Broadening the denotational semantics of linear logic

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Abstract

The proof-theoretic origins and specialized models of linear logic make it primarily operational in orientation. In contrast first-order logic treats the operational and denotational aspects of general mathematics quite evenhandedly. Here we show that linear logic has models of even broader denotational scope than those of first order logic, namely Chu spaces, the category of which Barr has observed to form a model of linear logic. We have previously argued that every category of n-ary relational structures embeds fully and concretely in the category of Chu spaces over 2^n. The main contributions of this paper are improvements to that argument, and an embedding of every small category in the category of Chu spaces via a symmetric variant of the Yoneda embedding.

1 Introduction

Linear logic makes good sense operationally. It is a substructural logic resembling relevance logic in lacking weakening, from $\vdash B$ infer $A \vdash B$, and differing from it in also lacking contraction, from $A, A \vdash B$ infer $A \vdash B$. These inference rules perform for logic the functions that reflexivity and transitivity perform for binary relations, which for example turn the eminently decidable single-step relation for Turing machine configurations into an undecidable reachability relation. Their omission from linear logic forces proofs into an unconvoluted normal form by preventing short proofs whose brevity is achieved by clever tanglings. No theorems need be lost in this way since Gentzen's cut elimination procedure can straighten out every forbidden tangled argument into an acceptably smooth one.

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Linear logic's denotational semantics is considerably more problematic. Girard has considered phase semantics and coherent spaces [Gir87], Hilbert spaces, and most recently Banach spaces [this proceedings]. Phase semantics resembles Birkhoff and von Neumann's quantum logic [BvN36], while the other three are instances of the Curry-Howard isomorphism [How80] whose principal *raison d'être* would appear to be the good match of their respective closed monoidal structures to the rules of linear logic. Blass [Bla92] and Abramsky and Jagadeesan [AJ94] have interpreted linear logic over sequential games, while Blute [Blu96] has taken Hopf algebras as a model of noncommutative linear logic.

The problem with all these denotational semantics is that they constitute relatively specialized corners of mathematics all lacking the sort of generality associated with the denotational semantics of classical first order logic. Without that generality, linear logic cannot convincingly be argued to be *about* mathematical objects in general, only about those objects that conform to the laws of linear logic. With so thin a connection between proof theory and Platonic mathematics, the foundational significance of linear logic would appear to be largely operational.

We pose the question, is this necessary? In particular *can linear logic treat the operational and denotational aspects of general mathematics as even-handedly as does first-order logic?*

Barr [Bar91] and Lafont and Streicher [LS91] have proposed yet another model of linear logic, namely Chu spaces, or games as Lafont and Streicher call them. Now the strongest claim made in those papers as to the generality of Chu spaces is that of Lafont and Streicher, who observe that the category $\mathbf{Vct}_k$ of vector spaces over the field k embeds fully in $\mathbf{Chu}(\mathbf{Set}, |k|)$ and those of coherent spaces and topological spaces embed fully in $\mathbf{Chu}(\mathbf{Set}, 2)$. This is far from the sort generality we are looking for.

But in fact $\mathbf{Chu}(\mathbf{Set}, K)$ *is* sufficiently general. In previous papers [Pra93, Pra95] we have demonstrated this by showing that the concrete category of k-ary relational structures embeds fully and concretely in the category of Chu spaces over 2^k. In this paper we repeat that argument from a fresh perspective, and then give a quite different Yoneda-like full embedding.

The fresh perspective is that of group theory, where we recast the usual algebraic notion of a group as an equivalent purely topological notion with the help of "fuzzy" open sets having eight degrees of membership. This amounts to our previous embedding [Pra93] disguised as topology and specialized to groups, which is enough to demonstrate the principle. An immediate application is to obtain concrete duals of (nonabelian) groups. Although the generalization to higher arity operations is obvious, the generalization from operations to relations is slightly less obvious and we supply an argument.

We then show that the category of Chu spaces over K fully embeds every category C having up to K arrows. The underlying principle of this embedding is that of the Yoneda embedding. However the target of our embedding depends only the cardinality of C, unlike the Yoneda embedding whose target depends nontrivially on the structure of C. Furthermore our embedding represents objects and morphisms simply as binary relations and pairs of functions respectively, in contrast to the functors and natural transformations used in the Yoneda embedding. And whereas there are two Yoneda embeddings, we have only one, as a sort of symmetric blend of the Yoneda embeddings.

