Algebras and Bialgebras

via categories with distinguished objects

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October 9, 2016

Two frameworks

Sol Feferman: What rests on what?

VP: Foundations are annoying. Cycles are ok.

- ZFC starts from the binary relation $\in$ of **set membership**.
- CT starts from the "binary" operation $\circ$ of **function composition**.

Both frameworks, ZFC as *set theory* and CT as *category theory*, satisfy certain well-known properties.

Which to pick?

Compromise: Associatively composable homsets

- A **category** $\mathcal{C}$ is a large directed graph.
- Its edges are labeled with homsets $\mathcal{C}(X, Y)$.
- For all triples (X, Y, Z) of vertices (objects) there is an associative multiplication $\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ (composition).
- For all objects X, $\mathcal{C}(X, X)$ contains a 2-sided identity for $\circ$.

Sets arise as homsets. (I.e. restrict to locally small categories.)

Functions arise in context as left or right actions of composition.

That's it! Dispense with the rest of category theory: no colimits, (hom)functors, natural transformations, adjunctions, monads, etc.

In our framework, CDO, the elements of homsets play half a dozen roles: elements of various sorts, states of various properties, operation symbols, predicate transformer symbols, transformations, and qualia.

Associativity plays several roles, each automating a familiar definition.

CDO: Categories with Distinguished Objects

In full generality, a **CDO** $(\mathcal{C}, J, L)$ is a category $\mathcal{C}$ together with sets $J, L \subseteq ob(\mathcal{C})$ of objects of $\mathcal{C}$ (the **distinguished** objects).

Interpretation: $\mathcal{C}$ is a category of multityped bialgebras and their maps. Types are of two kinds:

- sorts $j, j', \ldots$ belonging to J.
- properties $\ell, \ell', \ldots$ belonging to L.

The **base** of a CDO $(\mathcal{C}, J, L)$ is the **bipartite subcategory** $\mathcal{K}$ of $\mathcal{C}$ consisting of the following.

- The full subcategory $\mathcal{J}$ of $\mathcal{C}$ for which $ob(\mathcal{J}) = J$.
- The full subcategory $\mathcal{L}$ of $\mathcal{C}$ for which $ob(\mathcal{L}) = L$.
- The $(J \times L)$-indexed family of homsets $C_{j\ell}(j, \ell)$ from $j \in J$ to $\ell \in L$.

Categories with sorts

To fix ideas we begin with the case $L = \emptyset$ (no properties, just sorts $j \in J$). Bialgebras simplify to multisorted algebras and the base is simply $\mathcal{J}$.

An **element** of an object X in $\mathcal{C}$ of sort j is a morphism $x : j \to X$.

Parallel morphisms $f, g : X \to Y$ of a CDO are **equivalent** when they have the same left actions, that is, when $fx = gx$ for all elements x of X.

A CDO is **extensional** when equivalent morphisms are equal.

Set theory

What is a set? What is a function?

A **category of sets and functions**, or CSF, is a CDO $(\mathcal{C}, \{\mathbf{1}\})$ for which $|C(\mathbf{1}, \mathbf{1})| = 1$ ($\mathbf{1}$ is **rigid**).

A set X is an object of a CSF. Its elements $x \in X$ are the morphisms $x : \mathbf{1} \to X$.

A function $f : X \to Y$ is a morphism of a CSF. Its action on each $x \in X$ is by composition fx on the left (left action).

Rigidity of $\mathbf{1}$ asserts that $\mathbf{1}$ has one element.

Such a CDO need not be extensional, and need not represent all sets or all functions.

Observation: When $(\mathcal{C}, \{\mathbf{1}\})$ is extensional, $\mathbf{1}$ is final in the full subcategory of $\mathcal{C}$ whose objects have a morphism to $\mathbf{1}$.

The (naive) category **Set** as a complete CSF

Two modes of extension of any extensional CSF.

1. By morphisms, preserving sets (can't add morphisms from **1**) and extensionality (can't add morphisms whose actions duplicate those of an extant morphism).

2. By sets, subject to extension mode 1. (Elements fixed "at birth".)

An extensional CSF is **complete** when it is equivalent to its every extension. Complete CSFs are understood to be extensional.

Axiom. (Naive set theory) There exists a complete CSF, and all such are equivalent.

Definition. **Set** is a complete CSF.

Set is only defined up to equivalence.

Graph theory

A category $\mathcal{C}$ of graphs and graph homorphisms is a CDO with two distinguished objects $\mathbf{V}$ and $\mathbf{E}$ (the primitive no-edge and one-edge graphs) such that there are two morphisms (s, t) from $\mathbf{V}$ to $\mathbf{E}$.

A **graph** G (**homomorphism** $h : G \to G'$) is an object (morphism) of $\mathcal{C}$.

A **vertex** v (**edge** e) is a morphism $v : \mathbf{V} \to G$ $(e : \mathbf{E} \to G)$.

