Geodesic spaces: Euclid's five postulates as an equational theory, starting with the second

Vaughan Pratt
Stanford University

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1. The five postulates of Euclid Book I

I Two points determine a line segment.

II A line segment may be extended indefinitely.

III Two points determine a circle.

IV All right angles are congruent.

V Inclined lines meet.

2. Foundations of Geometry/Synthetic Geometry: The program of modernizing Euclid

Pasch (1882) *Vorlesungen Uber Neuere Geometrie*, Teubner. (Ten axioms for ordered geometry)

Hilbert (1899) *Grundlagen der Geometrie* (92 pp) Teubner (2nd order)

Veblen (1904) A system of axioms for geometry, *TAMS*.

Pieri (1908) La geometria elementare istituita sulle nozioni di punto e sfera, *Mem. di Mat.*

Huntington (1913) A set of postulates for abstract geometry, *Math. Ann.*

Forder, 1927, *The Foundations of Geometry*, CUP.

Tarski (1959), What is elementary geometry? In *The Axiomatic Method*, NH. (1st order theory of between(x, y, z) and congruent(w, x, y, z).)

Robinson (1959), *The Foundations of Geometry* U. Toronto P.

Common underlying framework: logic according to Frege.

3. Within last third of a century

- Szmielew (1981/3) *From affine to Euclidean geometry*, Springer.

- Avigad (2009) "A Formal System for Euclid's Elements," *RSL*

- Beeson (2012) "Euclidean Constructive Geometry (ECG)."
(intuitionistic, provable implies ruler-and-compass-constructible)

Szmielew's is closer to our work in that it emphasizes the affine part and also is more algebraic. *But* her notion of algebra is the same numeric one as informs linear algebra as developed in these seminal works.

- Grassmann (1844/1862) *Die Lineale Ausdehnungslehre*, Wiegand/Enslin

- Gibbs/Wilson (1881/1901) *Vector Analysis*, Yale

- Heaviside (1884) "Electromagnetic induction and its propagation", *The Electrician*

4. Our goal

The goal of the present work:

To make Euclid's synthetic geometry just as algebraic, hence *constructive*, as Descartes' analytic geometry has become,

while respecting Euclid's avoidance of numbers.

In particular, not allowed to *explicitly* postulate a field.

Rational numbers had presumably been in use in surveying, architecture, and other applications of geometry for thousands of years before Euclid. In an appendix to a 1926 edition of Pasch's *Vorlesungen* coauthored with Max Dehn, Dehn wrote that the absence of numbers in the first four books of the *Elements* may have been a reaction to the then-recent discovery of irrational numbers.

5. The five postulates of Euclid Book I

I Two points determine a line segment.

II A line segment may be extended indefinitely.

III Two points determine a circle

IV All right angles are congruent.

V Inclined lines meet.

5. The five postulates of Euclid Book I made more constructive with algebra

I Two points determine a line segment
namely their convex hull.

II A line segment A, B may be extended
to twice its length by making a copy.
(Hence extending AB to C creates *two* segments, AC and BC.

III Two points A, B determine a ~~circle~~ sphere consisting of either
all points C s.t. $|AC| = |AB|$ (radius AB, using a compass), or
all points C s.t. $AC \perp CB$ (diameter AB, using a set square).

IV All right angles are congruent
and arise at the foot of a perpendicular.

V Inclined lines meet
at the extension of the trapezium witnessing inclination.

NB: Continuity is not assumed! This departs from prior approaches.

6. Euclid's ordering of his postulates

Euclid's ordering seems somewhat *ad hoc.*

Uncertainty as to the independence of the fifth postulate would appear to have led him to put it last, and to rely on it as little as possible.

He postulated line segments before circles, but used them in the other order: proposition 1 constructs an equilateral triangle from one side by intersecting two sweeps of a compass.

Why postulate 3 precedes 4 is a good question. One possibility is that he saw it as the power tool among his postulates. Its constructive nature was also clearer: "all right angles are congruent" is the least constructive of all the postulates.

7. Our reordering: to affinity and beyond

Our algebraic formulation of Euclid's postulates forces them into a unique linear order.

