Euclid's Elements as an Equational Theory

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Historical background (*selon* Max Dehn 1926)

Fact 1. *The positive rationals* $\frac{m}{n}$ *are* **dense.**

That is, given rationals $a < c$, there exists some rational b satisfying $a < b < c$, for example $b = (a + c)/2$.

Fact 2. *The square of a positive integer, when written in binary, ends in an even number of zeros.*

True, very plausible, but (full disclosure) not entirely obvious.

Corollary 3. $\sqrt{2}$ *is irrational.*

For if not, i.e. if $\sqrt{2} = \frac{m}{n}$, then $m^2 = 2n^2$. But the two sides written in binary end in respectively an even and odd number of zeros, contradicting their equality.

Historical background (*selon* Max Dehn 1926, cont.)

The Pythagoreans were a school of mathematical thought that flourished in the 5th century BC.

Fact 1, density of the rationals, would be a plausible reason for them to take as axiomatic that every positive real was rational. (How could there possibly be any room for yet more numbers? And even if there were, what would be the point of such monsters?)

Corollary 3, incommensurability of $\sqrt{2}$ (the diagonal of the unit square), is traditionally attributed to the Pythagorean Hippasus of Metapontum.

This discovery would plausibly have been as great a shock to his fellow Pythagoreans as Cantor's discovery in the 19th century of the uncountability of the continuum was to his fellow mathematicians.

Historical background (*selon* Max Dehn 1926, cont.)

Euclid's *Elements* (c. 300 BC) consisted of 13 books on geometry, geometric algebra, and number theory.

Book 1 bases the planar geometry of lines and circles on five postulates governing geometric constructions limited to compass and straightedge.

Although numbers appear in books 5-13, *no numbers of any kind, rational or otherwise, are mentioned in Books 1-4*.

Yet given any two line segments, of whatever lengths, Euclid is able to construct a third segment whose length is that of their geometric mean. (So in particular $\sqrt{2}$ is **constructible** in that sense.)

Max Dehn has speculated (1926) that Book 1 was in response to the crisis precipitated by the discovery that the square had an incommensurable diagonal, or as we say today, $\sqrt{2}$ is irrational.

Foundations of Geometry; or, modernizing Euclid

Common underlying framework: logic according to Frege (1879).

Pasch (1882) *Vorlesungen Uber Neuere Geometrie*, Teubner. (Ten axioms for ordered geometry)

Hilbert (1899) *Grundlagen der Geometrie* (92 pp) Teubner (2nd order)

Veblen (1904) A system of axioms for geometry, *TAMS*.

Pieri (1908) La geometria elementare istituita sulle nozioni di punto e sfera, *Mem. di Mat.*

Huntington (1913) A set of postulates for abstract geometry, *Math. Ann.*

Forder, 1927, *The Foundations of Geometry*, CUP.

Tarski (1959), What is elementary geometry? In *The Axiomatic Method*, NH. (1st order theory of between(x, y, z) and congruent(w, x, y, z).)

Robinson (1959), *The Foundations of Geometry* U. Toronto Press

Within last third of a century

- Szmielew (1981/3) *From affine to Euclidean geometry*, Springer.
- Avigad (2009) "A Formal System for Euclid's Elements," *RSL*
- Beeson (2012) "Euclidean Constructive Geometry (ECG)" (intuitionistic, provable implies ruler-and-compass-constructible)

Of these, Szmielew's is closest to our goal in that it separates out the affine part and also is more algebraic.

But her notion of algebra is the same numeric one as informs linear algebra as developed in these older but seminal works:

- Grassmann (1844/1862) *Die Lineale Ausdehnungslehre*, Wiegand/Enslin
- Gibbs/Wilson (1881/1901) *Vector Analysis*, Yale
- Heaviside (1884) "Electromagnetic induction and its propagation", *The Electrician*

Our goal

To make Euclid's synthetic geometry just as algebraic as Descartes' analytic geometry has become, but without reference to numbers.

The modern understanding of analytic geometry is in terms of vector spaces defined by equations involving scalars drawn from a given field.

But Book 1 of the *Elements* makes no mention of any field, or even of numbers.

This raises the question, what is a Euclidean space in the sense understood by Euclid? In particular can it be defined equationally without bringing in numbers?

Our answers: The affine fragment (no angles or congruences) is easy, the full Euclidean notion less so but some progress is possible.

Analogy: Boolean algebra

A Boolean algebra can be defined equivalently as either

- a Boolean ring $(B, +, \times, 0, 1)$, namely a ring satisfying $x^2 = x$; or
- a complemented distributive lattice $(B, \wedge, \vee, \neg, 0)$ (14 equations).