The foundational significance of these embeddings is that each supports the thesis that $\mathbf{Chu}(\mathbf{Set}, -)$ as the K-indexed family $\mathbf{Chu}(\mathbf{Set}, K)$ of categories of Chu spaces over K can be taken as a universal category for mathematics. Other categories such as that of directed graphs, and of semigroups, are universal in the sense that they fully embed all known categories. The additional significance of Chu spaces is that the embeddings are *concrete*: viewing objects as sets with structure specified in one way or another, and morphisms as certain functions between those sets, the Chu space representation of concrete categories leaves both the objects and the morphisms unchanged. All that changes is the representation of the structure associated to those sets. If we think of the choice of representation of that structure as a mere implementation detail, then Chu spaces constitute a uniform representation of structure for *all* mathematical objects that permits them to inhabit the one universal category.

The connection with linear logic is that the structure of this universal category is very close to that of linear logic. It shares this property with the categories of Hilbert spaces, coherent spaces, etc. The difference is that whereas each of those categories represents a small fragment of mathematics, $\mathbf{Chu}(\mathbf{Set}, -)$ represents all of mathematics in much the same way as do relational structures as the models of first order logic. Unlike relational structures however Chu spaces can also represent topological spaces and more generally relational structures equipped with a topology.

2 The Dual of a Group

If we consider only finite structures, then sets are dual to Boolean algebras, posets are dual to distributive lattices, semilattices are dual to (other) semilattices, and some structures are self-dual such as free semilattices, chains with bottom, abelian groups, and vector spaces (finite in dimension). For infinite structures, Hilbert spaces and complete semilattices miraculously remain self-dual, but in general a dose of topology needs to be administered to one side or the other of the duality if it is to survive the passage to infinity. Thus Boolean algebras are dual to Stone spaces defined as totally disconnected compact topological spaces, locally compact abelian groups to other locally compact abelian groups, and so on.

But what is the dual of a nonabelian group?

A naive answer is that it is an object of $\mathbf{Grp}^\circ$, the category obtained simply by reversing the arrows of the category $\mathbf{Grp}$ of groups and their homomorphisms, those functions $f : A \to A'$ satisfying $f(ab) = f(a)f(b)$. This answer has two drawbacks. First, it trivializes the notion of "dual" by making every group its own dual, and moreover in a way that we could have applied to any of the structures above that came by their duality more honestly. Second, the reversed arrows do not have any obvious presentation as functions transforming a concrete structure, calling for a religious conversion to abstract category theory in order to accept this answer.

Although topology standardly understood is not powerful enough to produce directly the dual of a nonabelian group, it can if we bend the rules a little. We shall equip an arbitrary group A with a "quasitopology"[1] that exactly expresses the group structure of A, and from this obtain a concrete representation of the dual of A.

Definition 1 A *complex* of a group A is any subset of A. (So every subgroup of A is a complex of A but not vice versa.) A *tricomplex* of A is a triple[2] $x = (x_1, x_2, x_3)$ of complexes $x_i \subseteq A$. We shall call such a tricomplex *open* when for every $a \in x_1$ and $b \in x_2$, $ab \in x_3$, where ab is the group multiplication. ∎

An intuitive connection with topology can be made here if we think of each open tricomplex or *otc* of A as a sort of neighborhood of the group operation. Neighborhoods can be broadened by reducing the first two complexes and increasing the third, operations that preserve openness. The "tight" neighborhoods are those for which every element of the third complex is the product of elements from the first two.

Now A is determined by its otc's, indeed by just those consisting of singletons, since the group operation can be recovered from those otc's of the form $(\{a\}, \{b\}, \{c\})$ (with c necessarily being ab) simply by erasing the set braces.

But there is more to this quasitopology than just representing individual groups in isolation. The following definition and proposition show that we have captured the group structure in essentially topological terms.

