# ENRICHED CATEGORIES AND THE FLOYD-WARSHALL CONNECTION

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**Abstract**

We give a correspondence between enriched categories and the Gauss-Kleene-Floyd-Warshall connection familiar to computer scientists. This correspondence shows this generalization of categories to be a close cousin to the generalization of transitive closure algorithms. Via this connection we may bring categorical and 2-categorical constructions into an active but algebraically impoverished arena presently served only by semiring constructions. We illustrate these techniques by applying them to Birkoff's poset arithmetic, interpretable as an algebra of "true concurrency."

The Floyd-Warshall algorithm for generalized transitive closure [AHU74] is the code fragment

$$\textbf{for } v \textbf{ do } \textbf{for } u,w \textbf{ do } \delta_{uw} += \delta_{uv} \cdot \delta_{vw}.$$

Here $\delta_{uv}$ denotes an entry in a matrix $\delta$, or equivalently a label on the edge from vertex $u$ to vertex $v$ in a graph. When the matrix entries are truth values 0 or 1, with + and $\cdot$ interpreted respectively as $\vee$ and $\wedge$, we have Warshall's algorithm for computing the transitive closure $\delta^+$ of $\delta$, such that $\delta^+_{uv} = 1$ just when there exists a path in $\delta$ from $u$ to $v$. When the entries are nonnegative reals, with + as min and $\cdot$ as addition, we have Floyd's algorithm for computing all shortest paths in a graph: $\delta^+_{uv}$ is the minimum, over all paths from $u$ to $v$ in $\delta$, of the sum of the edges of each path.

Other instances of this algorithm include Kleene's algorithm for translating finite automata into regular expressions, and Gauss's algorithm for inverting a matrix, in each case with an appropriate choice of semiring.

Not only are these algorithms the same up to interpretation of the data, but so are their correctness proofs. This begs for a unifying framework, which is found in the notion of semiring. A semiring is a structure differing from a ring principally in that its additive component is not a group but merely a monoid, see AHU [AHU74] for a more formal treatment.

Other matrix problems and algorithms besides Floyd-Warshall, such as matrix multiplication and the various recursive divide-and-conquer approaches to closure, also lend themselves to this abstraction.

This abstraction supports mainly vertex-preserving operations on such graphs. Typical operations are, given two graphs $\delta, \epsilon$ on a common set of vertices, to form their pointwise sum $\delta + \epsilon$ defined as $(\delta + \epsilon)_{uv} = \delta_{uv} + \epsilon_{uv}$, their matrix product $\delta\epsilon$ defined as $(\delta\epsilon)_{uv} = \delta_{u-} \cdot \epsilon_{-v}$ (inner product), along with their transitive, symmetric, and reflexive closures, all on the same vertex set.

We would like to consider other operations that combine distinct vertex sets in various ways. The two basic operations we have in mind are the disjoint union and cartesian product of such graphs, along with such variations of these operations as pasting (as not-so-disjoint union), concatenation (as a disjoint union with additional edges from one component to the other), etc.

An efficient way to obtain a usefully large library of such operations is to impose an appropriate categorical structure on the collection of such graphs. In this paper we show how to use enriched categories to provide such structure while at the same time extending the notion of semiring to the more general notion of monoidal category. In so doing we find two layers of categorical structure:

enriched categories in the lower layer, as a generalization of graphs, and ordinary categories in the upper layer having enriched categories for its objects. The graph operations we want to define are expressible as limits and colimits in the upper (ordinary) categories.

We first make a connection between the two universes of graph theory and category theory. We assume at the outset that vertices of graphs correspond to objects of categories, both for ordinary categories and enriched categories. The interesting part is how the edges are treated.

The underlying graph $U(C)$ of a category $C$ consists of the objects and morphisms of $C$, with no composition law or identities. But there may be more than one morphism between any two vertices, whereas in graph theory one ordinarily allows just one edge. These "multigraphs" of category theory would therefore appear to be a more general notion than the directed graphs of graph theory.

A staple of graph theory however is the label, whether on a vertex or an edge. If we regard a homset as an edge labeled with a set then a multigraph is the case of an edge-labeled graph where the labels are sets. So a multigraph is intermediate in generality between a directed graph and an edge-labeled directed graph.