The former is arithmetical in character, the latter logical.

Halmos preferred the former definition for his *Lectures on Boolean algebras*.

Courses in logic tend to start with the latter.

Affine space

The variety $\mathbf{Aff}_{\mathbb{Q}}$ of affine spaces over the rationals can be defined equivalently with either an arithmetic or geometric signature:

Arithmetic: rational linear combinations whose coefficients sum to unity. Sublanguage of $\mathbf{Vct}_{\mathbb{Q}}$ (*all* linear combinations).

Presumes the prior definition of a specific field $\mathbb{Q}$.

Our contribution:

Geometric: (i) Extension $C = e(A, B)$ of segment AB
s.t. $|AC| = 2|AB|$.
(ii) Centroid $c_n(A_1, \ldots, A_n)$ for $2 \leq n < \omega$.

No mention of $\mathbb{Q}$*!*

Surely this contradicts the Löwenheim-Skolem theorem. . . ?

Geodesic spaces

A geodesic space, or just space, is an algebraic structure (S, e) with a binary operation $e(x, y)$, or just xy with xyz abbreviating $(xy)z$, satisfying

$$xx = x \tag{A1}$$
$$xyy = x \tag{A2}$$
$$xyz = xz(yz) \tag{A3}$$

A ***geodesic*** is a subspace (= subalgebra) properly generated by two points. (2 antipodal points on a sphere do not form a geodesic.)

Two points are ***connected*** when they belong to a common geodesic.

Geometric interpretation:

(11) Every point is a subspace.

(22) Every subspace is symmetric.

(33) Geodesics are preserved under reflection in z.

The category **Gsp**

Geodesic spaces and their homomorphisms standardly defined form a variety or algebraic category **Gsp**.

Examples.

Symmetric spaces: Affine, hyperbolic, and elliptic spaces.

Groups: Interpret $e(x, y)$ as $yx^{-1}y$ (abelian groups: $2y - x$). (AKA ***kei*** (Takahasi 1943) but not ***racks*** or ***quandles*** (Wraith-Conway 1959) which have a pair of operations interpreted as respectively yxy^{-1} and $y^{-1}xy$ and bear on knots.)

Number systems: Integers $\mathbb{Z}$, rationals $\mathbb{Q}$, Gaussian integers, complex rationals $\mathbb{Q}[i]$, as the free space on two generators 0,1 depending on any additional equationally axiomatizable operations such as $((n-1)x + y)/n \quad (\mathbb{Q})$, $(1 - i)x + iy \quad (\mathbb{C})$, etc.

Combinatorial structures: Sets, dice, $\ldots$.

Sets as discrete geodesic spaces

Theorem 9. For any space S the following are equivalent (TFAE):

- S contains no geodesics.
- The connected components of S are its points.
- $xy = x$ for all points x, y in S (e.g. Earth's two poles).

A ***set*** is a geodesic space satisfying any (hence all) of the above three conditions.

Theorem 10. **Set** is a full reflective subcategory of **Gsp** (the embedding preserves limits instead of colimits).

This permits founding set theory on geometry instead of vice versa.

Normal form terms and free spaces

A ***normal form*** geodesic algebra term over a set V of variables is one with no parentheses or stuttering, i.e. a finite nonempty list $x_1 x_2 \ldots x_n$ of variables $x_i \in V$ with no consecutive repetitions.

Theorem 11. All terms are reducible to normal form using axioms A1-A3.

Proof outline: Axiom A3 removes parentheses while A1 and A2 remove repetitions.

Theorem 12. The normal form terms over V form a space.

We denote this space $F(V)$ and call it the ***free space*** on V, consisting of V-ary operations.

$F(\{\}) = \{\}$
$F(\{x\}) = \{x\}$.

F(2)

$F(\{0,1\})$ or $F(2)$ is the *infinite*, *discrete* geodesic, namely

$$\ldots, 1010, 010, 10, 0, 1, 01, 101, 0101, \ldots$$

Call this *geodesimal notation*, amounting to tally notation with sign bit at the right and parity bit at the left.