Euler was the first to realize, two millennia after Euclid, that postulates 1, 2, and 5 define affine geometry, which 3 and 4 expand on by introducing notions of distance and angle. This creates a natural separation: 125+34.

Postulates 2 and 5 are independent of any notion of density and so logically come first, with 2 as a prerequisite for 5. We base them on a binary operation $e(A, B)$ of *extension* of a line segment AB yielding a point C such that $|AC| = 2|AB|$, thereby creating a sublanguage of that of abelian groups sufficient to express a discrete form of affine geometry

Postulate 1 introduces density in terms of n-ary centroid $c_n(A_1, \ldots, A_n)$, defined by induction on n but expanded out so as to define a variety, albeit without a finite axiomatization. We therefore reorder 1,2,5 as 2,5,1, bringing the affine portion to a natural conclusion.

8. Beyond affinity

Szmielew 1981: "From affine to Euclidean geometry."

Challenge: accomplish the affine-to-Euclidean passage
(i) algebraically
(ii) without introducing numbers.

Our language: four ternary operations p, n, t, c.

Associated notions: *p*erpendicular, *n*ormalize, *t*angent, *c*ircumcenter.

We *derive* the foregoing affine operations e and c_n as a sublanguage.

Since p concerns right angles while n, t, c concern circles, it is natural to associate p with Postulate 4 and n, t, c with Postulate 3.

The final order is then as follows, with the affine portion delimited by $+$.

$$2 \;\; 5 \;\; 1 \;\; + \;\; 4 \;\; 3$$

9. Geodesic spaces

A **geodesic space** is an algebraic structure (X, e) with a binary operation $e(x, y)$ satisfying the following equations. (Abbreviations: xy for $e(x, y)$, xyz for $(xy)z$.)

$$xx = x \tag{1}$$

$$xyy = x \tag{2}$$

$$xyz = xz(yz) \tag{3}$$

Canonical example: The underlying set of a group, with $e(x, y)$ taken to be $yx^{-1}y$ in the group.

(Caveat: Distinguish from the quandle operation pair $y^{-1}xy$, yxy^{-1}.)

10. The free geodesic spaces

Theorem

Every geodesic term over a given set of variables is equivalent to exactly one term containing neither parentheses nor adjacent repetitions.

Corollary

Terms over two variables x, y code the integers as

$$\ldots, yxyx, xyx, yx, x, y, xy, yxy, xyxy, \ldots$$

x and y are 0 and 1, parity on the left, sign on the right.

Write $e_n(x, y)$ for the n-th term in this sequence, where $e_2 = e$.

Hence e_0 and e_1 are the projections.

In general $e_{n+1}(A, B) = e(e_n(B, A), B)$, $e_{-n} = e_{n+1}(B, A)$.

In a free geodesic space every doubleton generates (a copy of) the affine integers.

In a set $(xy = x)$ every subset generates just itself.

11. Groves

A grove is a geodesic space satisfying

$$(wx)(yz) = (wy)(xz) \tag{3'}$$

This implies, and therefore replaces, the right distributivity axiom $xyz = xz(yz)$, so still only three equations.

Geometrical interpretation: flatness (analogous to abelian groups).

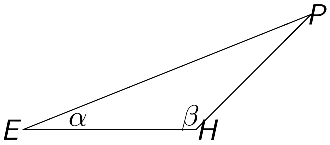
Canonical example: The underlying set of an additive (= abelian) group, with $e(x, y)$ taken to be $2y - x$ in the group.

Fifth postulate (strengthened to both directions) as a single equation!

12. Interchange as the Fifth Postulate

P5: Inclined lines intersect.

Euclid's criterion for "inclined":
$\alpha + \beta < 180^o$.



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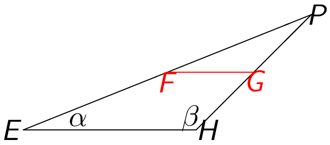

Our criterion: a *witness trapezium* $FGHE$ s.t. $FG||EH$ and $|EH| = 2|FG|$.
The axiom $e(E, F) = e(H, G)$ constructs the intersection of EF and HG.