So starting from graphs whose edges are labeled with sets, we may pass to categories by specifying identities and a composition law, *or* we may pass to edge-labeled graphs by allowing other labels than sets. What is less obvious is that we can elegantly and usefully do both at once, giving rise to enriched categories. The basic ideas behind enriched categories can be traced to Mac Lane [Mac65], with much of the detail worked out by Eilenberg and Kelly [EK65], with the many subsequent developments condensed by Kelly [Kel82]. Lawvere [Law73] provides a highly readable account of the concepts.

We require of the edge labels only that they form a *monoidal category.* Roughly speaking this is a set bearing the structure of both a category and a monoid. Formally a *monoidal category* $\mathcal{D} = \langle D, \otimes, I, \alpha, \lambda, \rho \rangle$ is a category $D = \langle D_0, m, i \rangle$, a functor $\otimes: D^2 \to D$, an object $I$ of $D$, and three natural isomorphisms $\alpha: c \otimes (d \otimes e) \to (c \otimes d) \otimes e$, $\lambda: I \otimes d \to d$, and $\rho: d \otimes I \to d$. (Here $c \otimes (d \otimes e)$ and $(c \otimes d) \otimes e$ denote the evident functors from $D^3$ to $D$, and similarly for $I \otimes d$, $d \otimes I$ and $d$ as functors from $D$ to $D$, where $c, d, e$ are variables ranging over $D$.) These correspond to the three basic identities of the equational theory of monoids. To complete the definition of monoidal category we require a certain coherence condition, namely that the other identities of that theory be "generated" in exactly one way from these, see Mac Lane [Mac71] for details.

A $\mathcal{D}$-*category*, or (small) *category enriched in* a monoidal category $\mathcal{D}$, is a quadruple $\langle V, \delta, m, i \rangle$ consisting of a set $V$ (which we think of as vertices of a graph), a function $\delta: V^2 \to D_0$ (the edge-labeling function), a family $m$ of morphisms $m_{uvw}: \delta(u,v) \otimes \delta(v,w) \to \delta(u,w)$ of $D$ (the composition law), and a family $i$ of morphisms $i_u: I \to \delta(u,u)$ (the identities), satisfying the following diagrams.

$$\begin{array}{ccccc}
(\delta(u,v) \otimes \delta(v,w)) \otimes \delta(w,x) & & \xrightarrow{\alpha_{\delta(u,v)\delta(v,w)\delta(w,x)}} & & \delta(u,v) \otimes (\delta(v,w) \otimes \delta(w,x)) \\
\downarrow {\scriptstyle m_{uvw} \otimes 1} & & & & \downarrow {\scriptstyle 1 \otimes m_{vwx}} \\
\delta(u,w) \otimes \delta(w,x) & \xrightarrow{m_{uwx}} & \delta(u,x) & \xleftarrow{m_{uvx}} & \delta(u,v) \otimes \delta(v,x)
\end{array}$$

$$
\begin{CD}
I \otimes \delta(u,v) @>{\lambda_{\delta(u,v)}}>> \delta(u,v) @<{\rho_{\delta(u,v)}}<< \delta(u,v) \otimes I \\
@V{i_u \otimes 1}VV @| @VV{1 \otimes i_v}V \\
\delta(u,u) \otimes \delta(u,v) @>{m_{uuv}}>> \delta(u,v) @<{m_{uvv}}<< \delta(u,v) \otimes \delta(v,v)
\end{CD}
$$

Inspection reveals the first of these as expressing abstractly the associativity of composition and the second as expressing the behavior of identities.

Associated with the notion of $\mathcal{D}$-category is that of $\mathcal{D}$-functor $F\colon A \to B$ where $A$ and $B$ are $\mathcal{D}$-categories. This is just like an ordinary functor for its object part, mapping objects of $A$ to objects of $B$ via $f\colon ob(A) \to ob(B)$. The usual morphism part of a functor now becomes a family $\tau_{uv}\colon \delta_A(u,v) \to \delta_B(fu,fv)$ of morphisms of $\mathcal{D}$:

$$
\begin{array}{ccccc}
u & & \xRightarrow{\quad \delta_A(u,v) \quad} & & v \\
f \Big\downarrow & & \tau_{uv} \Big\downarrow & & f \Big\downarrow \\
fu & & \xRightarrow{\quad \delta_B(fu,fv) \quad} & & fv
\end{array}
$$

which compose vertically in the obvious way.