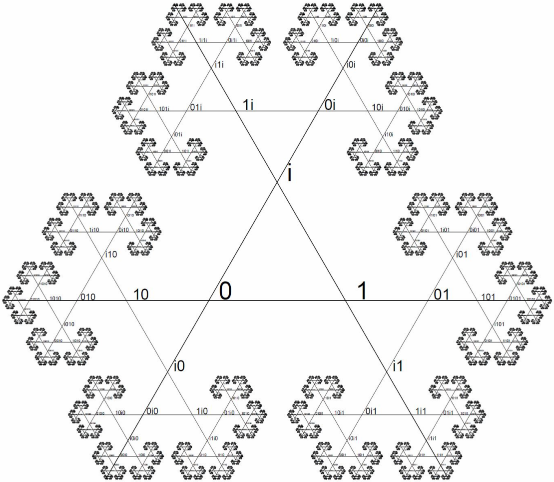
Define $e^n(x,y)$ so as to satisfy

$$\begin{aligned} e^0(x,y) &= x && (1)\\ e^{n+1}(x,y) &= e^{-n}(y,x) && (2)\\ e^{n+2}(x,y) &= e(e^n(x,y),y) && (3) \end{aligned}$$

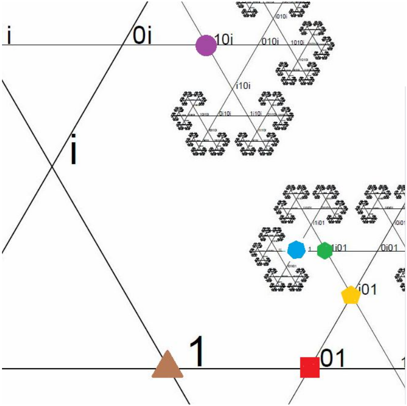
In particular $e^n(0,1)$ denotes n in geodesimal notation.

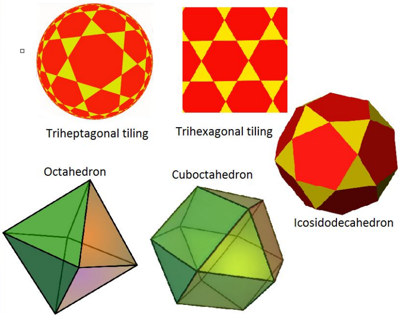
F(3)

Generators 0,1,i. All points out to ∞ are shown, suitably compressed. All triangles shown are congruent yet have undefined angles and sides.



The curvature hierarchy

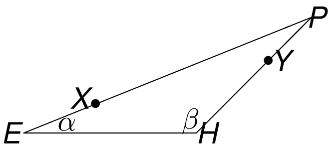




Equations anticlockwise from Octahedron: $wxyz = w$, $wxyzx = w$, $wxyzxy = w$, $wxyzxyz = w$ (or $wxyz = wzyx$), $wxyzxyzx = w$.

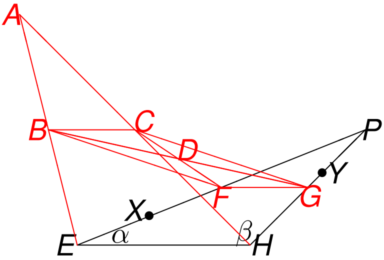
Not shown: Dice ($wxwx = w$), Sets ($wx = w$).

Fifth postulate + converse, as an equation



Euclid's fifth or parallel postulate: *EX* and *HY*, when inclined inwards, meet when *produced*. Euclid: "inclined" = $\alpha + \beta < 180^o$. Necessary for *similar* triangles (only congruent Δ's in curved space).

Fifth postulate + converse, as an equation



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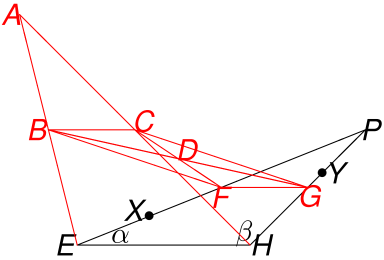
Our inclination condition: a *witness triangle* Δ*AEH* with parallelogram *BCGF* (centroid *D*) s.t *B*, *C* at midpoints of *AE*, *AH*.

Fifth postulate + converse, as an equation



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Our 5th postulate: *EF* and *HG*, when obtained by extending the four sides of the skew quadrilateral *ABDC*, meet when *extended*.

$$\underbrace{e(e(A,B), e(C,D))}_{E \quad F} = \underbrace{e(e(A,C), e(B,D))}_{H \quad G} \tag{4}$$

Equivalent fifth postulate equation

From the previous slide,

$wx(yz) = wy(xz)$ (middle-two interchange)

Apply Axiom A3 ($v(yz) = vzyz$, $v = wx$) twice to obtain

$wxzyz = wx(yz) = wy(xz) = wyzxz$.

Then use Axiom A2 ($wzz = w$) to cancel z on the right yielding

$wxzy = wyzx$ (interchange x and y; an "abelian" geodesic space)

or

$wxzyxzy = w$ (another form of the parallel postulate)

as per the curvature diagram for the trihexagonal tiling.