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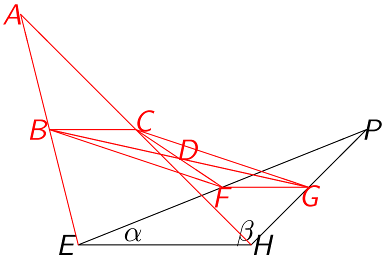

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The trapezium is further witnessed by a copy $ACHEB$ of $PGHEF$ defined by an arbitrary point A (not necessarily in the same plane), such that $E = e(A, B)$, $F = e(C, D)$, $H = e(A, C)$, and $G = e(B, D)$.

Here D is the midpoint of the parallelogram $BCGF$ obtained from the quadrilateral $AHPE$ by Varignon's theorem.

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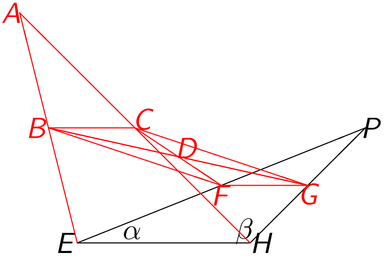

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Postulate 5, that inclined lines meet, then becomes

$$e(e(A, B), e(C, D)) = e(e(A, C), e(B, D)).$$

13. Centroids

Every finite nonempty list of points of a grove has a set of centroids defined inductively as follows.

1. A singleton is its own set of centroids.

2. Given a nonempty list Γ of $n-1$ points, Y is a centroid of Γ, C when Γ has a centroid X satisfying $e_n(X, Y) = C$.

Example: the centroid Y of a triangle ΔABC with apex C satisfies $e_3(X, Y) = C$ (i.e. $e(e(Y, X), Y) = C$) where X is the midpoint of AB.

Weighted centroids are possible via repetition: centroid(A, A, B) is a third of the way along AB.

In a set $(e(A, B) = A)$, only the constant lists $A, A, A, \ldots$ have any centroids, namely A itself.

14. Centroidal groves

A grove is **centroidal** when every finite nonempty set of elements has a unique centroid.

Theorem

The full subcategory of **Grv** *consisting of the centroidal groves is equivalent to the category* $\mathbf{Aff}_{\mathbb{Q}}$ *of affine spaces over the rationals.*

Theorem

n-ary centroid for each $n \geq 2$ *is definable as a family of operations* $c_n(x_1, \ldots, x_n)$ *(*$c = c_2$*) satisfying two equations per arity.*

Language of **affine spaces**: normally weighted linear combinations, e.g. $2y - x$, $x/2 + y/2$.

Language of **centroidal groves**: extension and centroid.

Analogy: Boolean rings $(\times, +, -, 1)$ and Boolean lattices $(\wedge, \vee, \neg, 1)$: two languages, axiomatized quite differently, yet defining equivalent categories.

16. From affine to Euclidean geometry

Whereas affine geometry forms a variety, Euclidean geometry cannot because $\mathbb{E}^2$ is not the direct square of $\mathbb{E}^1$. Partiality could cause this.

Of our four Euclidean operations p, n, c, t, only p and n are total:

$c(A, B, C)$ is undefined when A, B, C collinear and $A \neq B \neq C \neq A$;
$t(A, B, C)$ is undefined when $|AC| > |BC|$ and A, B, C are collinear, or when $|AC| < |AB|$.

Hence in dimension 1 or less,
$p(A, B, C) = C$;
$n(A, B, C) = e(A, B)$ when $\mathrm{Bet}(B, A, C)$, else B;
$c(A, A, B) = c(A, B, A) = c(B, A, A) = c_2(A, B)$, c undefined elsewhere;
$t(A, B, B) = t(A, e(B, A), B) = B$, t undefined elsewhere.

17. Perp

$p(A, B, C)$ yields the foot of the perpendicular dropped from C to the line AB.

Total because the line AA is a point, as before, so $p(A, A, B) = A$. (Picture AA as a line segment seen end-on.)

A fragment of the theory using p alone:

$$\begin{aligned} p(A, B, C) &= p(B, A, C) \\ p(A, B, B) &= B \\ p(A, A, B) &= A \\ p(C, p(A, B, C), A) &= p(A, B, C) \end{aligned}$$

18. Norm

$n(A, B, C)$ yields the point nearest to C on the sphere with center A and radius AB.

