Algebraically compact functors

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Abstract

In a previous paper, we investigated the relation between the initial algebra and terminal coalgebra for an endofunctor on the category of sets. In this one we study conditions on a functor to be algebraically compact, which means that the canonical comparison morphism between those objects is an isomorphism.

Introduction

Suppose $\mathcal{C}$ is a category and $T: \mathcal{C} \longrightarrow \mathcal{C}$ is a functor. In both [Barr, 1991] and [Freyd, 1991] it is shown that there is a canonical arrow between the initial T-algebra and terminal T-coalgebra and both papers study its properties in some special cases. Freyd has introduced the term **algebraically compact** to describe a category for which that arrow is always an isomorphism. He does not actually exhibit any non-trivial examples of such categories, although he claims that the realizable topos has a "small full reflective subcategory that is algebraically compact in the relevant sense, that is, the condition holds for every endofunctor that is definable as a functor in the topos." This suggests that it might be worth restricting attention to functors that are "relevant". For example, when dealing with categories enriched over some base category, it may be relevant to restrict to functors that preserve that enriched structure.

For these and other reasons, we define a *functor* to be **algebraically compact** if the canonical map is an isomorphism. Freyd also defined a category to be **algebraically complete** if every functor has an initial algebra. Clearly an algebraically compact category is also algebraically complete. However we wish to explore a condition closely related to algebraic compactness that is meaningful even in the non-algebraically-complete case.

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If T is an endofunctor, let us say that a fixed object for T is an object C with an isomorphism $TC \longrightarrow C$. This is a special kind of T algebra and, using the inverse isomorphism, it is also a special kind of T-coalgebra. An initial algebra, if one exists, is a fixed object and the initial fixed object and a terminal coalgebra, if one exists, is a fixed object and the terminal fixed object. One of the main interests is in the category of fixed objects. A functor need not have any fixed object. For example the covariant power set functor on the category of sets does not have any.

In general, not very many categories are algebraically compact. However, it may happen that every functor in some usefully large class of functors is algebraically compact. For example, the homsets might be ordered and we may restrict to functors that preserve the order. In that case, we say that that class of functors is **algebraically compact**. Finally, we define a class of functors to be **conditionally algebraically compact** if every functor in the class that has a fixed object is algebraically compact.

For various reasons, it appears that the category of **CPO**s (defined below) would prove to be a good source of examples. In fact, it is there that many of the models of invariant objects are found. And indeed we find a class of functors both on the that category and on the category of CPOs with bottom which are algebraically compact (Theorems 4.6 and 4.8).

A CPO is a partially ordered set in which every directed set of elements has a sup. It is equivalent (using the axiom of choice) to suppose that every ordinal indexed increasing chain of elements has a sup.

Among the motivations for Freyd's paper was the feeling (which I shared) that there was something *ad hoc* about the embedding/projection pairs that have been used to find invariant objects for functors that were contravariant or of mixed variance. (See [Smith & Plotkin, 1983] or [Barr & Wells, 1990] for an explanation.) It was thus of considerable surprise to me to find embedding/projection sequences arising naturally in this investigation. In retrospect, it perhaps should not have been so surprising. Among the results found in Freyd's paper are that invariant objects for covariant, contravariant and mixed variance functors are found under the same conditions. Originally, embedding/projection sequences were introduced to make variance irrelevant. In Freyd's treatment, all functors are converted to covariant endofunctors on an appropriate category. The price to be paid is that now one needs not just an initial fixed point, but a simultaneously initial and terminal fixed point. The search for this turns out to lead quite naturally to embedding/projection sequences. It may well be that these sequences are inevitable in this connection, rather than just being a feature of one way of looking at it.

1 Initial algebras and terminal coalgebras

In this section we construct a sequence that will usually lead to an initial algebra for an endofunctor, if it has one. It appears to be essentially the same as that given in Dubuc [1972]. We include it because it requires less machinery.

Let $\mathcal{C}$ be a category and suppose that T is an endofunctor. Suppose that $\mathcal{C}$ has an initial object and has colimits along ordinal indexed diagrams. Under those conditions we can construct what we call the **initial sequence** of T.

We begin with $A_0 = 0$, $A_1 = T0$ and $f_1^0 \colon A_0 \longrightarrow A_1$ the unique arrow. Let α be an ordinal and assume that we have defined objects A_β for all ordinals $\beta < \alpha$, and arrows $f_\beta^\gamma \colon A_\gamma \longrightarrow A_\beta$ for all $\gamma \leq \beta < \alpha$ such that the following hold for all $\delta \leq \gamma \leq \beta < \alpha$:

IS–1. $A_{\beta+1} = TA_\beta$.

IS–2. $f_{\beta+1}^{\gamma+1} = Tf_\beta^\gamma$.

IS–3. $f_\beta^\beta = \mathrm{id}$.

IS–4. $f_\beta^\gamma \circ f_\gamma^\delta = f_\beta^\delta$ for $\delta \leq \gamma \leq \beta$.

IS–5. If β is a limit ordinal, the cocone $\{f_\beta^\gamma \colon A_\gamma \longrightarrow A_\beta\}$ is a colimit.

We now have to construct the object A_α and the morphisms $f_\alpha^\beta \colon A_\beta \longrightarrow A_\alpha$ for $\beta \leq \alpha$ so that IS–1 to IS–5 hold for $\delta \leq \gamma \leq \beta \leq \alpha$. We consider three cases.

α is a limit ordinal: In this case we let $\{f_\alpha^\beta \colon A_\beta \longrightarrow A_\alpha\}$ be a colimit and $f_\alpha^\alpha = \mathrm{id}$. IS–1 and IS–2 are inapplicable for $\beta = \alpha$ and so remain valid, IS–3 and IS–4 hold by definition and IS–5 obviously continues to hold.

$\alpha = \beta + 1$ and β is a limit ordinal: Let $A_\alpha = TA_\beta$. Since A_β is the colimit of the A_γ for $\gamma < \beta$ we define f_α^β by giving the composites $f_\alpha^\beta \circ f_\beta^\gamma$ for $\gamma < \beta$ and proving them compatible. We define $f_\alpha^\beta \circ f_\beta^\gamma = Tf_\beta^\gamma \circ f_{\gamma+1}^\gamma$, which makes sense since $A_{\gamma+1} = TA_\gamma$. The compatibility follows from

$$Tf_\beta^\gamma \circ f_{\gamma+1}^\gamma \circ f_\gamma^\delta = Tf_\beta^\gamma \circ f_{\gamma+1}^\delta = Tf_\beta^\gamma \circ f_{\gamma+1}^{\delta+1} \circ f_{\delta+1}^\delta = Tf_\beta^\gamma \circ Tf_\gamma^\delta \circ f_{\delta+1}^\delta = Tf_\beta^\delta \circ f_{\delta+1}^\delta$$

Finally, we let f_α^α and $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$ for $\gamma < \beta$. IS–1 is obvious. IS–2 follows from

$$f_{\beta+1}^{\gamma+1} = f_\alpha^{\gamma+1} = Tf_\beta^{\gamma+1} \circ f_{\gamma+2}^{\gamma+1} = Tf_\beta^{\gamma+1} \circ Tf_{\gamma+1}^\gamma = Tf_\beta^\gamma$$

IS–3 and IS–4 hold by definition and IS–5 continues to be valid since this construction is not at a limit ordinal.

$\alpha = \beta + 2$ for some β: In this case, we let $f_\alpha^\alpha = \mathrm{id}$ and $f_\alpha^{\beta+1} = Tf_{\beta+1}^\alpha$. For $\gamma < \beta+1$, we let $f_\alpha^\gamma = f_\alpha^{\beta+1} \circ f_{\beta+1}^\gamma$. IS–1 through IS–4 are true by definition and no new case of IS–4 is created.

Thus we can continue to build this sequence through all ordinals. If it should ever happen that $f^{\alpha}_{\alpha+1}$ is an isomorphism, it is clear that we will have f^{α}_{β} is an isomorphism for all $\beta > \alpha$. In this case, we will say that the sequence **terminates** at α. In fact, it is sufficient that any f^{β}_{α} with $\beta < \alpha$ be an isomorphism.