Midway predicate vs. midpoint operation

Define $\mathrm{mid}(A, M, B)$ as $e(A, M) = B \quad (\equiv \ \mathrm{mid}(B, M, A))$.

For example $\mathrm{mid}(N, E, S)$ for N, S the north and south poles on the globe holds for all and only the points E on the equator.

Define ***midpoint*** $A \oplus B$ to be the maximal partial binary operation satisfying

$$A \oplus e(A, B) \ = \ B \ = \ e(A, (A \oplus B))$$

Theorem 13 *$A \oplus B$ is defined iff $|\{M \mid \mathrm{mid}(A, M, B)\}| = 1$.*

Corollary 5 $N \oplus S$ is undefined on the globe.

Theorem 14 *If $h(e(A, B)) = e(h(A), h(B))$ for all $A, B \in S$ (i.e. the category* **Gsp***) then $h(A \oplus B) = h(A) \oplus h(B)$ when $A \oplus B$ is defined.*

Theorem 15 *$\oplus$ is total-completable in torsion-free spaces.*

Centroids

Generalize midpoint $A \oplus B$ to centroid $A_1 \oplus \ldots \oplus A_n$ as a partial n-ary operation via

$$A_1 \oplus \ldots \oplus A_n = B \text{ iff } e^n(A_1 \oplus \ldots \oplus A_{n-1}), B) = A_n.$$

This axiom forces $e^n((A_1 \oplus \ldots \oplus A_{n-1}), B) = A_n$ to have at most one solution in B. A ***centroidal*** space is one for which n-ary centroid is total for all finite $n \geq 2$.

Theorem 16 *The full subcategory of* **Gsp** *consisting of the flat centroidal spaces is equivalent to the category* $\mathbf{Aff}_{\mathbb{Q}}$ *of affine spaces over the rationals.* (*pace* Löwenheim-Skolem)

Extends to $\mathbf{Vct}_{\mathbb{Q}}$ by adjoining a constant as the origin. A further expansion in the same vein permits $\mathbb{Q}$ to be extended to $\mathbb{Q}[i]$ (complex rationals).

The geodesic neighborhood

$$
\begin{tikzcd}
\color{blue}{\textit{Operations:}xy} & & \color{blue}{[yxz]} & & \color{blue}{xy, x^{-1}, e} & \\
& \mathbf{Gsp} \arrow[r, shift right, "xyxz"'] \arrow[dd, bend right=60, "x\ (\pi^2_1)"'] & \mathbf{Heap} \arrow[l, shift right, "{[yxy]}"'] & & \mathbf{Grp} \arrow[ll, "yx^{-1}z"'] & \\
& \mathbf{Grv} \arrow[u, "\text{G3: } xywz = xzwy"] \arrow[r, shift right, "xyxz"'] & \mathbf{Schar} \arrow[u, "{[yxz]=[zxy]}"'] \arrow[l, shift right, "{[yxy]}"'] & & \mathbf{Ab} \arrow[ll, "y-x+z"'] \arrow[u, "xy = yx"'] & \mathbf{Ring} \arrow[l, "x-y"'] \\
& \mathbf{Set} \arrow[u, "xy = x"'] & \mathbf{Cube} \arrow[l, "x\ (\pi^3_2)"'] \arrow[u, "{[yxy] = x}"'] & & \mathbf{Cube}_* \arrow[ll, "y+x+z"'] \arrow[u, "x+x=0"'] & \mathbf{Bool} \arrow[l, "x+y"'] \arrow[u, "x^2 = x"']
\end{tikzcd}
$$

Every path in this commutative diagram denotes a forgetful functor, hence one with a left adjoint. Vertical arrows *forget* the indicated *equation*, horizontal arrows *retain* the indicated *operation*. E.g. the left adjoint of the functor $U_{\mathbf{AbGrp}} : \mathbf{Ab} \to \mathbf{Grp}$ is abelianization, that of the functor $U_{\mathbf{SetGsp}} : \mathbf{Set} \to \mathbf{Gsp}$ gives the set $F_{\mathbf{GspSet}}(S)$ of connected components of S, and so on.

From affine to Euclidean geometry

Aff$_{\mathbb{F}}$ forms a variety for any field $\mathbb{F}$. So far $\mathbb{F} = \mathbb{Q}$, an Archimedean field.

But the plane $\mathbb{E}^2$ cannot be the direct square of the line $\mathbb{E}$.

Hence any definition of **Euc** as the theory of Euclidean space must involve either partial operations, other Boolean combinations of equations besides conjunction, or both.

