ (associativity), $\mu(\eta(), a) = a$ ($\eta()$ is a left identity), and $\mu(a, \eta()) = a$ (right identity).

This state of bliss also suffices for some useful coalgebras, such as the final coalgebra of sort $A \to N \times A$ where A is a set constituting the final coalgebra in question and $N = \{0, 1, 2, \ldots\}$ is the set of natural numbers, shown by Pavlović and the present author in the 1999 incarnation of CMCS to form a coinductive basis for the continuum [7].

However monoids are too broadly applicable to belong exclusively to **Set**. A ring is a monoid in the monoidal category **Ab** of abelian groups, a monad or triple on C is a monoid in the monoidal category C^C of endofunctors on C (taking the tensor product as composition of functors), and so on.

A monoid (A, μ, η) in a monoidal category $(C, \otimes, I)$ [2] consists of an object A and morphisms $\mu : A \otimes A \to A$, $\eta : I \to A$, such that the following diagrams commute.

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "A \otimes \mu"] \arrow[d, "\mu \otimes A"] & A \otimes A \arrow[d, "\mu"'] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{r} I \otimes A \\ = A \\ = A \otimes I \end{array} \arrow[r, shift left=2, "\eta \otimes A"] \arrow[r, shift right=2, "A \otimes \eta"'] & A \otimes A \arrow[l, "\mu" description]
\end{tikzcd}$$

Can comonoids be developed pedagogically in the same way as for monoids?

Now a comonoid in $(C, \otimes, I)$ is simply a monoid in $(C^{\mathrm{op}}, \otimes, I)$. The only impact of replacing C by C^{op} is to reverse the arrows in the sorts of μ and η; the monoidal structure itself remains unchanged. After the reversal we rename μ to δ and η to ε.

With these changes, the defining monoid equations of associativity and the unit laws become the defining comonoid equations of coassociativity and the counit laws. These are expressed in the first instance as the following commuting diagrams.

[2] Nothing herein conflicts with taking the customary natural transformations $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda : I \otimes A \to A$, and $\rho : A \otimes I \to A$ normally associated with a monoidal category to be "on the nose," namely the identifications $A \otimes (B \otimes C) = A \otimes B \otimes C = (A \otimes B) \otimes C$ and $I \otimes A = A = A \otimes I$, allowing them to be suppressed.

$$
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "A \otimes \delta"'] \\
A \otimes A \arrow[u, "\delta \otimes A"'] & A \arrow[l, "\delta"'] \arrow[u, "\delta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{array}{c} I \otimes A \\ = A \\ = A \otimes I \end{array} \arrow[r, "\delta" description] & A \otimes A \arrow[l, shift right=3, "\varepsilon \otimes A"'] \arrow[l, shift left=3, "A \otimes \varepsilon"]
\end{tikzcd}
$$

Now, can this dual notion of a comonoid (A, δ, ε) be explained, or at least illustrated initially, in terms of comonoids in **Set**?

In Set, A is a set, $A \otimes A$ is cartesian product $A \times A$, and I is the singleton $\{0\}$. In this category we can write $(a0, a1)$ for $\delta(a)$, $(a10, a11)$ for $\delta(a1)$, and so on. Coassociativity can then be defined as the equation $(a00, a01, a1) = (a0, a10, a11)$, that is, $a00 = a0$, $a01 = a10$, and $a1 = a11$. And the two equations that go with the counit are $a = a0$, $a = a1$. These two force δ to be the diagonal function $\delta(a) = (a, a)$, and make coassociativity redundant ($a01 = a0 = a = a1 = a10$, and $a00 = a0$ and $a1 = a11$ are even easier).

So every set is a comonoid in a unique way, and these are the only comonoids.

In Set, a morphism of comonoids $f : (A, \delta, \varepsilon) \to (A', \delta', \varepsilon')$ is a function $f : A \to A'$ such that $\delta'(f(a)) = (f(a0), f(a1))$ and $\epsilon'(f(a)) = 0$. The former reduces to $(f(a), f(a)) = (f(a), f(a))$ and the latter to $0 = 0$, both vacuous. So every function is a comonoid morphism in a unique way, and these are the only comonoid morphisms.