1.1 Proposition. *Suppose $\beta < \alpha$ are such that f^{β}_{α} is an isomorphism. Then $f^{\beta}_{\beta+1}$ is an isomorphism.*

Proof. If f^{β}_{α} is an isomorphism, so is

$$Tf^{\beta}_{\alpha} = f^{\beta+1}_{\alpha+1} = f^{\alpha}_{\alpha+1} \circ f^{\beta+1}_{\alpha}$$

from which it follows that $f^{\beta+1}_{\alpha}$ is a split monomorphism. On the other hand, from $f^{\beta}_{\alpha} = f^{\beta+1}_{\alpha} \circ f^{\beta}_{\beta+1}$ being an isomorphism, it follows that $f^{\beta+1}_{\alpha}$ is a split epimorphism and hence an isomorphism. But then $f^{\beta}_{\beta+1}$ is also an isomorphism. □

1.2 Theorem. *Suppose the initial sequence terminates at α. Then $(A_{\alpha}, (f^{\alpha}_{\alpha+1})^{-1})$ is an initial T-algebra.*

Proof. Let (B, b) be a T-algebra. We begin with the unique arrow $h^0: A_0 = 0 \longrightarrow B$. Suppose $\beta \le \alpha$ is an ordinal and we have, for all $\gamma < \beta$ an arrow $h^{\gamma}: A_{\gamma} \longrightarrow B$ such that $h^{\gamma} \circ f^{\delta}_{\gamma} = h^{\delta}: A_{\delta} \longrightarrow B$ for $\delta \le \gamma < \beta$ and $b \circ Th^{\gamma} \circ f^{\gamma}_{\gamma+1} = h^{\gamma}$ for $\gamma < \beta$. We want to construct $h^{\beta}: A_{\beta} \longrightarrow B$ to continue the induction. We consider three cases:

β is a limit ordinal: Then A_{β} is a colimit of the f^{β}_{γ} for $\gamma < \beta$ and we can let h^{β} be the unique arrow such that $h^{\beta} \circ f^{\gamma}_{\beta} = h^{\gamma}$.

$\beta = \gamma + 1$ and γ is a limit ordinal: Let $h^{\beta} = b \circ Th^{\gamma}$. We claim that $b \circ Th^{\gamma} \circ f^{\beta}_{\gamma} = h^{\gamma}$. We show this by composing with f^{δ}_{γ} for $\delta < \gamma$. We have

$$b \circ Th^{\gamma} \circ f^{\gamma}_{\beta} \circ f^{\delta}_{\gamma} = b \circ Th^{\gamma} \circ f^{\delta}_{\beta} = b \circ Th^{\gamma} \circ f^{\delta+1}_{\gamma+1} \circ f^{\delta}_{\delta+1} = b \circ Th^{\gamma} \circ Tf^{\delta}_{\gamma} \circ f^{\delta}_{\delta+1}$$

$$= b \circ Th^{\delta} \circ f^{\delta}_{\delta+1} = b \circ h^{\delta+1} \circ f^{\delta}_{\delta+1} = h^{\delta} = h^{\gamma}\ f^{\delta}_{\gamma}$$

This gives immediately that $h^{\beta} \circ f^{\gamma}_{\beta} = h^{\gamma}$ from which we have for $\delta \le \gamma$ that

$$h^{\beta} \circ f^{\delta}_{\beta} = h^{\beta} \circ f^{\gamma}_{\beta} \circ f^{\delta}_{\gamma} = h^{\gamma} \circ f^{\delta}_{\gamma} = h^{\delta}$$

$\beta = \gamma + 2$ for some γ: In this case, we let $h^{\beta} = b \circ Th^{\gamma+1}$. Then

$$b \circ Th^{\gamma+1} \circ f^{\gamma+1}_{\gamma+2} = b \circ Th^{\gamma+1} \circ Tf^{\gamma}_{\gamma+1} = b \circ Th^{\gamma} = h^{\gamma+1}$$

from which it is immediate that $h^{\beta} \circ f^{\gamma+1}_{\beta} = h^{\gamma+1}$ and that, just is in the previous case, for $\delta \le \gamma + 1$,

$$h^{\beta} \circ f^{\delta}_{\beta} = h^{\beta} \circ f^{\gamma}_{\beta} \circ f^{\delta}_{\gamma} = h^{\gamma} \circ f^{\delta}_{\gamma} = h^{\delta}$$

It follows that if $f^\alpha_{\alpha+1}$ is an isomorphism, h^α is a morphism of T-algebras. Thus $(A_\alpha,(f^\alpha_{\alpha+1})^{-1})$ is at least weakly initial. Suppose $k{:}\,A_\alpha \longrightarrow B$ is another morphism of T-algebras. For $\beta\le\alpha$, let $k^\beta = k\circ f^\beta_\alpha$. We will show by induction that $h^\beta = k^\beta$ for all $\beta\le\alpha$. Certainly $h^0 = k^0$ since their domain is the initial object. Assuming $h^\beta = k^\beta$, we have

$$k\circ f^{\beta+1}_\alpha = k\circ (f^\alpha_{\alpha+1})^{-1}\circ f^{\beta+1}_{\alpha+1} = b\circ Tk\circ Tf^\beta_\alpha = b\circ T(k\circ f^\beta_\alpha) = b\circ T(k^\beta) = b\circ T(k^\beta)$$

which reduces by the same argument to $h^\alpha\circ f^{\beta+1}_\alpha$. Suppose β is a limit ordinal and if for every $\gamma<\beta$ we have $k\circ f^\gamma_\alpha = h^\alpha\circ f^\gamma_\alpha$. The fact that $A_\beta = \operatorname{colim}_{\gamma<\beta} A_\gamma$ implies that $k\circ f^\beta_\alpha = h^\alpha\circ f^\beta_\alpha$. This shows that $k = h^\alpha$ and demonstrates uniqueness. □

By dualizing the above argument, we get the following.

1.3 Theorem. *If T is an endofunctor on a category $\mathcal{C}$ for which the requisite limits exist, there is a terminal T-sequence*

$$B_0 \xleftarrow{g^1_0} B_1 \xleftarrow{g^2_1} \cdots B_\beta \xleftarrow{g^\alpha_\beta} B_\alpha \cdots$$

defined for all ordinals with $B_0 = 1$, $B_{\alpha+1} = TB_\alpha$ and $B_\alpha = \lim_{\beta<\alpha} B_\beta$ for a limit ordinal α. If $g^{\alpha+1}_\alpha$ is an isomorphism, $(B_\alpha,(g^{\alpha+1}_\alpha)^{-1})$ is a terminal T-coalgebra.

1.4 An example. Let $\mathcal{C}$ be the category whose objects are all ordinals, ordered by inclusion, plus one more object ∞ greater than all the ordinals. Then $\mathcal{C}$ is complete and cocomplete. Let T be the endofunctor defined by $T\alpha = \alpha+1$ when α is an ordinal and $T\infty = \infty$. Then the initial sequence for T consists of all the ordinals and never stabilizes. There is only one T-algebra and that is ∞ and it is initial. Thus the initial T-algebra need not be reachable from the initial sequence.

1.5 If $a{:}\,TA \longrightarrow A$ is an algebra and $b{:}\,B\longrightarrow TB$ is a T-coalgebra, then a morphism $f{:}\,A\longrightarrow B$ is called a **relational T-morphism** if

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ TB & B \arrow[l, "b"] \end{tikzcd}$$

commutes.

The reason for the name is that it is a morphism in the category of relational T-algebras whose objects are relations $TA \leftarrow R \rightarrow A$. A morphism from that one to $TB \longleftarrow S \longrightarrow B$ is a morphism $f: A \longrightarrow B$ for which there is a (necessarily unique) $g: R \longrightarrow S$ such that

$$\begin{tikzcd} TA \arrow[d, "Tf"'] & R \arrow[l] \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"] \\ TB & S \arrow[l] \arrow[r] & B \end{tikzcd}$$

commutes. This category includes both the algebras and coalgebras as full subcategories. However, the inclusions do not preserve initial and terminal objects. More precisely, the inclusion of the algebras preserves terminal algebras but not initial ones and vice versa for the coalgebras. These are precisely the ones we are not interested in.