Neither $n(A, B, C)$ nor $t(A, B, C)$ is total, and it is convenient to condition their properties on both positive and negative equations.

Define **Euc** as the Π^0_1 theory of $\mathbb{R}^n$ in this language, namely the universal Boolean combinations of equations between terms in the language e, c, p, n, t that are valid in $\mathbb{R}^n$ for all nonnegative integers n.

Absent existential quantifiers and constants, the empty universe is a model of **Euc**, as is the singleton universe $\mathbb{R}^0$.

Claim Models of **Euc** = the inner product spaces over arbitrary (including non-Archimedean) ordered extensions of the constructible field.

The definition of Euclidean space

A ***Euclidean space*** is

- a partial algebra $(\mathbf{E}, c, a, b, t)$
- with signature 3-3-3-3 (four ternary partial operations),
- satisfying the equations that hold of $\mathbb{R}^n$ for all finite n
- between terms built from c, a, b, t and subjunction.

Variables $A, B, C, \ldots$ range over the points of the space **E**.

Operation c: Line-line intersection via circumcenter

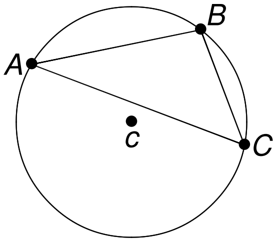


Figure 1. Circumcenter $c(A, B, C)$

Parameters: vertices of a triangle ΔABC.
Circumcenter is the intersection of the perpendicular bisectors of any two sides *in the plane of the triangle*.
Domain of c: nondegenerate triangles. (A, B, C may not be collinear.)
Exception: $c(A, A, A) = A$ (subjunction can't express $c(A, A, A) = \perp$).
Uses: multiplication, division, flatness (Euclid's 5th Postulate).

Operations a, b, t: Line-sphere intersection

Sphere B_A [Pieri 1908]:

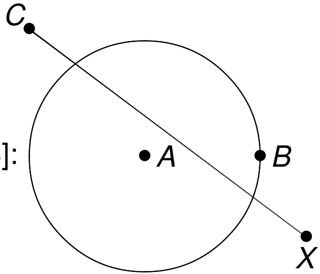


Figure 2. Line-sphere intersection $f(A, B, C, X)$

We reduce the four parameters to three by taking X to be one of A, B, or a tangent point T.
This specializes f to one of a, b, or t respectively.
Benefits:

- Eliminates the case of line CX completely missing sphere B_A.
- Parameters coplanar: ABC plane cuts sphere in great circle.
- Each special case has special properties.

Operation a: Normalization

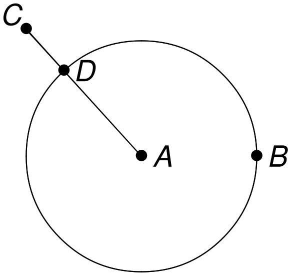


Figure 3. Normalization $D = a(A, B, C)$

D is the point on sphere B_A closest to C. Or, $D =$ ray $AC \cap B_A$.
Domain: Defined everywhere except $C = A \neq B$. ($t(A, A, C) = A$.)

AD is AC normalized to the same length as AB.
Any motion of C induces a rigid motion of AD about A.

Uses: metric, order (ray AC).

Operation b: Generic chord

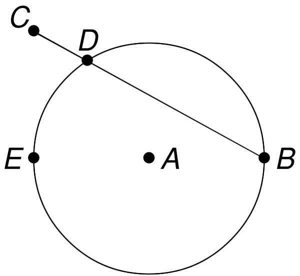


Figure 4. Generic chord $D = b(A, B, C)$

D is the intersection point that makes BD a chord.
Generic in the sense that any chord length from 0 to the diameter is possible depending on C.
Domain: Defined everywhere except $C = B \neq A$.
Important special case: $b(A, B, A) = e(B, A) = E$
(extend radius BA to a diameter BE). E total, affine. $ED \perp BC$.
Uses: projection (E onto BC, extension (BA to BE).

Operation t: Tangent point

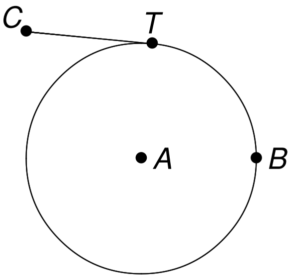


Figure 5. Tangent point $T = t(A, B, C)$

T is the tangent point from C nearest B.
Domain: Defined except when C is strictly inside B_A (no tangent point), or strictly outside and on line AB (ambiguous). (No ambiguity when C is on B_A, i.e. $|CA| = |BA|$.)

$$CT \perp AT$$

Main use: square root.