Edge e has

- A **source vertex** $es : \mathbf{V} \xrightarrow{s} \mathbf{E} \xrightarrow{e} G$ (right action of s)
- A **target vertex** $et : \mathbf{V} \xrightarrow{t} \mathbf{E} \xrightarrow{e} G$ (right action of t)

Graph homomorphisms respect s and t as a consequence of associativity of composition: $h(es) = (he)s$ where $hes : \mathbf{V} \xrightarrow{s} \mathbf{E} \xrightarrow{h} G'$.
(Conventionally, $h(s(e)) = s(h(e))$, i.e. img of source = source of img.)

The category **Grph** is defined analogously to **Set**.

Theories

The **theory** $\mathcal{T}$ of $(\mathcal{C}, J)$ is $\mathcal{J}^{op}$.

Set theory: $\mathbf{1}$ (or just $\bullet$) (simpler than ZFC set theory.)

Graph theory: $\mathbf{V} \overset{s}{\underset{t}{\Longleftarrow}} \mathbf{E}$ (note opposite)

Each object of $\mathcal{T}$ is understood as a **sort**, e.g. $\mathbf{1}$, $\mathbf{V}$, $\mathbf{E}$.

Each morphism $f : s \to t$ of $\mathcal{T}$ is understood as an **operation symbol**, e.g. $s(e)$, $t(e)$.

Each object X of $\mathcal{C}$ interprets each operation symbol $f : s \to t$ as its right action xf in $\mathcal{C}$ on elements x of X of sort s.

Example: A graph G interprets s as the operation mapping each edge e to its source vertex es.

Variants of graph theory

Reflexive graphs: Expands the base $\mathcal{J}$ for graphs with three distinct morphisms $i : \mathbf{E} \to \mathbf{V}$, $si : \mathbf{E} \to \mathbf{E}$, $ti : \mathbf{E} \to \mathbf{E}$.

This can understood as the category of the positive finite ordinals ≤ 2 and their monotone functions.

By Morita equivalence omitting "positive" ($\{0, 1, 2\}$) makes no difference.

Simplicial sets: Omit "≤ 2" (all finite ordinals Δ). $\mathcal{T} = \Delta^{op}$.

Cubical sets: $\mathcal{T} = \mathbf{Fbip}$. **Fbip** is the category of finite bipointed sets, dual to the category of primitive finite cubical sets. (By Morita equivalence it is not necessary to add "skeletal".)

Presheaves

A **presheaf** (morphism thereof) is an object (morphism) of a CDO $(\mathcal{C}, J)$.

In the language of category theory a presheaf is a functor $A : \mathcal{J}^{op} \to \mathbf{Set}$ where $\mathcal{J}$ is the base of $(\mathcal{C}, J)$.

(We don't *need* this larger language of category theory since the CDO framework does not depend on it, but it's convenient notationally, conceptually, and (higher) algebraically.

A complete CDO is a category of presheaves, namely

$$\mathbf{Set}^{\mathcal{J}^{op}} = \mathbf{Psh}(\mathcal{J}).$$

Yoneda embedding: $\mathcal{J}$ fully embeds in $\mathbf{Set}^{\mathcal{J}^{op}}$. This is the sense in which the objects of $\mathcal{J}$, or sorts, can be understood as primitive presheaves or unary algebras, and its morphisms as homomorphisms.

Every complete CDO is a topos.

Chu spaces, untyped and typed

Thus far homomorphisms have transformed algebras covariantly.

The CDO framework can be taken much further, for example the notion and role of topos, higher arities than unary, coalgebras, etc.

The main goal here to show how the CDO framework generalizes easily to incorporate the open sets of point set topology, the functionals of linear algebra, etc. These transform contravariantly.

Distinguished objects are now of two kinds, sorts as before, and properties.

Morphisms *from* sorts are elements as before, or **points**.

Morphisms *to* properties are **states**.

This methodology was first employed in point set topology. It can also be usefully observed in linear algebra, and generalizes in a number of ways.

Chu spaces

Basic case: Chu spaces over $2 = \{0, 1\}$.

The base category is a category $\mathbf{1} \overset{0}{\underset{1}{\rightrightarrows}} \perp$.

A category $\mathcal{C}$ of Chu spaces $\mathcal{A}, \mathcal{B}, \ldots$ over 2 and their transformations is a category with rigid distinguished objects $\mathbf{1}$ and $\perp$ and morphisms

$$\mathbf{1} \overset{0}{\underset{1}{\rightrightarrows}} \perp.$$

A **point** a is a morphism $a : \mathbf{1} \to \mathcal{A}$. Notation: $a \in A$.

A **state** x is a morphism $x : \mathcal{A} \to \perp$. Notation: $x \in X$.

Composition xa of a state x with a point a is 0 or 1. It defines an $A \times X$ matrix $c : A \times X :\to \{0, 1\}$ of 0's and 1's, where $\mathcal{A}$ is understood as a triple (A, c, X).