1.6 Theorem. *Let T be an endofunctor on $\mathcal{C}$ and let $\{A_\alpha, f_\alpha^\beta\}$ and $\{B_\alpha, g_\beta^\alpha\}$ be the initial and terminal T-sequences respectively. Then there is a unique family of morphisms $\{h_\alpha^\alpha: A_\alpha \longrightarrow B_\alpha\}$ such that for all β, $h_{\beta+1}^{\beta+1} = Th_\beta^\beta$ and such that for all $\beta \le \alpha$, $g_\beta^\alpha \circ h_\alpha^\alpha \circ f_\alpha^\beta = h_\beta^\beta$. Suppose, moreover, that $a: TA \longrightarrow A$ is a T-algebra, that $\{k^\alpha: A_\alpha \longrightarrow A\}$ is the sequence constructed in 1.2, that $b: B \longrightarrow TB$ is a T-coalgebra, $\{l_\alpha: B \longrightarrow B_\alpha\}$ is the dual sequence and that $m: (A, a) \longrightarrow (B, b)$ is a relational morphism. Then $h_\alpha^\alpha = l_\alpha \circ m \circ k^\alpha$.*

Proof. The morphism $h_0^0: A_0 \longrightarrow B_0$ is the unique arrow from 0 to 1. Assuming that $h_\alpha^\alpha: A_\alpha \longrightarrow B_\alpha$ is given, then define $h_{\alpha+1}^{\alpha+1} = Th_\alpha^\alpha: A_{\alpha+1} \longrightarrow B_{\alpha+1}$. If α is a limit ordinal, then $A_\alpha = \operatorname{colim}_{\beta<\alpha} A_\beta$ and $B_\alpha = \lim_{\beta<\alpha} B_\beta$. Then h_α^α is an element of

$$\operatorname{Hom}(\operatorname*{colim}_{\beta<\alpha} A_\beta, \lim_{\gamma<\alpha} B_\gamma) = \lim_{(\beta,\gamma)\in(\alpha\times\alpha)} \operatorname{Hom}(A_\beta, B_\gamma)$$

But this limit is taken over $\alpha \times \alpha$ and it follows from [Lawvere, 1963], page 36 that for a filtered diagram, the diagonal is cofinal in the square and so that limit is the same as $\lim_{\beta<\alpha} \operatorname{Hom}(A_\beta, B_\beta)$ and the family $\{h_\beta^\beta\}$ is an element of the limit. Thus at the limit ordinal α, there is a unique $h_\alpha^\alpha: A_\alpha \longrightarrow B_\alpha$ such that $g_\beta^\alpha \circ h_\alpha^\alpha \circ f_\alpha^\beta = h_\beta^\beta$. The uniqueness of the sequence of h_α^α subject to those two conditions is clear.

For the second part, we note that the sequences of k^α and l_α are defined so that the upper and lower squares, respectively, of the diagrams

$$\begin{tikzcd}
A_\alpha \arrow[r, "f^\alpha_{\alpha+1}"] \arrow[d, "k^\alpha"'] & A_{\alpha+1} \arrow[d, "Tk^\alpha"] \\
A & TA \arrow[l, "a"] \arrow[d, "Tm"] \\
B \arrow[u, phantom] \arrow[r, "b"] \arrow[d, "l_\alpha"'] & TB \arrow[d, "l_{\alpha+1}"] \\
B_\alpha & B_{\alpha+1} \arrow[l, "g^{\alpha+1}_\alpha"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_\beta \arrow[r, "f^\beta_\alpha"] \arrow[d, "k^\beta"'] & A_\alpha \arrow[d, "k^\alpha"] \\
A & A \arrow[l, "="] \arrow[d, "Tm"] \\
B \arrow[r, "="] \arrow[d, "l_\beta"'] & B \arrow[d, "l_\alpha"] \\
B_\beta & B_\alpha \arrow[l, "g^\alpha_\beta"]
\end{tikzcd}$$

(In both diagrams the left vertical arrow from A to B is m.)

commute, the first for all α and the second for all limit ordinals α. It follows by induction that the family $\{l_\alpha \circ m \circ k^\alpha\}$ satisfy the same hypotheses as the $\{h^\alpha_\alpha\}$ and the uniqueness makes the two families equal. □

If both the initial and terminal T-sequences stabilize, then so does the sequence of h^α_α and is the canonical map from the initial algebra to the terminal coalgebra.

1.7 Theorem. *Suppose the endofunctor T has the property that there is some ordinal α_0 for which h^α_α is an isomorphism for all $\alpha > \alpha_0$. Suppose there is a relational morphism m from some T-algebra $a{:}\,TA \longrightarrow A$ to some T-coalgebra $b{:}\,B \longrightarrow TB$. Then T is algebraically compact.*

We remark that it is sufficient that there be an invariant object, that is an $A \cong TA$ since then A is both an algebra and coalgebra and the identity function is a relational morphism.

Proof. It will simplify the argument to suppose, as we may without loss of generality, that $A_\alpha = B_\alpha$ and $h^\alpha_\alpha = \mathrm{id}$ for all α. Then we can construct the families $k^\alpha{:}\,A_\alpha \longrightarrow A$ and $l_\alpha{:}\,B \longrightarrow A_\alpha$ as above and they will satisfy $l_\alpha \circ m \circ k^\alpha = \mathrm{id}$ for all α. Then $k^\alpha \circ l_a \circ m$ is an idempotent endomorphism of A for each α. Since an object has only a set of endomorphisms, there are ordinals $\beta < \alpha$ such that $k^\beta \circ l_\beta \circ m = k^\alpha \circ l_\alpha \circ m$. But then

$$\begin{aligned}
f^\beta_\alpha \circ g^\alpha_\beta &= l_\alpha \circ m \circ k^\alpha \circ f^\beta_\alpha \circ g^\alpha_\beta \circ l_\alpha \circ m \circ k^\alpha \\
&= l_\alpha \circ m \circ k^\beta \circ l_\beta \circ m \circ k^\alpha = l_\alpha \circ m \circ k^\alpha \circ l_\alpha \circ m \circ k^\alpha = \mathrm{id}
\end{aligned}$$

As we have seen in Proposition 1.1, as soon as f^β_α is an isomorphism, $(A_\beta, (f^\beta_{\beta+1})^{-1})$ is an initial T-algebra. Obviously $(A_\beta, (g^{\beta+1}_\beta)^{-1})$ is a terminal T-coalgebra and T is therefore algebraically compact. □

2 Partial monomorphisms

We illustrate these points with the category $\mathcal{PM}$ whose objects are sets and arrows are partial monomorphisms. If $f: X \longrightarrow Y$ is a partial monomorphism, so is the converse relation, which we will denote $f^*: Y \longrightarrow X$. A moment's thought will convince the reader of the following fact.

2.1 Proposition. *Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ be morphisms of $\mathcal{PM}$ such that $g \circ f = \mathrm{id}_X$. Then $g = f^*$.*

Note that although we call the arrows partial monomorphisms, the only ones that are monomorphisms in $\mathcal{PM}$ itself are the total ones and they are split.

If $T: \mathcal{PM} \longrightarrow \mathcal{PM}$ is an endofunctor, the fact that $\mathcal{PM}$ is pointed—the empty set is both initial and final—implies that the initial and terminal sequences begin the same. We next show:

2.2 Proposition. *Suppose $\{f_\beta^\gamma: A_\gamma \longrightarrow A_\beta \mid \gamma < \beta\}$ is a chain based on the ordinals less than α and $\{g_\gamma^\beta: A_\beta \longrightarrow A_\gamma \mid \gamma < \beta\}$ is a cochain based on the same set. Suppose also that $g_\gamma^\beta \circ f_\beta^\gamma = \mathrm{id}$ for $\gamma < \beta$. Then $\operatorname{colim} A_\beta$ and $\lim A_\beta$ both exist and the induced map between them is an isomorphism.*

Proof. An arrow in the category with a left inverse is a monic function. If A_α is the colimit in the category of sets, with the cocone given by the functions $f_\alpha^\beta: A_\beta \longrightarrow A_\alpha$, I claim it is also the colimit in $\mathcal{PM}$.