Subjunction: a domain-independent operation

Binary case: ***subjunction*** $A \# B$ is

- idempotent
- commutative
- unit is $\perp$ (undefined): $A \# \perp = A$
- $A \neq B \;\rightarrow\; A \# B = \perp$

Convention: Variables $A, B, \ldots$ range over points of the space and are therefore always defined. $X, Y, \ldots$ are metavariables denoting terms that may be undefined.
Higher arities: If a term $X(A)$ has a nonempty domain
on which it is constant,
then $\Sigma_A X(A)$ is that constant.
Otherwise $\Sigma_A X(A)$ is undefined.

Applications of subjunction

- Strengthens equational logic without logical connectives
- Complete a partial operation to a total one.
- More generally, combine compatible partial operations.
- Specify and compare domains.
- Formalize notion of *general position*

Subjunction: Completion to a total operation

- $a(A, B, C)\#(A\#C)$ extends a with $a(A, B, A) = A$.
- $b(A, B, C)\#(B\#C)$ extends b with $a(A, B, B) = B$.

Euclidean spaces

Recall: a ***Euclidean space*** is

- a partial algebra (E, c, a, b, t)
- with signature 3-3-3-3 (four ternary partial operations),
- satisfying the equations that hold of $\mathbb{R}^n$ for all finite n
- between terms built from c, a, b, t and subjunction.

Dimension defined as for vector and affine spaces, with no restriction on cardinality of dimension.

Now what?

We’ve set up the machinery.

What would Euclid do?

Proposition 1': Gram-Schmidt orthogonalization

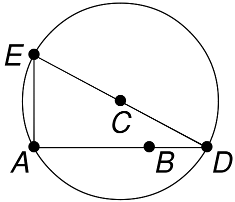


Figure 6. Gram-Schmidt: $E = e(b(C, A, B), C)$

Proposition 1'. *Given a segment AB and a point C not on the line AB, to erect a perpendicular AE to AB at A in the plane of* $\triangle ABC$.

Let $D = b(C, A, B)$, $E = e(D, C)$ $(= b(C, D, C))$.
Then $AE \perp AB$.

For *orthogonalization* the operation b sufficed.

Proposition 1: Gram-Schmidt orthonormalization

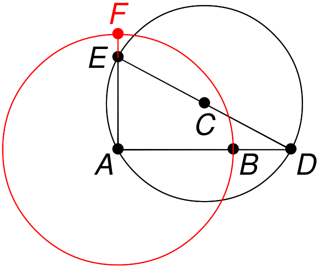


Figure 7. Gram-Schmidt: $F = a(A, B, E)$

Proposition 1. *Given a segment AB and a point C not on the line AB, to erect a perpendicular AF to AB at A s.t.* $|AF| = |AB|$.

Let $D = b(C, A, B)$, $E = e(D, C)$ $(= b(C, D, C))$, and $F = a(A, B, E)$. Then $AF \perp AB$, and $|AF| = |AB|$.

For *orthonormalization* we supplemented b with operation a.

The unit circle B_A

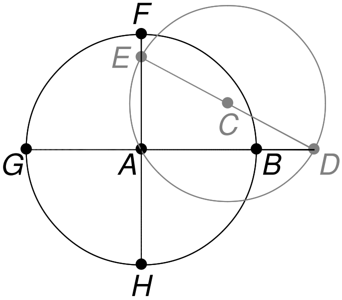


Figure 8. Unit circle B, F, G, H (taking $|AB| = 1$)

Complete radii BA and FA to respective diameters BG and FH.

$$G = e(B, A)$$

$$H = e(F, A)$$

Reciprocation $1/x$

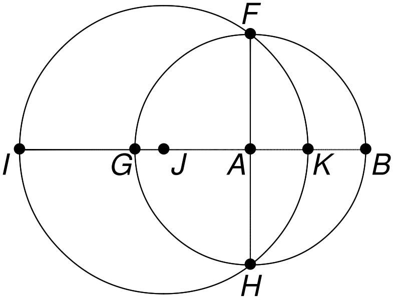


Figure 9. If $|AB| = 1$ then $|AK| = 1/|AI|$

Let I be any point on AB to the left of A.
Let $J = c(F, I, H)$ and $K = e(I, J)$ (the diameter). (Uses operation c.)
If $I = G$ then $K = B$.
If $I = e(A, G)$ (as shown) then $K = m(A, B)$, the ***midpoint*** of AB.