Definition 2 A function $f : A \to A'$ is *otc-continuous* when for every otc (x_1, x_2, x_3) of A', $(f^{-1}(x_1), f^{-1}(x_2), f^{-1}(x_3))$ is an otc of A. ∎

Proposition 1 *f is otc-continuous if and only if f is a group homomorphism.*

Proof: *Only if.* Let $a, b \in A$. Then $(\{f(a)\}, \{f(b)\}, \{f(a)f(b)\})$ is an otc of A'. By continuity, $(f^{-1}\{f(a)\}, f^{-1}\{f(b)\}, f^{-1}\{f(a)f(b)\})$ is an otc of A. But $a \in f^{-1}\{f(a)\}$ and $b \in f^{-1}\{f(b)\}$, whence $ab \in f^{-1}\{f(a)f(b)\})$, that is, $f(ab) = f(a)f(b)$.

If. Let (x_1, x_2, x_3) be an otc of A'. We wish to show that $(f^{-1}(x_1), f^{-1}(x_2), f^{-1}(x_3))$ is an otc of A. Let $a \in f^{-1}(x_1)$ and $b \in f^{-1}(x_2)$. Hence $f(a) \in x_1$ and $f(b) \in x_2$, whence $f(a)f(b) \in x_3$. But $f(a)f(b) = f(ab)$, so $ab \in f^{-1}(x_3)$. ∎

This result made no use of the group axioms, a matter we will take up shortly.

Define a *concrete dual group* to be the set of open tricomplexes of some group A, which will be the group to which this dual group is dual. We denote this dual

[1] As in *quasiwabbit*.

[2] In using x rather than X we depart from the usual convention of capital letters for sets in anticipation of the schizophrenic passage from $x \subseteq A$ to $a \subseteq X$.

of A by $A^\perp$. Given two concrete dual groups $A^\perp$ and $A'^\perp$, define a dual-group transformation between them to be a function $f : A^\perp \to A'^\perp$ such that f is the inverse image (applied coordinatewise to the otc's as in the definition of otc-continuity) of some group homomorphism from A' to A.

The category of concrete dual groups and their transformations form a concrete category isomorphic to $\mathbf{Grp}^\circ$, the category of groups with their morphisms (namely group homomorphisms) reversed, by Proposition 1.

A brute force way of passing to the corresponding abstract notion is to define a dual group, or *puorg*, to be any set standing in a specified bijection with a concrete dual group, with the evident notion of dual-group transformation via their respective concrete dual groups, accessed via the respective bijections. The category of dual groups and their transformations is equivalent (but not isomorphic) to that of concrete dual groups, and hence equivalent to $\mathbf{Grp}^\circ$, i.e. dual to $\mathbf{Grp}$.

Abstraction by fiat is of course not in the spirit of abstract algebra, and we may ask whether this notion of dual group has a more traditional abstract definition. A more important question for this paper is how all this ties in with the Chu construction and in particular our claimed universality of it. Fortunately these questions are closely linked and we can answer them together.

Let $x = (x_1, x_2, x_3)$ range over the otc's of the group A, and let X denote the set of all otc's of A. The membership relation between group elements $a \in A$ and otc's $x \in X$ is given by the truth values of $a \in x_1$, $a \in x_2$, and $a \in x_3$, which collectively have $2^3 = 8$ possible outcomes. Let us view these outcomes as the eight possible values of membership of a in x, making x an "8-fuzzy" set. We can therefore think of the group as a pair of sets A, X together with an 8-valued binary relation between them, defined as a function $A \times X \to 8$. Generalizing 8 to K then leads to the following notion of a Chu space.

Definition 3 A *Chu space* $(A, \models\!\!\!\!\dashv, X)$ *over* a set K consists of a set A of *points*, a set X of *states*, and a function $\models\!\!\!\!\dashv : A \times X \to K$. ∎

We write $\models\!\!\!\!\dashv(a, x)$ as either $a \models\!\!\!\!\dashv x$ or $x \models a$. The latter is intended to suggest x as a model or interpretation, a as a proposition, and $x \models a$ as the truth value of proposition a in state x, where K is the set of possible truth values.

Next we generalize the notion of otc-continuous function to the notion of Chu transform, as follows. Begin with the observation that, if we identify the subsets of a set A with their characteristic functions $x : A \to 2$, then the inverse image function $f^{-1} : 2^{A'} \to 2^A$ can be defined simply as $f^{-1}(x) = x \circ f$, that is, $f^{-1}(x)(a) = x(f(a))$. Passing to tricomplexes changes this to $f^{-1}(x_1, x_2, x_3)(a) = (x_1(f(a)), x_2(f(a)), x_3(f(a)))$, but then writing x for (x_1, x_2, x_3) restores the equation to $f^{-1}(x)(a) = x(f(a))$, except that now each side ranges over eight values instead of two. Writing g for the restriction of f^{-1} to the set X' of open tricomplexes, switching the two sides around, and using the $\models\!\!\!\!\dashv$ notation for 8-valued membership, turns the equation into $f(a) \models\!\!\!\!\dashv' x \;=\; a \models\!\!\!\!\dashv g(x)$.