It follows that no insight can be had into what makes comonoids different from sets if one starts with examples of comonoids in **Set**.

Starting instead with comonoids in **chu** has the benefit that the notion has a simple elementary definition in terms just of sets and functions independent of **chu** itself. Further as we have seen the the previous section, even comonoids over **2** have the richness of topological spaces, as witnessed by their position at the top right of the Hasse diagram, making **Com** sibling to **Top**.

Let us now reconcile the elementary **chu**-independent definition of an "ordinary" comonoid with the formal notion of a comonoid in C for the case $C = \mathbf{chu}$.

The notion of a biextensional Chu space can be understood by analogy to that of T_0 topological space, namely as a set A together with a set X of subsets of A, which we may continue to call the open sets of A. Furthermore the morphisms are defined as though they were continuous functions: a function from (A, X) to (B, Y) is a Chu morphism just when the inverse image of each element of Y under f^{-1} is an element of X.

Tensor product $\mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A} = (A, X)$ is the biextensional collapse of $(A \times A, F)$ where F is the set of all crosswords on $A \times A$ such that the rows and columns are all drawn from X. The biextensional collapse of a Chu space is obtained by identifying equal rows, and identifying equal columns. The tensor unit I (or 1 in the notation of linear logic) is the discrete singleton, that is, the Chu space with one point and two open sets, namely the empty set and the whole (singleton) space. We treat $((a, b), c)$, $(a, (b, c))$, and (a, b, c) as identical, corresponding to associativity being on the nose, and $(a, 0)$ and $(0, a)$ as identical when 0 is the unique element of the tensor unit I.[3]

We can now describe $\varepsilon : \mathcal{A} \to I$ and $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

Now the inverse image of the empty set must be empty, while the inverse image of the whole space must be the whole space. Since these are the only open sets in I, it follows that $\mathcal{A}$ must have the empty set and the whole space (the set A) among its open sets.

The inverse image δ^{-1} must map any given open set y of $\mathcal{A} \otimes \mathcal{A}$ to an open set of $\mathcal{A}$. But this amounts to collecting the set of those a for which $f(a, a)$ is in y. Since y is a crossword having rows and columns drawn from the open sets of $\mathcal{A}$, by the definition of tensor product $\mathcal{A} \otimes \mathcal{A}$, this amounts to requiring that the diagonal of this crossword, as a function from A to 2, be an open set of $\mathcal{A}$.

So continuity of ε and δ are equivalent to the elementary conditions on a comonoid that we started out with.

4 Casuistries

Nearly a decade ago Francois Lamarche [5] developed a notion of ***casuistry*** that specializes the above development in a way that makes the associated comonoids topological spaces, while retaining the feature of biextensional Chu spaces that they are closed under matrix transposition.

A ***casuistry*** is a biextensional Chu space whose rows and columns are closed under directed unions. (We are here regarding each row of (A, X) as a subset of X, and each column as a subset of A.)

(A set B of rows of a Chu space $\mathcal{A}$ is called ***directed*** when it has the property that for any rows a, b in B, B also contains a row c such that $a \leq c$ and $b \leq c$. This holds vacuously if $a \leq b$ or $b \leq a$, so it only has any force when a and b are incomparable. Directed union is the union of a directed set, analogously to finite union being the union of a finite set.)

Write **Cas** for the category of casuistries and their continuous functions. Write **Lam** (for Lamarche) for the category of comonoids in **Cas**.

An elementary fact about directed sups is that a poset closed under finite sups and directed sups is also closed under arbitrary sups. This is because an arbitrary subset Y and the closure Z of that set under finite sups have the same set of upper bounds. But Z is directed and so has a least upper bound $\bigvee Z$, whence this is also the least upper bound on Y.