Euclid's Proposition 1

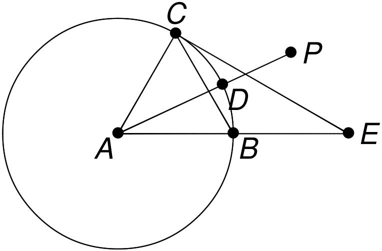


Figure 10. Find C making ABC equilateral.

Euclid P1. *Given points A, B, to construct an equilateral triangle ΔABC.*

Vague: what half-plane to draw C in?
Determine this with an additional point P lying properly within the desired half-plane. This time we use operation t.

Take $C = t(A, \underbrace{a(A, B, P)}_{D}, \underbrace{e(A, B)}_{E})$.

Euclid's Proposition 2

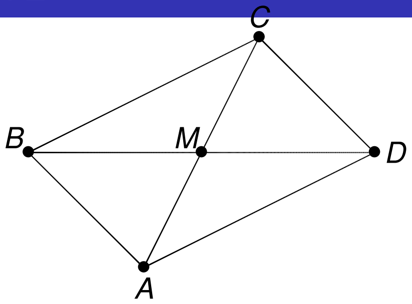


Figure 11. Translate BC to A: $D = p2(A, B, C)$

Euclid P2. To place a straight line equal to the given straight line BC with one end at the point A.
Approach: translate BC to AD.
Equivalently, add vectors BA and BC (B a local origin) to give BD.
Construction: $M = m(A, C)$, $D = e(B, M)$.
So $p2(A, B, C) = e(B, m(A, C))$.
Besides $1/x$ we now have $x + y$.

Metric

Any choice of distinct points 0,1 determines a distance metric $|AB|_{01}$ on the space in terms of points on the ray 01, as follows.

$$|AB|_{01} = a(0, p2(0, A, B), 1)$$

$p2(0, A, B) = D$ translates AB to $0D$.
$a(0, D, 1) = E$ rotates $0D$ to $0E$ in the ray 01.

Congruence of line segments AB, CD is then definable in the obvious way as $|AB| = |CD|$.

Congruence can be shown to satisfy all the axioms for Tarski's quaternary congruence relation.

Order

Euclidean space is ordered by the relation $Between(A, B, C)$, definable in our language as

$$a(A, B, C) = a(C, B, A) = B$$

The second equation is redundant except when $A = C$, where it forces B to equal A.

When B is between A and C the condition is obviously met, even when $A = C$ (since $a(A, A, A) = A$). To see the converse, assume $A \neq C$ and let D be the common value of $a(A, B, C)$ and $a(C, B, A)$. D must lie on the rays AC and CA and therefore on the segment AC, showing that D is between A and C. Now D must also lie on the spheres B_A and B_C, whence the spheres must share a tangent plane at D. But since A and C are on opposite sides of D the spheres cannot intersect elsewhere. But B lies on both spheres and must therefore be their unique intersection, Hence $B = D$ whence B is between A and C.

Operations p, r, i

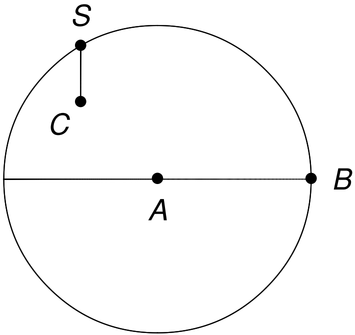
$p(A, B, C)$ denotes the foot of the *perpendicular* to AB from C.

$r(A, B, C)$ denotes the result of sliding C parallel to AB in order to *rectify* ΔABC at A, that is, to make $AC \perp AB$.

$i(A, B, C)$ *inverts* (reflects) C in AB, thinking of AB as a mirror.

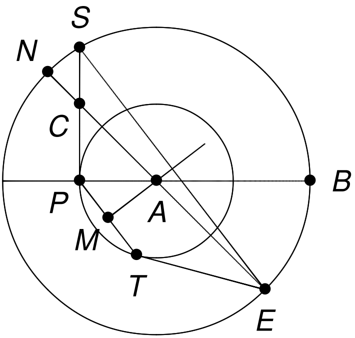
$$
\begin{array}{ll}
p(A, B, C) = b(m(B, C), B, A) & b(A, B, C) = p(B, C, e(B, A)) \\
r(A, B, C) = p2(A, p(A, B, C), C) & p(A, B, C) = p2(A, r(A, B, C), C) \\
i(A, B, C) = e(C, p(A, B, C)) & p(A, B, C) = m(C, i(A, B, C))
\end{array}
$$

Operation $s(A, B, C)$



$S = s(A, B, C)$ is the point on sphere B_A in the plane of ΔABC, such that $CS \perp AB$.