Definition 4 Given two Chu spaces $(A, \models\!\!\!\!\dashv, X)$, $(A', \models\!\!\!\!\dashv', X')$, a *Chu transform* between them is a pair (f, g) of functions $f : A \to A'$, $g : X' \to X$ such that for all $a \in A$ and $x \in X'$,

$$f(a) \dashv' x \;=\; a \dashv g(x).$$

We refer to this condition on (f, g) as the *adjointness condition.* ∎

Chu transforms compose according to $(f', g')(f, g) = (f'f, gg')$, easily seen to be a Chu transform itself, with the evident identities and associativity property. Hence Chu spaces over K and their Chu transforms form a category, denoted **Chu**(**Set**, K).

Chu(**Set**, K) is evidently self-dual. The dual of $(A, \dashv, X)$ is the Chu space $(X, \models, A)$, and the corresponding dual of any Chu transform (f, g) is simply (g, f). That is, if we regard $(A, \dashv, X)$ as an $A \times X$ matrix then duality reduces to mere matrix transposition.

This completely symmetric view of Chu spaces in terms of two sets A and X connected by a K-valued binary relation $\dashv$ indicates how to define puorgs more naturally. We had taken A as the carrier and X as a certain set of 8-fuzzy subsets of A. But by symmetry we could just as well have started with X and taken A to consist of 8-fuzzy subsets of X, or equivalently triples of ordinary subsets of X. To translate the specification of an A-based object to that of an X-based one, reinterpret every formula $a \in x_i$ as $x \in a_i$ leaving its truth value unchanged. Just as we viewed otc's $x = (x_1, x_2, x_3)$ as the "open sets" of a group, so may we view the elements $a = (a_1, a_2, a_3)$ of A as the "open sets" of a puorg.

The condition we shall impose on A is that it form a group. This might seem to bring us right back where we started, except that now A is not an arbitrary set but a subset of 8^X. Ordinary topological spaces supply a precedent for this: the open sets are required to form a concrete frame. Now a *frame* is a distributive lattice having all joins including infinite joins and the empty join or bottom, with meets distributing over all joins. A *concrete* frame is a frame whose elements are sets and whose join and meet operations are realized as union and intersection. To complete this analogy for concrete groups as the "open sets" of a puorg, we need to realize the binary operation of a group as some binary operation on a set of 8-fuzzy sets.

Now when we were constructing a concrete dual group starting from the group A, the defining property of an otc (x_1, x_2, x_3) was that, for all $a_1a_2 = a_3$, $a_1 \in x_1$ and $a_2 \in x_2$ implies $a_3 \in x_3$. The transpose of this condition is, $x \in a_{11}$ and $x \in a_{22}$ implies $x \in a_{33}$, which is to say, $a_1a_2 = a_3$ implies $a_{11} \cap a_{22} \subseteq a_{33}$. Moreover the converse holds because if $a_3 \neq a_1a_2$ then $a_{11} \cap a_{22}$ contains the otc $(\{a_1\}, \{a_2\}, \{a_1a_2\})$ but a_{33} does not.

This suggests that, in our new definition of an abstract dual group or puorg starting from a set X, with A consisting of triples of subsets of X, we take the inclusion $a_{11} \cap a_{22} \subseteq a_{33}$ to define a ternary relation on A. The analogy with topological spaces then leads us to the definition of a puorg as a set X with a set A of triples of subsets of X such that this ternary relation on A is a binary operation making A a group. A puorg morphism $(X, A), (X', A')$ is a function $f : X \to X'$ such that for every triple $a \in A'$, $f^{-1}(a) = (f^{-1}(a_1), f^{-1}(a_2), f^{-1}(a_3))$ is a triple in A.