Normalizes AC to the same length as AB, independently of the direction of AC.

Undefined when $C = A \neq B$.

More theory fragments.

When C is moved around, norm(A, B, C) traces out the sphere with radius AB.

$$\begin{aligned} n(A, A, B) &= A \\ n(A, B, B) &= B \\ n(A, B, A) &= \perp \\ n(A, B, n(A, B, C)) &= n(A, B, C) \\ e(n(A, B, C), A) &= n(A, B, e(C, A)) \end{aligned}$$

19. Tangent

tangent(A, B, C) is the nearest point T to B satisfying any (hence all) of the following equivalent conditions:

(i) T is a tangent point from C on the sphere AB;

(ii) T is on the sphere formed as the intersection of the sphere with radius AB and the sphere with diameter AC;

(iii) AB is the hypotenuse of a right triangle ATB with side AT equal to AB.

21. Independence and basis

Subspace = subalgebra (same as in linear algebra).

Isometry = isomorphism (different from linear algebra)

A set of points is

• **independent** when no proper subset generates the same space (note constructivity), and

• **spherical** when its members lie on a common sphere; formally, when there exist points A, B such that for all members C, $n(A, B, C) = C$ (radius AB), equivalently $t(A, C, B) = C$ (diameter AB).

A (Euclidean) **basis** is a spherical independent set.

(In this algebraic treatment, we are always working in some subspace of space, and have no need of the concept of "the whole space".)

22. Barycentric and cartesian bases

The difference between barycentric and cartesian coordinates is a language one, namely the presence of a constant O (the origin) in the language of the latter.

The rectangular shape of the Pyramid at Djoser (2700 BCE), and of most city buildings since, suggests that the latter has been in common use for at least five thousand years. Our axiomatization works equally well for both *with no change in axioms!* The following assumes the latter as being more familiar.

An n-element basis generates an n-dimensional space.

An axis for a basis is the space generated by an element of the basis serving as the unit vector for that axis.

All axes are isometric.

23. The Gram-Schmidt process

Given a basis $B_1, \ldots, B_n$, the i-th **coordinate** of a point A is the projection $p(O, B_i, A)$ of A onto the axis OB_i.

Since we are always working in a subspace, any new point arising as e.g. the value of a variable not previously considered needs to be tested for whether it is the true point A' at those coordinates.

If not, the new point A is used to expand the basis with $n(O, B, e(A', c(O, A)))$ where B is any extant member of the basis. This is the difference between A and A' translated to O and normalized to unit length, constituting the Gram-Schmidt orthonormalization process as adapted to our circumstances.

24. Euclidean metric

Since all axes are isometric, any one of them can serve as a ruler to establish a metric. In the following 1 refers to the unit vector in an axis performing that duty. We assume at least two dimensions, equivalent to the existence of a point Z not on the axis $O1$.

Theorem

O, 1, and Z generate the constructible line containing O and 1.

Proof.

O and 1 freely generate the rational line using just the affine fragment. The affine operations suffice to keep the line closed under addition, and to negate as needed. If A, B realize $a, b > 0$ then $n(O, p(O, t(O, n(O, A, Z), B), 1), 1)$ realizes a/b. For $a \geq 1$, if B realizes $(a+1)/2$ then $n(O, t(B, n(B, 1, Z), O), 1)$ realizes $\sqrt{a}$. For $0 < a < 1$ obtain $\sqrt{a}$ as $1/\sqrt{1/a}$. □

Note the use of both n and t in the realizations of both a/b and $\sqrt{a}$.

25. Non-Euclidean geometry

Idea: (i) restate our 5th postulate, interchange $(wx)(yz) = (wy)(xz)$, as the equivalent form $wxyzxyz = w$.