For the purpose of this argument, let us suppose that all the f_α^β are inclusions and, therefore, so are the f_β^γ for $\gamma \leq \beta \leq \alpha$. A compatible family of partial monomorphisms $A_\beta \longrightarrow B$ is given by a family of subsets $U_\beta \subseteq A_\beta$ and monomorphisms $u^\beta: U_\beta \longrightarrow B$ such that $U_\gamma = A_\gamma \cap U_\beta$ and $u^\beta | A_\gamma = u^\gamma$ for $\gamma \leq \beta$. Let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$ and let $u^\alpha: U_\alpha \longrightarrow B$ be the unique function such that $u^\alpha | U_\beta = u^\beta$ for $\beta < \alpha$. Then the partial monomorphism represented by u^α is the desired arrow.

The duality ensures that if A_α and $\{f_\alpha^\beta\}$ is the colimit of the $\{f_\beta^\gamma\}$, then A_α with the $\{(f_\alpha^\beta)^*\}$ is also the limit of the $\{(f_\beta^\gamma)^*\}$. But $(f_\beta^\gamma)^* = g_\gamma^\beta$ and if we let $g_\beta^\alpha = (f_\alpha^\beta)^*$ we have continued the isomorphism between the initial sequence to the limit ordinal α. Thus h_α^α is an isomorphism (which can be taken to be the identity) for all α. □

2.3 Corollary. *In $\mathcal{PM}$ the class of all functors is conditionally algebraically compact.*

2.4 The category $\mathcal{PM}_\kappa$. For a cardinal κ, let $\mathcal{PM}_\kappa$ denote the full subcategory of $\mathcal{PM}$ consisting of the sets of cardinality at most κ. Suppose T is an endofunctor on $\mathcal{PM}_\kappa$. The first thing we observe is that a colimit of an increasing chain of sets of cardinality at most κ is still at most κ. In fact, under those circumstances, the entire chain and hence its union can be embedded into a set of cardinality exactly κ. Thus $\mathcal{PM}_\kappa$ admits all the colimits necessary to carry out the construction of the initial, and hence the terminal, T-chains. In particular, in $\mathcal{PM}$ the class of all functors is conditionally algebraically compact.

For a cardinal λ, let X_λ denote a set of cardinality λ (which might be λ itself, depending on your model of cardinals). It may happen that $T(X_\kappa) \cong X_\kappa$. If not, it has smaller cardinality. Since $T(X_0)$ does not have cardinality less than 0 and since the cardinals are well-ordered, there is a least cardinal λ such that $\#(T(X_\lambda)) = \mu < \lambda$. Then there is a split monic $X_\mu \longrightarrow X_\lambda \longrightarrow X_\mu$. Applying T we get a split sequence $T(X_\mu) \longrightarrow T(X_\lambda) \longrightarrow T(X_\mu)$ so that $\#(T(X_\mu)) \le \#(T(X_\lambda)) = \mu$. On the other hand, from the choice of λ it is not possible that $\#(T(X_\mu)) < \mu$ so that we conclude that $T(X_\mu) \cong X_\mu$. Thus there is a fixed point for every functor and so we conclude that $\mathcal{PM}_\kappa$ is algebraically compact for every κ.

3 Hilbert spaces

We consider the category $\mathcal{H}$ of hilbert spaces and linear maps of norm at most 1. Nothing we say depends on whether the ground field is $\mathbf{R}$ or $\mathbf{C}$.

Every map $f: H \longrightarrow K$ in $\mathcal{H}$ has an adjoint $f^*: K \longrightarrow H$. This defines a contravariant endofunctor on $\mathcal{H}$ that is the identity on objects and whose square is the identity functor. It is characterized by the property that $fu \cdot v = u \cdot f^*v$ for $u \in H$ and $v \in W$.

The main property we need is,

3.1 Proposition. *Let $f: H \longrightarrow K$ and $g: K \longrightarrow H$ be morphisms such that $g \circ f = \mathrm{id}$. Then f is an isometric embedding and $g = f^*$.*

Proof. Since g cannot increase norm, f cannot decrease it and so must be an isometry. The arrow $p = f \circ g$ is an idempotent endomorphism of K.

We claim that $\mathrm{im}(p) = \ker(p)^\perp$. In fact, let $p(v) \in \mathrm{im}(p)$. Write $p(v) = v_1 + v_2$ with $v_1 \in \ker(p)^\perp$ and $v_2 \in \ker(p)$. Then

$$v_1 + v_2 = p(v) = p^2(v) = p(v_1) + p(v_2) = p(v_1)$$

so that

$$\|p(v_1)\| = \|v_1 + v_2\| = \sqrt{\|v_1\|^2 + \|v_2\|^2} \ge \|v_1\| \ge \|p(v_1)\|$$

which means that the inequalities are equalities and, in particular, that $\|v_2\| = 0$, whence $v_2 = 0$ and $p(v) \in \ker(p)^\perp$. This gives that $\mathrm{im}(p) \subseteq \ker(p)^\perp$. Since p is idempotent, $H = \mathrm{im}(p) \oplus \ker(p)$, while the idempotence and continuity of p imply that $H = \mathrm{im}(p) \oplus \mathrm{im}(p)^\perp$, from which we have that $\mathrm{im}(p) = \ker(p)^\perp$.

We now see that for $u \in H$ and $v \in K$,

$$fu \cdot v = fu \cdot pv = fu \cdot f \circ gv = u \cdot gv$$

since f is an isometry and hence $g = f^*$. $\square$

This shows that the category has many properties in common with the category $\mathcal{PM}$. For example, $\mathcal{H}$ has colimits along chains of isometric embedding. This is done by taking the colimit as vector spaces and then completing. Dimension replaces the cardinality used in the preceding example; the exact same considerations are valid. Two hilbert spaces of the same dimension are isomorphic and one of smaller dimension has an isometric embedding into one of larger dimension. Let $\mathcal{H}_\lambda$ denote the full subcategory of hilbert spaces of dimension at most λ.

3.2 Theorem. *The category $\mathcal{H}$ is conditionally algebraically compact and the categories $\mathcal{H}_\lambda$ are algebraically compact.*

Although the results are independent of it, it is interesting to note that there is a canonical embedding of $\mathcal{PM}$ into $\mathcal{H}$ that restricts to an embedding of $\mathcal{PM}_\lambda$ into $\mathcal{H}_\lambda$. If X is a set, $l_2(X)$ is the set of square summable families $\{a_x \mid x \in X\}$ with $\left(\sum_{x\in X} |a_x|^2\right)^{1/2}$ as norm. This is not a functor on the category of sets, even to the category of continuous linear maps. It is a functor on sets and monomorphisms and, more to the point, it is also a functor on $\mathcal{PM}$. If $f: X \longrightarrow Y$ is a partial monomorphism, define

$$l_2(f)(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom}(f) \\ 0 & \text{otherwise} \end{cases}$$

Then l_2 preserves all the constructions used in showing that these categories are conditionally algebraically compact.

4 CPOs and CPOs with bottom

One commonly supposes of CPOs that they contain a least or bottom element, denoted $\perp$. One does not suppose that morphisms are to preserve this bottom. This convention, although well motivated by the results that follow therefrom, makes for an ill-behaved category. There is no initial object; in fact no categorical sums at all.

On the one hand, if one does not suppose a bottom element, such things as Tarski's fixed point theorem fail. On the other hand, if you suppose there is a bottom and it is preserved, then Tarski's theorem is trivial and uninteresting, since the bottom element is the least fixed point. The solution we will adopt is to work with the category of CPOs that do not necessarily have a bottom element, but state the Tarski theorem only for those objects of the category that do.

There is an obvious category of CPOs with bottom and morphisms that preserve the directed sups *and* bottom. Since, as Freyd points out in his 1991 preprint, an algebraically compact category is pointed, it seems likely that more complete results are to be found in that category. We let $\mathbf{CPO}_\perp$ denote the category of CPOs with bottom.