Now consider the group A of triples of subsets of X defined in this way. We might expect to recover X up to isomorphism as the otc's of A. And indeed every element $x \in X$ must arise as an otc of A, namely the otc (x_1, x_2, x_3) such that $a \in x_i$ just when $x \in a_i$, as seen from the symmetrical view of X and A. For let $a_1 a_2 = a_3$ in the group A, and let $x = (x_1, x_2, x_3)$ be a tricomplex of A such that $a_1 \in x_1$ and $a_2 \in x_2$. Transposing, $x \in a_{11}$ and $x \in a_{22}$, whence $x \in a_{33}$ (being part of requirement for $a_1 a_2 = a_3$). That is, $a_3 \in x_3$, making x an open tricomplex of A.

There is however no reason why the converse should hold, that is, X may lack otc's of A. We may well have an object that transforms dually to A, but if it lacks even one otc of A, it cannot be isomorphic to the concrete dual group $A^\perp$ formed as the otc's of A. This is because dual groups are constructed to transform concretely, i.e. via functions, but isomorphisms in any concrete category must be bijections.

This incompleteness in our definition of dual group can be rectified with one more condition: that X be *saturated* in the sense that every otc x of the group A of opens correspond suitably to an element $x' \in X$.

Definition 5 A *puorg* (X, A) is a set X and a set A of triples of subsets of X, such that (i) the ternary relation on A consisting of those triples $((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33}))$ satisfying $a_{11} \cap a_{22} \subseteq a_{33}$ is the binary operation of a group, and (ii) for every otc (x_1, x_2, x_3) of that group A there exists $x \in X$ such that for all $a = (a_1, a_2, a_3) \in A$, $a \in x_i$ if and only if $x \in a_i$, $i = 1, 2, 3$. ∎

Puorg morphisms can now be defined in exactly the same way as for otc-continuity, thanks to Chu duality.

Definition 6 A function $f : X \to X'$ between two puorgs (X, A), (X', A') is a puorg homomorphism just when for every $a = (a_1, a_2, a_3) \in A'$, $f^{-1}(a) = (f^{-1}(a_1), f^{-1}(a_2), f^{-1}(a_3)) \in A$. ∎

Saturation is expensive: a quick computer check revealed the puorgs dual to the permutation groups S_2 and S_3 to have respectively 41 and 20750 elements. A systematic way of selecting "enough" otc's would yield a more succinct representation, but this takes us too far afield.

3 Generalizing to n-ary Relations

Nowhere did the previous section make any use of the group axioms. We could just as well have been constructing a dual monoid, or for that matter the dual of any set with a binary operation. Moreover the arity played no essential role, and we could have been treating n-ary operations for any n, even infinite or zero.

In fact the method generalizes to n-ary relational structures (A, R), $R \subseteq A^n$, which we have proved elsewhere [Pra93, Pra95], but this generalization is slightly less obvious. We give the proof here in a form that makes clear the connection with the (somewhat smoother) version of the proof for algebras.

Now a tricomplex $x = (x_1, x_2, x_3)$ of a group A can be defined as open just when, for all $a_1, a_2, a_3 \in A$ such that $a_1 a_2 = a_3$, either $a_1 \notin x_1$ or $a_2 \notin x_2$ or $a_3 \in x_3$. To make this more symmetric, uniformly replace the first and second components of every tricomplex by its complement, so that the openness condition becomes, there exists i such that $a_i \in x_i$. This change does not require any modification to the definition of otc-continuous because $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ (indeed f^{-1} commutes with all finitary and even infinitary Boolean operations).

Definition 7 The n-ary relational structure (A, R) has for its set X of open n-tuples of subsets of A those $x = (x_1, \ldots, x_n)$ such that for all $(a_1, \ldots, a_n) \in R$, there exists i such that $a_i \in x_i$. ∎

Definition 8 A function $f : (A, X) \to (A', X')$ is *quasicontinuous* when for every
$x = (x_1, \ldots, x_n) \in X'$, $f^{-1}(x) = (f^{-1}(x_1), \ldots, f^{-1}(x_n)) \in X$. ∎

A homomorphism of relational structures (A, R), (A', R') is a function $f : A \to A'$ such that for all $(a_1, \ldots, a_n) \in R$, $(f(a_1), \ldots, f(a_n)) \in R'$. We denote by $\mathbf{Str}_n$ the category of all n-ary relational structures and their homomorphisms. **Grp** for example is a full subcategory of $\mathbf{Str}_3$.