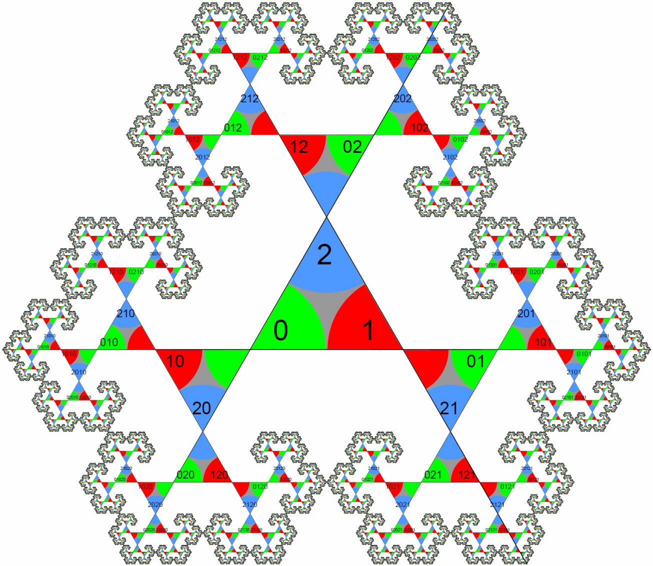
(ii) Vary the 5th as follows:

$$wxyz = w \tag{3}$$
$$wxyzx = w \tag{4}$$
$$wxyzxy = w \tag{5}$$
$$wxyzxyz = w \tag{6}$$
$$wxyzxyzx = w \tag{7}$$

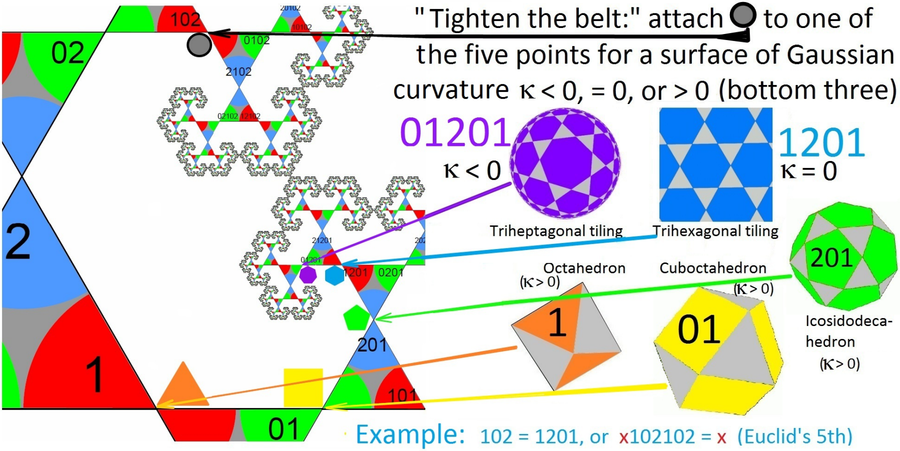
Sign of Gaussian curvature = sign(6 - eqn.no.).

Equation 6 expresses flatness of space just as done by commutativity for groups.

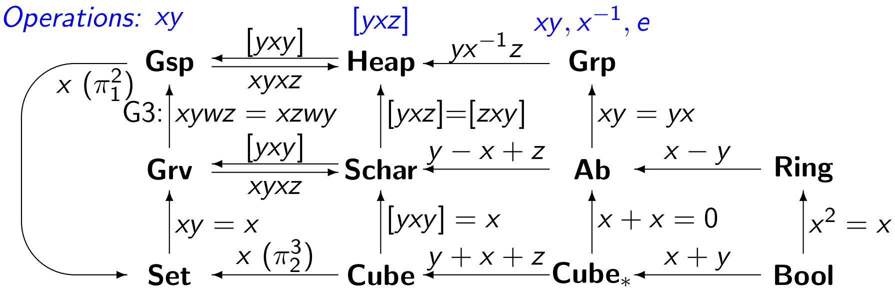
26. Geodesic plane = free geodesic space on 3 generators



27. Non-Euclidean geometry in the geodesic plane



28. The geodesic neighborhood



Every path in this commutative diagram denotes a forgetful functor, hence one with a left adjoint. Vertical arrows *forget* the indicated *equation*, horizontal arrows *retain* the indicated *operation*. E.g. the left adjoint of the functor $U_{\mathbf{AbGrp}} : \mathbf{Ab} \to \mathbf{Grp}$ is abelianization, that of the functor $U_{\mathbf{SetGsp}} : \mathbf{Set} \to \mathbf{Gsp}$ gives the set $F_{\mathbf{GspSet}}(S)$ of connected components of S, and so on.