4.1 CPO- and CPO$_\perp$-enriched categories. A category $\mathcal{C}$ is **CPO**-enriched, resp. **CPO**$_\perp$-enriched, if its Homsets are CPOs, resp. with bottom, such that composition preserves the sups of ordinal indexed chains, resp. and bottom.

4.2 Proposition. *Let $\mathcal{C}$ be a category enriched over* **CPO** *. Consider a diagram defined for $i \in \mathbf{N}$:*

$$
\begin{tikzcd}
A_0 \arrow[r, "f_1^0"] \arrow[d, "h_0^0"'] & A_1 \arrow[r, "f_2^1"] \arrow[d, "h_1^1"] & A_2 \arrow[r, "f_2^3"] \arrow[d, "h_2^2"] & \cdots \arrow[r, "f_i^{i-1}"] & A_i \arrow[r, "f_{i+1}^i"] \arrow[d, "h_i^i"] & \cdots \\
B_0 \arrow[ur, "l_1^0"'] & B_1 \arrow[l, "g_0^1"] \arrow[ur, "l_2^1"'] & B_2 \arrow[l, "g_1^2"] \arrow[ur, "l_3^2"'] & \cdots \arrow[l, "g_2^3"] \arrow[ur, "l_i^{i-1}"] & B_i \arrow[l, "g_{i-1}^i"] \arrow[ur, "l_{i+1}^i"'] & \cdots \arrow[l, "g_i^{i+1}"]
\end{tikzcd}
$$

Suppose the following identities are satisfied for all $i \in \mathbf{N}$

EP–1. $g_i^{i+1} \circ h_{i+1}^{i+1} \circ f_{i+1}^i = h_i^i$.

EP–2. $l_{i+1}^i \circ h_i^i = f_{i+1}^i$.

EP–3. $h_{i+1}^{i+1} \circ l_{i+1}^i \circ g_i^{i+1} \leq \mathrm{id}$.

EP–4. $g_i^{i+1} \circ h_{i+1}^{i+1} \circ l_{i+1}^i = \mathrm{id}$.

Suppose $f_\omega^i \colon A_i \longrightarrow A_\omega$ is a colimit of the upper sequence. Then there are arrows $g_i^\omega \colon A_\omega \longrightarrow B_i$ that define a limit cone over the lower sequence and such that $g_i^\omega \circ f_\omega^i = h_i^i$ for all $i \in \mathbf{N}$.

Proof. Define $f_j^i \colon A_i \longrightarrow A_j$ for $i \leq j$ as the composite $f_j^{j-1} \circ f_{j-1}^{j-2} \circ \cdots \circ f_{i+1}^i$ and similarly $g_j^i \colon B_i \longrightarrow B_j$ for $i \geq j$ as the composite $g_j^{j+1} \circ g_{j+1}^{j+2} \circ \cdots \circ g_{i-1}^i$. Empty composites are, as usual, defined to be identity maps. It is evident that for $i < j$, we have $f^j \circ f_j^i = f^i$, that for $i < j < k$, we have $f_k^j \circ f_j^i = f_k^i$ and that for $i > j > k$, we have $g_k^j \circ g_j^i = g_k^i$.

We define $h_j^i \colon A_i \longrightarrow A_j$ for all $i \neq j$ by the formulas

$$
h_j^i = \begin{cases} h_j^j \circ f_j^i & \text{if } i < j \\ g_j^i \circ h_j^i & \text{if } i > j \end{cases}
$$

4.3 Lemma. *The following identities hold*

EP–5. $h_j^i \circ f_i^k = h_j^k$ *for all j and all $k < i$.*

EP–6. $g_k^i \circ h_j^i = h_k^i$ *for all i and all $j > k$.*

Proof.

EP–5. We have to consider the cases $i \le j$ and $i > j$ separately. In the first case,

$$h^i_j \circ f^k_i = h^j_j \circ f^i_j \circ f^k_i = h^j_j \circ f^k_j = h^k_j$$

In the second case, we use induction on $i - j$. When $i - j = 0$, it is the first case. For $i - j > 0$,

$$h^i_j \circ f^k_i = g^i_j \circ h^i_i \circ f^k_i = g^{i-1}_j \circ g^i_{i-1} \circ h^i_i \circ f^{i-1}_i \circ f^k_{i-1} = g^{i-1}_j \circ h^{i-1}_{i-1} \circ f^k_{i-1} = h^k_j$$

We have used EP–1 and induction.

EP–6. Except for the l's, the diagram in the dual category would look the same, interchanging the A's and B's and f's and g's and leaving the h's the same. It would leave EP–1 fixed and would interchange EP–5 and EP–6, none of which involve the l's. □

We can now return to the proof of Proposition 4.2. The universal mapping properties of the colimit, together with EP–5 imply that for each j there is a map $g^\omega_j : A_\omega \longrightarrow A_j$ such that $g^\omega_j \circ f^i_\omega = h^i_j$. EP–6 and uniqueness of maps from a colimit imply that for $j > k$, we have $g^j_k \circ g^\omega_j = g^\omega_k$ so that the g^ω_j give a cone over the lower sequence. Suppose we have another cone given by $m_i : C \longrightarrow B_i$.

First we see that for all i,

$$\begin{aligned} f^{i+1}_\omega \circ l^i_{i+1} \circ m_i &= f^{i+2}_\omega \circ f^{i+1}_{i+2} \circ l^i_{i+1} \circ g^{i+1}_i \circ m_{i+1} \\ &= f^{i+2}_\omega \circ l^{i+1}_{i+2} \circ h^{i+1}_{i+1} \circ l^i_{i+1} \circ g^{i+1}_i \circ m_{i+1} \\ &\le f^{i+2}_\omega \circ l^{i+1}_{i+2} \circ m_{i+1} \end{aligned}$$

Thus the family $\{f^{i+1}_\omega \circ l^i_{i+1} \circ m_i\}$ gives a increasing sequence of maps in $\mathrm{Hom}(C, A)$ and we let $m = \bigvee_i f^{i+1}_\omega \circ l^i_{i+1} \circ m_i : C \longrightarrow A$. Now $g^\omega_j \circ m = \bigvee_i g^\omega_j \circ f^{i+1}_\omega \circ l^i_{i+1} \circ m_i$ is the sup of an increasing sequence so we need consider only its tail. But as soon as $i > j$, we have

$$\begin{aligned} g^\omega_j \circ f^{i+1}_\omega \circ l^i_{i+1} \circ m_i &= h^{i+1}_j \circ l^i_{i+1} \circ m_i \\ &= g^{i+1}_j \circ h^{i+1}_{i+1} \circ l^i_{i+1} \circ m_i \\ &= g^i_j \circ g^{i+1}_i \circ h^{i+1}_{i+1} \circ l^i_{i+1} \circ m_i \\ &= g^i_j \circ m_i = m_j \end{aligned}$$

Thus we see that A_ω with the g^ω_i is at least a weak limit. In order to prove uniqueness, we need some lemmas.

4.4 Lemma.

1. *The sequence* $\{f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega\}$ *is an increasing sequence of endomorphisms of* A.
2. $f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega \circ f_\omega^j = f_\omega^j$ *for all* $j > i$.
3. $\bigvee_i f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega \circ f_\omega^j = f_\omega^j$.
4. $\bigvee_i f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega = \mathrm{id}$.

Proof.