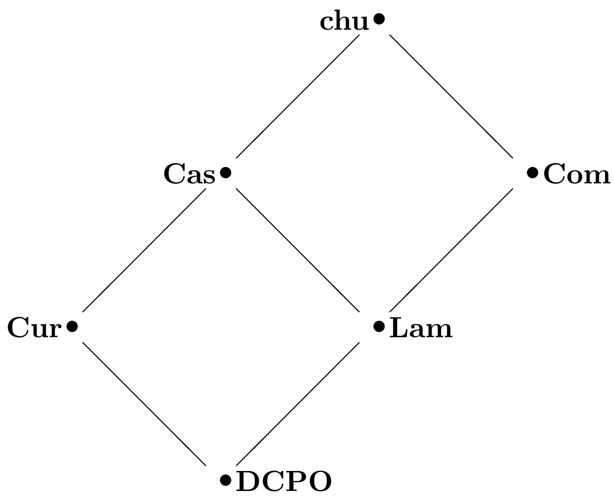
Now the open sets of comonoids in **Cas** are closed under finite union and finite intersection, as a special case of comonoids in **chu**. But they are also closed under directed sups, being casuistries, whence they are closed under arbitrary sups. But this makes comonoids in **Cas** topological spaces!

Lamarche [5] credits P.-L. Curien with a further shrinking of **Cas** via the requirement that the columns (open sets) form a subbasis for the Scott topology on

[3] This can be arranged if the carrier or set of points of a Chu space is taken to be an ordinal, the functions between carriers ignores that order, cartesian product is defined as ordinal or lexicographic product $\kappa \cdot \kappa'$, and symmetry of product is the evident isomorphism (not an identity in general, witness $\omega \cdot 2 \neq 2 \cdot \omega$) pairing up (a, b) with (b, a), again ignoring the ordinal order. Treating all sets as ordinals amounts to accepting the Axiom of Choice, the alternative being the gory details of natural transformations $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda : I \otimes A \to A$, and $\rho : A \otimes I \to A$ and their coherence conditions.

the specialization order of the given Chu space. If we call the resulting category **Cur**, then the comonoids in **Cur** turn out to be exactly the DCPOs (with the Scott topology, but this is what we mean by a DCPO when represented as a Chu space, as distinct from a poset which is furnished with the Alexandroff topology).

We then obtain the following embeddings.



The two steps along the top left edge constitute the embeddings defined by the successive restrictions to casuistries and Curien's objects. The three steps down and to the right represent the restriction to comonoids. Since **Cur**, **Cas**, and **chu** are symmetric monoidal closed categories, all with monoidal and faithful forgetful functors to **Set**, their comonoid counterparts below, namely **DCPO**, **Lam**, and **Com**, all form cartesian closed categories (CCCs).

As observed above, the objects of **Lam** are topological spaces, unlike those of **Com**. It follows that **DCPO** as a full subcategory of **Lam** consists of topological spaces. However we already know what **DCPO** is since it has an independent and historically much earlier definition as the CCC of directed-complete partial orders with the Scott topology and their continuous functions.

Lamarche's paper treats the portion of this Hasse diagram below **Cas**. focusing mainly on **Cas** and **Lam**. The two main theorems of the paper seem to be that **DCPO** $\subseteq$ **Lam**, and that the objects of **Lam** are topological spaces. Combining these into one thought, in the world of those CCCs that consist of certain topological spaces and their continuous functions, **Lam** is a proper generalization of **DCPO**.

Lamarche also proves that every T_1 casuistry comonoid is discrete.

The present paper adds **chu** and **Com** to Lamarche's picture. Like **Lam**, **Com** is a CCC. Unlike it however it contains some objects that are not topological spaces, such as the dual DCPO $-\infty < \ldots < -2 < -1 < 0$.

5 Open Problems

The most interesting aspect to us of comonoids in **chu** is the question of whether they are merely a blend of posets and biDCPOs (with DCPOs and dual DCPOs being considered as intermediate cases of these two extremes) or include other structures.

One intriguing possibility is that Lamarche's result, that every T_1 casuistry comonoid is discrete, might not hold for all chu comonoids. This question is Puzzle 1.5 at `http://thue.stanford.edu/puzzle.html`. To generate additional interest in the problem, a small cash prize is offered (small by the standards of the prizes being offered for such questions as P=NP and the Riemann hypothesis).

This property is at least arithmetic, in the sense that it holds for countable comonoids. The situation becomes murky at uncountable comonoids, see `http://groups.yahoo.com/g` and following messages.

More generally, is every dyadic comonoid some weakening of the Alexandroff topology on the specialization order of $\mathcal{A}$? If so, which such weakenings are comonoids? If not, what applications exist for the counterexamples?

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