Proposition 2 *$f : (A, X) \to (A, X')$ is quasicontinuous if and only if f is a homomorphism of the corresponding n-ary relational structures (A, R), (A, R').*

Proof: *Only if.* Suppose f is not a homomorphism, i.e. there exists $(a_1, \ldots, a_n) \in R$ but $(f(a_1), \ldots, f(a_n)) \notin R'$. Then $(\overline{f(a_1)}, \ldots, \overline{f(a_n)}) \in X'$ where $\overline{a}$ denotes the cosingleton $A' - \{a\}$. But $a_i \notin f^{-1}(\overline{f(a_i)})$ for any i whence $(f^{-1}(\overline{f(a_i)}), \ldots, f^{-1}(\overline{f(a_i)})) \notin X$. Hence f is not quasicontinuous.

If. Let $(x_1, \ldots, x_n)$ be in X', and let $(a_1, \ldots, a_n) \in R$. Then $(f(a_1), \ldots, f(a_n)) \in R'$. Hence there exists i such that $f(a_i) \in x_i$, that is, $a_i \in f^{-1}(a_i)$. Since this holds for every element of R, it follows that $(f^{-1}(x_1), \ldots, f^{-1}(x_n))$ is in X. ∎

(The passage from operations to relations seemed to complicate the only-if direction, for which it seemed best to argue the contrapositive.)

This theorem can be stated in more categorical language as a full embedding $F : \mathbf{Str}_n \to \mathbf{Chu}(\mathbf{Set}, 2^n)$. But unlike many other such full embeddings of "all algebraic categories" in a universal category, this embedding is concrete in the sense that the representing object has the same carrier A as that of the represented object. That is, F commutes with the respective forgetful functors to **Set**. This makes Chu spaces a more useful representation because one can continue to reason about objects in terms of their ordinary elements.

A natural generalization of this embedding is to topological relational structures (A, R, O), where $R \subseteq A^n$ and $O \subseteq 2^A$ is a set of subsets of A constituting the open sets of a topology on A. (R itself may or may not be continuous with respect to O in some sense, but this is independent of the embedding proved here.)

Such a structure has a straightforward representation as a Chu space over 2^{n+1}, as follows. Take $X = X' \times O$ where $X' \subseteq (2^A)^n$ is the quasitopology on A determined by R as in the previous section. Hence $X \subseteq (2^A)^{n+1}$. The quasicontinuous functions will then respect both the relational structure and the topological structure, in the sense that they will be precisely the functions that are both homomorphisms with respect to the relational structure and continuous functions with respect to the topological structure. For example the category of topological groups embeds fully and concretely in $\mathbf{Chu}(\mathbf{Set}, 16)$.

This is an instance of a more general technique for combining two structures on a given set A. Let $(A, \models_1, X_1)$ and $(A, \models_2, X_2)$ be Chu spaces over K_1, K_2 respectively, having A in common. Then $(A, \models, X_1 \times X_2)$ is a Chu space over the product $K_1 \times K_2$, where $a \models (x_1, x_2) = (a \models_1 x_1, a \models_2 x_2)$. If $(A', \models', X'_1 \times X'_2)$ is formed similarly from $(A', \models'_1, X'_1)$ over K_1 and $(A', \models'_2, X'_2)$ over K_2, then it is easily seen that $f : A \to A'$ is (the first coordinate of) a Chu transform from $(A, \models, X_1 \times X_2)$ to $(A', \models', X'_1 \times X'_2)$ if and only if it is Chu transform from $(A, \models_1, X_1)$ to $(A', \models'_1, X'_1)$ and also a Chu transform from $(A, \models_2, X_2)$ to $(A', \models'_2, X'_2)$. For if (f, g_1) and (f, g_2) are the latter two Chu transforms, with $g_1 : X'_1 \to X_1$ and $g_2 : X'_2 \to X_2$, then the requisite $g : X'_1 \times X'_2 \to X_1 \times X_2$ is simply $g(x_1, x_2) = (g_1(x_1), g_2(x_2))$.

4 Symmetrizing the Yoneda embedding

We turn from the problem of embedding standard large categories in $\mathbf{Chu}(\mathbf{Set}, K)$ to that of embedding arbitrary small categories. The former was a concrete embedding, preserving the underlying sets. Although the objects of an arbitrary category don't in general have an underlying set, we may interpret the arrows to an object as its elements.