1. $$\begin{aligned} f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega &= f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega \\ &= f_\omega^{i+2} \circ f_{i+2}^{i+1} \circ l_{i+1}^i \circ g_i^\omega \\ &= f_\omega^{i+2} \circ l_{i+2}^{i+1} \circ h_{i+1}^{i+1} \circ l_{i+1}^i \circ g_i^{i+1} \circ g_{i+1} \\ &\le f_\omega^{i+2} \circ l_{i+2}^{i+1} \circ g_{i+1} \end{aligned}$$
2. For $i > j$, $$\begin{aligned} f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega \circ f_\omega^j &= f_\omega^{i+1} \circ l_{i+1}^i \circ h_i^j \\ &= f_\omega^{i+1} \circ l_{i+1}^i \circ h_i^i \circ f_i^j \\ &= f_\omega^{i+1} \circ f_{i+1}^i \circ f_i^j = f_\omega^j \end{aligned}$$
3. This is now immediate.
4. This follows since the $\{f_\omega^j\}$ are a colimit cocone. □

Now we are in a position to prove the uniqueness, which will complete the proof of Proposition 4.2. Suppose $n: C \longrightarrow A$ is a map with $g_i^\omega \circ n = m_i$ for all i. Then

$$\begin{aligned} m &= \bigvee f_\omega^{i+1} \circ h_{i+1} \circ m_i = \bigvee f_\omega^{i+1} \circ h_{i+1} \circ g_i^\omega \circ n \\ &= \left(\bigvee f_\omega^{i+1} \circ h_{i+1} \circ g_i^\omega\right) \circ n = n \end{aligned}$$

□

4.5 Proposition. *Let* α *be a limit ordinal and suppose we have an ordinal indexed diagram*

$$A_0 \underset{g_0^1}{\overset{f_1^0}{\rightleftarrows}} A_1 \underset{g_1^2}{\overset{f_2^1}{\rightleftarrows}} \cdots A_\gamma \underset{g_\beta^\gamma}{\overset{f_\gamma^\beta}{\rightleftarrows}} A_\beta \rightleftarrows \cdots$$

defined for all ordinals $\beta < \alpha$ *suppose there is a cardinal* α_0 *such that the following identities are satisfied for all cardinals* β *and* γ *such that* $\alpha_0 \le \gamma \le \beta < \alpha$:

EP′–1. $g^\beta_\gamma \circ f^\gamma_\beta = \mathrm{id}$.

EP′–2. $f^\gamma_\beta \circ g^\beta_\gamma \le \mathrm{id}$.

Then if $\{f^\beta_\alpha\colon A_\beta \longrightarrow A_\alpha\}$ is a colimit of the increasing sequence, there are maps $g^\alpha_\beta\colon A_\alpha \longrightarrow A_\beta$ such that $g^\alpha_\beta \circ f^\beta_\alpha = \mathrm{id}$, $f^\beta_\alpha \circ g^\alpha_\beta$ and the cone defined by the $\{g^\alpha_\beta\}$ is a limit. Moreover, the above (in)equalities are true at α as well.

Proof. Since neither the limit nor the colimit depend on the early part of the sequence, we can assume, without loss of generality, that the conditions of the proposition are satisfied for all $\gamma \le \beta$.

The maps g^α_β are maps out of a colimit and so they are defined by giving their composite with every f^α_γ and proving it compatible. We define it by

$$g^\alpha_\beta \circ f^\gamma_\alpha = \begin{cases} f^\gamma_\beta & \text{if } \gamma \le \beta \\ g^\gamma_\beta & \text{if } \beta \le \gamma \end{cases}$$

To show that this family is compatible, we have to choose a $\delta \ge \beta$ and compose on the right with f^β_δ. We then have three cases:

$\delta \le \gamma \le \beta$: We have $f^\gamma_\beta \circ f^\delta_\gamma = f^\gamma_\beta$.

$\delta \le \beta \le \gamma$: We have $g^\gamma_\beta \circ f^\delta_\gamma = g^\gamma_\beta \circ f^\beta_\gamma \circ f^\delta_\beta = f^\delta_\beta$.

$\beta \le \delta \le \gamma$: We have $g^\gamma_\beta \circ f^\delta_\gamma = g^\delta_\beta \circ g^\gamma_\delta \circ f^\delta_\gamma = g^\delta_\beta$.

This defines the arrows g^α_β and shows, incidentally, that $g^\alpha_\beta \circ f^\beta_\alpha = \mathrm{id}$. Next we wish to show that this family of arrows is a cone, that is $g^\beta_\delta \circ g^\alpha_\beta = g^\alpha_\delta$ for $\delta \le \beta$. Since these arrows are determined uniquely by their composites with all f^γ_α we must show that this equation is true when composed with all f^γ_α. Again, we consider three cases:

$\delta \le \beta \le \gamma$: We have

$$g^\beta_\delta \circ g^\alpha_\beta \circ f^\gamma_\alpha = g^\beta_\delta \circ g^\gamma_\beta = g^\beta_\delta = g^\alpha_\delta \circ f^\beta_\alpha$$

$\delta \le \gamma \le \beta$:

$$g^\beta_\delta \circ g^\alpha_\beta \circ f^\gamma_\alpha = g^\beta_\delta \circ f^\gamma_\beta = g^\gamma_\delta \circ g^\beta_\gamma \circ f^\gamma_\beta = g^\gamma_\delta = g^\alpha_\delta \circ f^\beta_\alpha$$

$\gamma \le \delta \le \beta$:

$$g^\beta_\delta \circ g^\alpha_\beta \circ f^\gamma_\alpha = g^\beta_\delta \circ f^\gamma_\beta = g^\beta_\delta \circ f^\beta_\delta \circ f^\gamma_\beta = f^\gamma_\beta = g^\alpha_\delta \circ f^\beta_\alpha$$

Now suppose an object B is given and arrows $h_\beta\colon B \longrightarrow A_\beta$ such that for $\gamma < \beta$, $g^\beta_\gamma \circ h_\beta = h_\gamma$. I claim that the sequence $g^\beta_\alpha \circ h_\beta$ is an increasing sequence of arrows from B to A_α. In fact, for $\gamma \le \beta$, we have

$$f^\gamma_\alpha \circ h_\gamma = f^\gamma_\alpha \circ g^\beta_\gamma \circ h_\beta = f^\beta_\alpha \circ f^\gamma_\beta \circ g^\beta_\gamma \circ h_\beta \le f^\beta_\alpha \circ h_\beta$$

since $f^\gamma_\beta \circ g^\beta_\gamma \le \mathrm{id}$. Thus we can let $h_\alpha : B \longrightarrow A_\alpha = \bigvee f^\beta_\alpha \circ h_\beta$. For fixed β the sequence $g^\alpha_\beta \circ f^\gamma_\alpha \circ h_\gamma$ is also an increasing sequence and so we have

$$g^\alpha_\beta \circ h_\alpha = g^\alpha_\beta \circ \left(\bigvee_g f^\gamma_\alpha \circ h_\gamma \right) = \bigvee_\gamma g^\alpha_\beta \circ f^\gamma_\alpha \circ h_\gamma$$

$$= \bigvee_{\gamma > \beta} g^\alpha_\beta \circ f^\gamma_\alpha \circ h_\gamma = \bigvee_{\gamma > \beta} g^\gamma_\beta \circ h_\gamma = \bigvee_{\gamma > \beta} h_\beta = g_\beta$$

For uniqueness, we begin by showing that $\bigvee_{\beta < \alpha} f^\beta_\alpha \circ g^\alpha_\beta = \mathrm{id}$. In fact, fix a $\gamma < \alpha$. Then

$$\left(\bigvee_{\beta < \alpha} f^\beta_\alpha \circ g^\alpha_\beta \right) \circ f^\gamma_\alpha = \bigvee_{\beta < \alpha} f^\beta_\alpha \circ g^\alpha_\beta \circ f^\gamma_\alpha = \bigvee_{\gamma < \beta < \alpha} f^\beta_\alpha \circ g^\alpha_\beta \circ f^\gamma_\alpha$$

$$= \bigvee_{\gamma < \beta < \alpha} f^\beta_\alpha \circ g^\gamma_\beta \circ g^\alpha_\gamma \circ f^\gamma_\alpha = \bigvee_{\gamma < \beta < \alpha} f^\beta_\alpha \circ g^\gamma_\beta$$

$$= \bigvee_{\gamma < \beta < \alpha} f^\gamma_\alpha = f^\gamma_\alpha$$

Then if $k_a : B \longrightarrow A_a$ is another arrow such that $h_\beta = g^\alpha_\beta \circ k_\alpha$ for all $\beta < \alpha$, then $f^\beta_\alpha \circ h_\beta = f^\beta_\alpha \circ g^\alpha_\beta \circ k_\alpha$ so that

$$h_\alpha = \bigvee f^\beta_\alpha \circ h_\beta = \bigvee f^\beta_\alpha \circ g^\alpha_\beta \circ k_\alpha = \left(\bigvee f^\beta_\alpha \circ g^\alpha_\beta \right) \circ k_\alpha = k_\alpha$$