The class of all $\mathcal{D}$-categories and $\mathcal{D}$-functors then forms a (large) category, called $\mathcal{D}$-**Cat**.

The category **Cat** of all small categories can now be seen to be **Set**-**Cat**. Rendering this abstraction more accessible and appealing is the very pretty case $\mathcal{D} = \mathbf{R}^{op}_{\geq 0} = \langle\langle R_{\geq 0}, \geq\rangle, +, 0\rangle$, reverse-ordered nonnegative reals under addition, for which **R**-**Cat** becomes the category of (generalized) metric spaces, with the composition law as the triangle inequality and functors as contracting maps [Law73]. Enriched categories first appeared in computer science with $\mathcal{D} = \mathbf{Poset} = \langle Poset, \times, 1\rangle$ [Wan79] yielding order-enriched categories, a natural notion for domain theory. **Poset** itself is definable as (the antisymmetric subcategory of) $\langle\langle\{0,1\}, \to\rangle, \wedge, 1\rangle$-**Cat**, categories enriched in truth-values.

We may now make the connection with semirings. The enriching monoidal category $\langle D, \otimes, I, \alpha, \lambda, \rho\rangle$ has for $D_0$ the set of edge labels, for $\otimes$ the semiring multiplication, and for its coproduct (which therefore needs to exist in $\mathcal{D}$) the semiring addition. The usual requirement of distributivity of multiplication over addition is met when when $\mathcal{D}$ is *biclosed*—$\otimes$ has a right adjoint in both arguments—with $\mathcal{D}$ *closed* corresponding to one-sided distributivity. (In these situations $\mathcal{D}$ *cartesian* closed is the exception rather than the rule.)

Although the literature has tended to make enriched categories seem if anything more abstract and forbidding than ordinary categories to most computer scientists, this perspective puts enrichment in quite a different light for those familiar with the Floyd-Warshall connection. For $\mathcal{D}$ a preorder with finite coproducts, enriched categories simply become the reflexive and transitive edge-labeled graphs output by the Gauss-Kleene-Warshall-Floyd algorithm. For $\mathcal{D}$ not a preorder, such as **Set** or **Cat**, yielding respectively ordinary categories and 2-categories, the notion becomes more involved (to which a categoriphobe might say "Ah, so that's the problem") but necessarily so for Gauss's algorithm, whose semiring addition is not idempotent.

This is a nice perspective in its own right, but it becomes considerably more useful when the 2-categorical structure of $\mathcal{D}$-**Cat** is brought to bear on the description of particular algebras. We illustrate this by applying it to the categorical treatment of Birkhoff's arithmetic of posets [Bir42] and its generalization to other metrics besides the truth-valued metric used for posets. This arithmetic provides a nice abstraction of the sort of concurrency operations we have been advocating [Pra86] to make the "true concurrency" or partially-ordered-time approach more algebraic

Birkhoff defines six operations on posets: addition, multiplication, and exponentiation, each in a cardinal and an ordinal version, as a way of unifying cardinal and ordinal arithmetic. (In the concurrency connection cardinal vs. ordinal corresponds to parallel vs. sequential.) The cardinal operations are conveniently described as universals in **Poset**, the ordinals not quite so conveniently categorically, but 2-categorically ordinal addition becomes just cocomma, indicating that the move from parallel to sequential can usefully be accompanied by a move from categories to 2-categories.

Birkhoff arithmetic admits useful generalizations to other semirings *qua* monoidal categories, suitable for modelling real-valued time in various forms: upper bounds, lower bounds, intervals, and arbitrary sets of reals, each associated with a specific monoidal category, *but with the definitions of the associated arithmetic operations unchanged*. These generalizations in turn suggest additional constructs, also definable universally, that would have been meaningless or degenerate in Birkhoff's original framework, but that have useful applications to the specification of real-time processes.

The prospect of a connection with Girard's linear logic obliges us to point out that as both an expansion and a nonconservative extension of the above theory, linear logic with negation is too strong for the purposes of making the connections of this paper, which are more appropriately described as aspects of a fragment of linear logic.

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