If very small K were the goal, we could achieve $K = 4$ by using the category $\mathbf{Str}_2$ of all binary relational structures (A, R), $R \subseteq A^2$, and their homomorphisms, and composing the above embedding of $\mathbf{Str}_2$ in $\mathbf{Chu}(\mathbf{Set}, 4)$ with Kuĉera and Hedrlín's embedding of an arbitrary small category in $\mathbf{Str}_2$ [Hed71]. But the latter embedding involves an intricate and somewhat arbitrary combinatorial representation that undermines the foundational relevance of this representation.

Instead we shall give a direct embedding involving no combinatorics, in return for giving up small K. This embedding will be seen to be a symmetric variant of the Yoneda embedding, with Chu spaces and their transforms in place of functors and their natural transformations.

There are actually two Yoneda embeddings, obtained by transposing the homfunctor $\mathrm{Hom} : C^\circ \times C \to \mathbf{Set}$ either as $Y : C \to \mathbf{Set}^{C^\circ}$ or $Y' : C \to (\mathbf{Set}^C)^\circ$ (aka $C^\circ \to \mathbf{Set}^C$), called respectively the covariant and contravariant Yoneda embedding. Each is in fact a (covariant) full embedding of C in a functor category, $(\mathbf{Set}^C)^\circ$ being isomorphic to the functor category $(\mathbf{Set}^\circ)^{C^\circ}$. The embedding represents objects as functors and morphisms as natural transformations. For the covariant embedding, the representing functors as functors to $\mathbf{Set}$ are *presheaves.*

Both $\mathbf{Set}^{C^\circ}$ and $(\mathbf{Set}^C)^\circ$ incorporate the structure of C in a nontrivial way. One might reasonably presume that some such dependence on C would be an inevitable feature of the target given that C is a completely arbitrary category. But in fact our embedding has almost no such dependence: only the cardinality of C plays any role in determining the target, which will be the category $\mathbf{Chu}(\mathbf{Set}, ar(C))$ of Chu spaces over the set of arrows of C, And even that dependence can be eliminated if the target is instead taken to be $\mathbf{Chu}(\mathbf{Set}, -)$ defined as consisting of Chu spaces $(A, =\!|, X, K)$ each furnished with its own K, with Chu transforms between $(A, =\!|, X, K)$ and $(A', =\!|', X', K')$ defined by regarding both as Chu spaces over $K \cup K'$. This category has a tensor product but no tensor unit.

Our embedding represents each object b of C as the Chu space $(A, ;, X)$ where $A = \{f : a \to b \mid a \in ob(C)\}$, $X = \{h : b \to c \mid c \in ob(C)\}$, and $f; g = g \circ f$, the converse of composition. That is, the points of this space are all arrows into b, its states are all arrows out of b, and the matrix entries $f; h$ are all composites $a \xrightarrow{f} b \xrightarrow{h} c$ of arrows in with arrows out.

We represent each morphism $g : b \to b'$ of C as the pair (φ, ψ) of functions $\varphi : A \to A'$, $\psi : X' \to X$ defined by $\varphi(f) = f; g$, $\psi(h) = g; h$. This is a Chu transform because the adjointness condition $\varphi(f); h = f; \psi(h)$ for all $f \in A$, $h \in X'$ has $f; g; h$ on both sides. In fact the condition expresses associativity.

If the set of arrows to b is understood as forming the carrier of b then the domain of φ is simply that carrier. Under that interpretation this representation is concrete (faithfulness will be seen momentarily).

At this point we have constructed a functor $F : C \to \mathbf{Chu}(\mathbf{Set}, ar(C))$.

Proposition 3 *F is full and faithful.*

Proof: For faithfulness, consider $g, g' : b \to b'$. Let $F(g) = (\varphi, \psi)$, $F(g') = (\varphi', \psi')$. If $F(g) = F(g')$ then $g = 1_b; g = \varphi(1_b) = \varphi'(1_b) = 1_b; g' = g'$.