□

4.6 Theorem. *Suppose $\mathcal{C}$ is a category enriched over* **CPO** *. Let T be an endofunctor on $\mathcal{C}$ that preserves the order relation. Suppose there is a morphism $l : 1 \longrightarrow T0$ such that the composite $T1 \longrightarrow 1 \xrightarrow{l} T0 \xrightarrow{Th} T1 \le \mathrm{id}_{T1}$, where $h : 0 \longrightarrow 1$ is the unique arrow. Assume that $\mathcal{C}$ has colimits along ordinal indexed sequences. Then the canonical map from the initial T-sequence to the terminal T-sequence is an isomorphism for indices ω and above and $\mathcal{C}$ is conditionally algebraically compact.*

Proof. We construct the initial and final sequences and the map between them as above. In addition we have a sequence of maps $l^i_{i+1} = T^i l : B_i = T^i 1 \longrightarrow A_{i+1} = T^{i+1} 0$. From the fact that $h^1_1 \circ l \circ g^1_0 \le \mathrm{id}$, it follows by applying T^i that $h^{i+1}_{i+1} \circ l^i_{i+1} \circ g^{i+1}_i \le \mathrm{id}$. It then follows that $\mathrm{colim}\, A_i$ is also a limit of the B_i. Thus B_ω exists and h^ω_ω is an isomorphism. Beyond that point, h^α_α remains an isomorphism. At limit ordinals, it is so by the preceding proposition and at non-limit ordinals, by applying T to the preceding ones. □

4.7 Proposition. *A non-empty* $\mathbf{CPO}_\perp$*-enriched category that has colimits along countable chains is pointed.*

Proof. Since each homset has a bottom and the bottom elements are preserved by composition, the category has the requisite class of morphisms and we need only find an initial (or a terminal) object. Let A be any object and consider the sequence

$$A \xrightarrow{\perp} A \xrightarrow{\perp} A \xrightarrow{\perp} \cdots$$

where $\perp$ is the bottom map in $\mathrm{Hom}(A, A)$. If $f_i: A \longrightarrow A_0$ is the map from the nth term of the sequence to the colimit, we have $f_i = f_{i+1} \circ \perp = \perp$. If B is any object of the category and $g: A_0 \longrightarrow B$ any arrow, we have $g \circ f_n = g \circ \perp = \perp = \perp \circ f_n$ for each n and, from the uniqueness of arrows from a colimit, that $g = \perp$. Thus $\perp$ is the unique arrow from A_0 to B, so that A_0 is initial. $\square$

4.8 Theorem. *Let* $\mathcal{C}$ *be a* $\mathbf{CPO}_\perp$ *enriched category and let* T *be an endofunctor on* $\mathcal{C}$ *that preserves the order structure (but not bottom and not the sups). Suppose* $\mathcal{C}$ *has colimits along ordinal chains. Then* T *is conditionally algebraically compact.*

Proof. This is an immediate consequence of the preceding results. In this case $0 = 1$ so we have the initial map $f: 1 \longrightarrow T0$. The map Th is the identity and we have $Th \circ f \circ g \leq \mathrm{id}$ because the composite is the bottom element of $\mathrm{Hom}(T1, T1)$. $\square$

5 ω-CPO and ω-$\mathbf{CPO}_\perp$ enriched categories

Many of the results on CPOs are also valid if we suppose only that countable chains have a sup, but only for functors that preserve colimits along countable chains. But one can argue that only such functors have computational meaning. For that condition is equivalent to the statement that every element of TX depends on a finite amount of data from X.

An ω-**CPO** is a poset in which every countable chain has a sup. This gives a category of ω-**CPO**s and another of ω-**CPO**s with bottom, which will be denoted ω-**CPO** and ω-$\mathbf{CPO}_\perp$, respectively. With appropriate assumptions of preservations of colimits along countable chains, the preceding results are valid in for categories enriched over these categories too. Since the proofs are strictly easier than the proofs above, we simply record the results.

5.1 Theorem. *Let* $\mathcal{C}$ *be an* ω-**CPO***-enriched category with an initial and a terminal object and colimits along* ω*-chains and let* T *be an endofunctor on* $\mathcal{C}$ *that preserves the order relation on the homsets and* ω*-indexed colimits. Suppose there is a map* $l: 1 \longrightarrow T0$ *such that the composite*

$$T1 \xrightarrow{g} 1 \xrightarrow{l} T0 \xrightarrow{Th} T1 \leq \mathrm{id}_{T1}$$

where $g: T1 \longrightarrow 1$ and $h: 0 \longrightarrow 1$ are the unique maps. Then T is algebraically compact.

5.2 Theorem. *Let $\mathcal{C}$ be an ω-$\mathbf{CPO}_\perp$ enriched category and let T be an endofunctor on $\mathcal{C}$ that preserves the order structure (but not bottom and not the ω sups). Suppose $\mathcal{C}$ has and T preserves colimits along ω chains. Then T is algebraically compact.*

5.3 Functors that preserve directed sups. If we suppose that a functor preserves directed sups on homsets, even just of countable sets, then we can prove that it is algebraically compact by showing that the initial/terminal fixed point is reached already at the countable stage.

5.4 Theorem. *Let $\mathcal{C}$ be a an ω-$\mathbf{CPO}$ category that has colimits along ordinal indexed sequences. Then the class of endofunctors that preserves countable directed sups and for which there is a morphism $l: 1 \longrightarrow T0$ such that the composite $T1 \longrightarrow 1 \xrightarrow{l} T0 \xrightarrow{Th} T1 \leq \mathrm{id}_{T1}$, where $h: 0 \longrightarrow 1$ is the unique arrow, is algebraically compact.*

Proof. Let T be an endofunctor of that class. One easily sees that a functor that preserves countable directed sups also preserves finite ones and hence preserves the order. It follows from the results of Section 1 and of Proposition 4.2 that there is a diagram

$$\begin{tikzcd}
A_0 \arrow[r, "f_1^0"] \arrow[d, "\mathrm{h}_0^0"'] & A_1 \arrow[r, "f_2^1"] \arrow[d, "\mathrm{h}_1^1"] & A_2 \arrow[r, "f_2^3"] \arrow[d, "\mathrm{h}_2^2"] & \cdots \arrow[r] & A_\omega \arrow[d, "\mathrm{h}_\omega^\omega"] \\
B_0 \arrow[ur, "l_1^0"'] & B_1 \arrow[l, "g_0^1"] \arrow[ur, "l_2^1"'] & B_2 \arrow[l, "g_1^2"] \arrow[ur, "l_3^2"'] & \cdots \arrow[l, "g_2^3"] & B_\omega \arrow[l]
\end{tikzcd}$$

in which the top row is a colimit, the bottom a limit and h_ω^ω is an isomorphism. It is sufficient to show that

$$TA_0 \xrightarrow{Tf_1^0} TA_1 \xrightarrow{Tf_2^1} TA_2 \longrightarrow \cdots \longrightarrow TA_\omega$$

is a colimit, for the colimit is clearly A_ω and $f_{\omega+1}^\omega$ is the induced map.

It follows (with a slight change of notation) from Lemma 4.4 that

$$\{f_\omega^{i+1} \circ l_{i+1}^i \circ g_i^\omega \circ h_\omega^\omega\}$$

is an increasing sequence of endomorphisms of A_ω whose sup is the identity. If we suppose that T preserves sups of countable chains, then we also have that

$$\{Tf_\omega^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega\}$$

is an increasing sequence of endomorphisms of TA_ω whose sup is the identity.