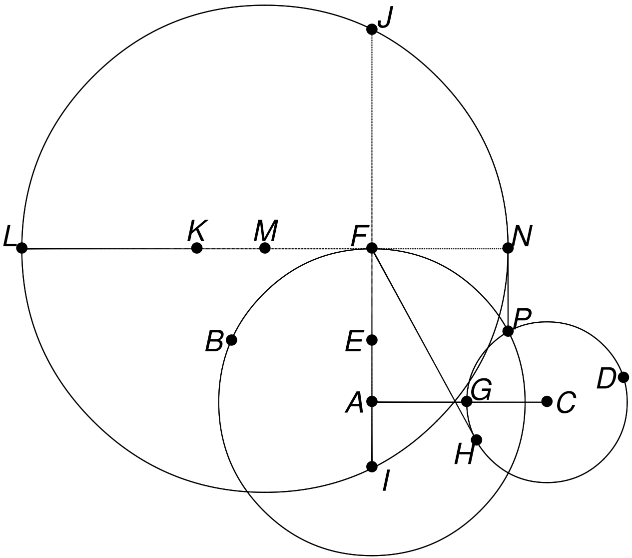
s is derivable from t as follows.



$$\begin{array}{llll} P = p(A,B,C) & M = m(P,T) \\ N = a(A,B,C) & S' = i(A,M,E) \\ E = e(N,A) & S'' = e(E, A\#P) \\ T = t(A,P,E) & S = S'\#S'' \end{array}$$

Figure 12. s from t

Sphere-sphere intersection: Mohr-Mascheroni$^{\perp}$



$$E_B = r(A, C, B)$$
$$F_B = a(A, B, E_B)$$
$$E_D = r(A, C, D)$$
$$F_D = a(A, B, E_D)$$
$$F = (F_B \# F_D) \# F_B$$
$$G = a(C, D, A)$$
$$H = t(C, G, F)$$
$$I = a(F, H, A)$$
$$J = e(I, F)$$
$$K = p2(F, C, A)$$
$$L = e(F, K)$$
$$M = c(I, J, L)$$
$$N = e(L, M)$$
$$P = s(C, G, N)$$

Domains of a, b, t, c

Definition: $X \restriction \mathrm{dom}(Y) = e(e(X, Y), Y)$.
a: $B \restriction \mathrm{dom}(a(A, B, C)) = B\#(A\#C)$
b: $A \restriction \mathrm{dom}(b(A, B, C)) = A\#(B\#C)$
t: $C \restriction \mathrm{dom}(t(A, B, C)) = X\#(Y\#Z)$
where $W = a(A, B, C)\#(B \restriction \mathrm{dom}(A\#C))$
$X = a(W, C, e(A, W))\#C$
$Y = b(A, B, C)\#e(A, B)$
$Z = a(B, C, e(B, A))\#a(e(B, A), C, B)$
$c(A, a(A, B, C)\#(A\#C), C) = A\#C$ (so $c(A, A, A) = A$)

Axioms

(Note: Substitution of terms for variables $A, B, C, \ldots$ is permitted only for total terms.)
The foregoing domain axioms.
Circumcenter $D = c(A, B, C)$.

$$c(A, B, C) = c(B, A, C) = c(A, C, B)$$

$$|DA| = |DB| = |DC|$$

$$\mathrm{coplanar}(A, B, C, D)$$

Normalize $D = a(A, B, C)$

$$\mathrm{collinear}(A, C, D)$$
$$|AD| = |AB|$$
$$a(A, A, C) = A$$

Axioms (cont.)

Generic chord: $D = b(A, B, C)$

$$\text{collinear}(B, C, D)$$

$$|AD| = |AB|$$

$$t(e(B, A), D, C) = D$$

$$b(A, A, C) = A$$

$$b(A, B, D) = D$$

$$e(e(A, B), B) = A$$

$$e(e(A, B), e(C, D)) = e(e(A, C), e(B, D))$$

Axioms (cont.)

Tangent: $T = t(A, B, C)$

$$|AT| = |AB|$$
$$b(m(A, C), A, T) = T$$
$$t(A, A, C) = A$$
$$t(A, B, T) = T$$

Subjunction:

$$X\#X = X$$
$$X\#Y = Y\#X$$
$$(A\#B)\#A = A$$
$$(((X\#Y)\#X)\#Y)\#X = (X\#Y)\#X$$

(Convention: $A, B, \ldots$ range over points, hence always defined, X over terms and may therefore be undefined.)