For fullness, let (φ, ψ) be any Chu transform from $F(b)$ to $F(b')$. We claim that (φ, ψ) is the image under F of $\varphi(1_b)$. For let $F(\varphi(1_b)) = (\varphi', \psi')$. Then $\varphi'(f) = f; \varphi(1_b) = f; \varphi(1_b); 1_{b'} = f; 1_b; \psi(1_{b'}) = f; \psi(1_{b'}) = \varphi(f)$, whence $\varphi' = \varphi$. Dually $\psi' = \psi$. ∎

The adjointness condition can be more succinctly expressed as the dinaturality in b of composition $m_{abc} : C(a,b) \times C(b,c) \to C(a,c)$. The absence of b from $C(a,c)$ collapses the three nodes of the right half of the dinaturality hexagon to one, shrinking it to the square

$$\begin{array}{ccc} C(a,b) \times C(b',c) & \xrightarrow{1\times\psi_g} & C(a,b) \times C(b,c) \\ \varphi_g \times 1 \big\downarrow & & \big\downarrow m_{abc} \\ C(a,b') \times C(b',c) & \xrightarrow{m_{ab'c}} & C(a,c) \end{array}$$

Here $1 \times \psi_g$ abbreviates $C(a,b) \times C(g,c)$ and $\varphi_g \times 1$ abbreviates $C(a,g) \times C(b',c)$. Commutativity of the square asserts $\varphi_g(f); h = f; \psi_g(h)$ for all f :

$a \to b$ and $h : b' \to c$. By letting a and c range over all objects of C we extend this equation to the full force of the adjointness condition for the Chu transform representing g.

Had we not been talking about Chu transforms, we would have interpreted the dinaturality in b of composition m_{abc} as merely expressing associativity. But the calculation of associativity from the diagram essentially passes through the adjointness condition, making that connection the prior one.

Comparing this embedding with the covariant Yoneda embedding of C in $\mathbf{Set}^{C^\circ}$, we observe that the latter realizes φ_g directly while deferring ψ_g via the machinery of natural transformations. The contravariant embedding is just the dual of this, realizing ψ_g directly and defers φ_g. Our embedding in Chu avoids functor categories altogether by realizing both simultaneously.

5 Conclusion

We have exhibited embeddings in **Chu** of two quite different notions of "general" category. One is that of relational structures and their homomorphisms, possibly with topological structure and the requirement that the homomorphisms be continuous. The other embedding mirrors the Yoneda embedding in some key details yet is more elementary (if one accepts that Chu transforms are more elementary than natural transformations), and its target is independent of any property of the embedded category except possibly its cardinality if we use $\mathbf{Chu}(\mathbf{Set}, ar(C))$ instead of $\mathbf{Chu}(\mathbf{Set}, -)$.

Both embeddings are concrete in a reasonable sense. The first is concrete in the ordinary sense of the representing object $(A, \models, X)$ having as its underlying set A the carrier of the represented relational structure. The second is concrete with respect to arrows-to as elements.

Quite a few categories are known that are universal to the extent of fully embedding all small categories, as well as all algebraic categories. However those embeddings are highly artificial, relying on the ability of such objects as graphs and semigroups to code the compositional structure of morphisms that compose at an object to be so represented. Any representation based on clever coding introduces irrelevant complexity into the mathematics of objects so represented. Furthermore the coding obscures the ordinary elements of concrete objects, further undermining our intuitions about concrete objects.

These embeddings provide a sense in which the denotational semantics of linear logic can be understood to be at least as general as that of first-order logic. This is not to say that the generality is achieved at the same level. A model of first order logic is a relational structure, and the models of a given theory form a category. A model of linear logic on the other hand is the category itself, whose objects are the denotations of mere formulas.

This is the basic difference between first order or elementary logic and linear logic. First order logic reasons about the interior of a single object, the domain of discourse being the elements or individuals that exist in that object together with the relationships that hold between them. Linear logic reasons instead about how things appear on the outside, understanding the structure of objects externally in terms of how they interact rather than internally in terms of what they might contain. The fundamental interaction is taken to be that of transformation of one object into another. Elements and their relationships are not discussed explicitly, but their existence and nature is inferred from how the objects containing them interact.

This being the essence of the categorical way of doing mathematics, linear logic so construed must therefore be the categorical logic of general mathematics. As such it is sibling to intuitionistic categorical logic, whose domain of discourse is confined to cartesian closed mathematics, having as its exemplar category **Set**. The thesis we have defended here is that the examplar category of general mathematics is **Chu**(**Set**, $-$).

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