Now suppose that $\{m^i : TA_i \longrightarrow C\}$ is a family of arrows such that $m^i \circ Tf_i^j = m^j$ for $j \le i$. I claim that $\{m^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega\}$ is an increasing family of morphisms $TA_\omega \longrightarrow C$. We have

$$\begin{aligned}
m^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega &= m^{i+2} \circ Tf_{i+2}^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega \\
&= m^{i+2} \circ Tl_{i+2}^{i+1} \circ Th_{i+1}^{i+1} \circ Tl_{i+1}^i \circ Tg_i^{i+1} \circ Tg_{i+1}^\omega \circ Th_\omega^\omega \\
&\le m^{i+2} \circ Tl_{i+2}^{i+1} \circ Tg_{i+1}^\omega \circ Th_\omega^\omega
\end{aligned}$$

Let $m^\omega = \bigvee m^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega$. Then

$$\begin{aligned}
\left(\bigvee_{i=0}^{\infty} m^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega\right) \circ Tf_\omega^j &= \bigvee_{i>j} m^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega \circ Tf_\omega^j \\
&= \bigvee_{i>j} m^{i+1} \circ Tl_{i+1}^i \circ Th_i^i \circ Tf_i^j \\
&= \bigvee_{i>j} m^{i+1} \circ Tf_{i+1}^i \circ Tf_i^j = \bigvee_{i>j} m^{i+1} \circ Tf_{i+1}^j \\
&= \bigvee_{i>j} m^j = m^j
\end{aligned}$$

This shows that m^ω has the right composite with each Tf_ω^j. Suppose $m: TA_\omega \longrightarrow C$ has the property that $m \circ Tf_\omega^i = m^i$ for all i. Then we have that

$$\begin{aligned}
m &= m \circ \left(\bigvee Tf_\omega^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega\right) \\
&= \bigvee m \circ Tf_\omega^{i+1} \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega \\
&= \bigvee m^i \circ Tl_{i+1}^i \circ Tg_i^\omega \circ Th_\omega^\omega = m^\omega
\end{aligned}$$

which shows uniqueness of m^ω. □

6 Some ω-continuous functors on ω-**CPO** and ω-**CPO**$_\perp$

The categories ω-**CPO** and ω-**CPO**$_\perp$ are not $\aleph_0$-accessible (they are $\aleph_1$-accessible), so there is no ready supply of ω-continuous functors. Therefore it is of interest to know that there is an interesting class of such functors.

Obviously, both constant functors and the identity functor are ω-continuous. It is clear from the fact that colimits commute with colimits that the disjoint union of ω-continuous functors is again ω-continuous. For similar reasons, on ω-$\mathbf{CPO}_\perp$, the smash product of ω-continuous functors is again ω-continuous.

If T_1 and T_2 are endofunctors on a category of posets, we let $T_1 \dot{+} T_2$ denote the functor defined by $(T_1 \dot{+} T_2)(A) = T_1 A \dot{+} T_2 A$ where $A \dot{+} B$ is the sum of A and B with every element of A below every element of B.

6.1 Theorem. *Let $\mathcal{C}$ denote either ω-$\mathbf{CPO}$ or ω-$\mathbf{CPO}_\perp$. Suppose T_1 and T_2 are ω-cocontinuous endofunctors on $\mathcal{C}$. Then both $T_1 \dot{+} T_2$ and $T_1 \times T_2$ are ω-cocontinuous.*

Proof. We begin with $T_1 \dot{+} T_2$. It is clearly sufficient to show that if

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow \cdots \longrightarrow A$$

and

$$B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_i \longrightarrow \cdots \longrightarrow B$$

are colimits, then

$$A_0 \dot{+} B_0 \longrightarrow A_1 \dot{+} B_1 \longrightarrow \cdots \longrightarrow A_i \dot{+} B_i \longrightarrow \cdots \longrightarrow A \dot{+} B$$

is a colimit. Suppose $f^i_j\colon A_i \longrightarrow A_j$ for $i \le j$ and $f^i\colon A_i \longrightarrow A$ and $g^i_j\colon B_i \longrightarrow B_j$ for $i \le j$ and $g^i\colon B_i \longrightarrow B$ are the arrows in the diagrams and the maps to the colimits. Suppose $\{h^i = \langle k^i, l^i\rangle\colon A_i \dot{+} B_i \longrightarrow C\}$ is a cocone. If D_i and E_i are the images of k^i and l^i, respectively, then we have $D_i \subseteq D_{i+1}$ and $E_i \subseteq E_{i+1}$. Moreover, every element of D_i precedes every element of E_i. Let $D' = \bigcup D_i$ and $E' = \bigcup E_i$. Then every element of D' precedes every element of E'. If we now let D and E be the ω-chain completion of D' and E' respectively, it is immediate that ever element of D precedes every element of E. The universal mapping properties of A and B give unique maps $k\colon A \longrightarrow D$ and $l\colon B \longrightarrow E$ such that $k \circ f^i = k^i$ and $l \circ g^i = l^i$. Then the map h defined as the composite $A \dot{+} B \xrightarrow{k \dot{+} l} D \dot{+} E \longrightarrow C$ is the unique map such that $h \circ (f^i \dot{+} g^i) = h^i$.

For products it is sufficient to show that if

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow \cdots \longrightarrow A$$

and

$$B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_i \longrightarrow \cdots \longrightarrow B$$

are colimits, then so is

$$A_0 \times B_0 \longrightarrow A_1 \times B_1 \longrightarrow \cdots \longrightarrow A_i \times B_i \longrightarrow \cdots \longrightarrow A \times B$$

We first do this in ω- **CPO** , which is a cartesian closed category. It follows that the rows and right hand column of

$$\begin{tikzcd}
A_0 \times B_0 \arrow[r] \arrow[d] & A_0 \times B_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_0 \times B_j \arrow[r] \arrow[d] & \cdots \arrow[r] & A_0 \times B \arrow[d] \\
A_1 \times B_0 \arrow[r] \arrow[d] & A_1 \times B_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_1 \times B_j \arrow[r] \arrow[d] & \cdots \arrow[r] & A_1 \times B \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] \\
A_i \times B_0 \arrow[r] \arrow[d] & A_i \times B_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_i \times B_j \arrow[r] \arrow[d] & \cdots \arrow[r] & A_i \times B \arrow[d] \\
\vdots & \vdots & & \vdots & & \vdots \arrow[d] \\
 & & & & & A \times B
\end{tikzcd}$$

are colimits. But then $A \times B$ is the colimit of the double sequence. Now it follows from the dual of the lemma on page 36 of [Lawvere, 1963] that the diagonal sequence is a colimit as well. This completes the proof for ω- **CPO** .

As for ω- $\mathbf{CPO}_\perp$, we observe that the inclusion ω- $\mathbf{CPO}_\perp \subseteq \omega$- **CPO** has a left adjoint and hence preserves products. It is easy to see directly that the inclusion preserves connected colimits since the colimit in **CPO** of a connected diagram in $\mathbf{CPO}_\perp$ has a bottom element and so lies in $\mathbf{CPO}_\perp$. It is immediate that it is the colimit there. □

From this we see that power functors and both finite discrete and finite ordinal sums of such functors are examples.

7 Examples

Here are two contrasting examples that show that the existence of a morphism l as described in Theorem 4.6 (or some other condition, at least) is necessary. Both are on the category ω- **CPO** . In the first we let $T_\perp X = 1 \dot{+} X$, the ordinal sum of a single point and the ω- **CPO** X, with the single point at the bottom. In this case $T_\perp 0 = 1$ and so the map $l: 1 \longrightarrow T_\perp 0$ is the identity, the only thing it can be. The map $h: T_\perp 0 \longrightarrow T_\perp 1$ takes the added point to added point, meaning the bottom. Thus the composite $T_\perp h \circ l \circ g$ takes both elements of $T_\perp 1$ to the bottom so that the inequality $T_\perp h \circ l \circ g < \mathrm{id}$ is immediate. The conditions of the theorem are satisfied and it is not hard to see that the initial algebra and terminal coalgebra are each the one point compactification of $\mathbf{N}$.

The second example is given by $T_{\top}X = X \dot{+} 1$ which is like $T_{\perp}$ except that the added point is put on top. There is still only one morphism $l: 1 \longrightarrow T_{\top}0$, but $T_{\top}h$ is the top map and it is not true that $T_{\perp}h \circ l \circ g \le \mathrm{id}$. And, sure enough, the initial algebra in this case is the negative integers, while the terminal coalgebra turns out to be its one point compactification, that is the negative integers with a point $-\infty$ added.

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