Transformations of Chu spaces

A **transformation** $h : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A} = (A, c, X)$, $\mathcal{B} = (B, d, Y)$ acts

- covariantly on a point $a \in A$ yielding a point $ha \in B$ (as before).
- contravariantly on a state $y \in Y$ yielding a state $yh \in X$.

Picture: $\mathbf{1} \xrightarrow{a} \mathcal{A} \xrightarrow{h} \mathcal{B} \xrightarrow{y} \perp$. Here h can be interpreted as a pair (f, g) of functions $f : A \to B$, $g : Y \to X$ defined by the actions of h on a and y respectively.

By associativity $y(ha) = (yh)a$. This is read more conventionally as an adjointness condition

$$\forall a \in A. \forall y \in Y. d(h(a), y) = c(a, g(y)).$$

In point set topology, the definition of continuity based on preservation of open sets under inverse image can be restated as such an adjointness requirement.

Comonoids in Chu2

In 1995 Michael Barr defined a comonoid in $\mathbf{Chu}_2$ as a triple $(\mathcal{A}, \delta, \epsilon)$ where $\mathcal{A}$ is a Chu space, $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a comultiplication (necessarily coassociative), with counit $\epsilon : \mathcal{A} \to \mathbf{1}$.

Barr observed that the comonoids in $\mathbf{Chu}_2$ formed a CCC.

The states can be viewed as forming a dictionary D over $\{0, 1\}$ of words of length $|A|$ with the properties
(i) every crossword formed from D has its main diagonal in D; and
(ii) both constant words are in D,

"T1" and "discrete" are defined as for topological spaces.

Problem (VP, 1995): Is every T1 comonoid discrete? Answer: No.

Bergman and Nielsen, On Vaughan Pratt's crossword problem, J. London Math. Soc. (2016) 93 (3): 825-845.

Generalizing Chu2 to ChuK

A simple generalization is to replace $\mathbf{1} \overset{0}{\underset{1}{\Longrightarrow}} \perp$ by $\mathbf{1} \overset{K}{\Longrightarrow} \perp$ where K is an arbitrary set whose elements supply the matrix entries or values of xa.

A complete CDO on such a base forms the category $\mathbf{Chu}_K$.

One of a number of master theorems about Chu categories:

Theorem

Every elementary class of total arity k and its relation-preserving homomorphisms fully embeds in $\mathbf{Chu}_{2^k}$.

More at `http://chu.stanford.edu`.

Generalizing to typed Chu spaces

The $\mathbf{Chu}_K$ base $\mathbf{1} \stackrel{K}{\Longrightarrow} \perp$ where K is an arbitrary set can be understood as two one-object categories $\mathbf{1}$ and $\perp$ and a set K of morphisms from $\mathbf{1}$ to $\perp$.

Generalize $\mathbf{1}$ to a small category $\mathcal{J}$, $\perp$ to a small category $\mathcal{L}$, and K to a doubly indexed family $K_{j\ell}$ of morphisms from $j \in ob(J)$ to $\ell \in ob(L)$.

Graph theorists call this a bipartite graph. Category theorists call it a bimodule or profunctor, notating it (counterintuitively) as $\mathcal{L} \not\to \mathcal{J}$.

A typed Chu space consists of a presheaf on $\mathcal{J}$, a dual presheaf on $\mathcal{L}$, and what I shall call qualia $k : j \to \ell$ in $K_{j\ell}$.

Requisite modifications to the covariant case (presheaves)

Mod 1. (Equivalence of parallel morphisms) Two transformations $f, g : \mathcal{A} \to \mathcal{B}$ are **equivalent** when they have the same left action on every point of $\mathcal{A}$ *and the same right action on every state of* $\mathcal{B}$.

Extensionality (equivalence is equality) is then understood for this more general notion.

Mod 2. Extensions must preserve both points *and states* of Chu spaces. (Note that qualia are simultaneously points of L and states of J.)

Everything else works in the same way.

A category of typed Chu spaces and their transformations is a complete CDO.

Problem

A topos is a category with all finite limits and power objects.

Theorem

For any category $\mathcal{C}$ TFAE.
(i) $\mathcal{C}$ is a cocomplete topos.
(ii) $\mathcal{C}$ is equivalent to a presheaf category.

That is, up to equivalence the notion of cocomplete topos axiomatizes presheaf categories.

(cf. B is a Boolean algebra iff it is isomorphic to a field of sets.)

Problem. Axiomatize typed Chu spaces up to equivalence.

APPENDIX: Three Questions of Philosophy

1. In Descartes's mind-body dichotomy, is mind an equal partner to body, as opposed to just a convenient conceptual artifact?
2. Can we speak of the extension of a property like color or weight by analogy with the extension of a sort (cat, truck) as the set of its members?
3. Are C.I. Lewis's qualia logically consistent?

Answers informed by (typed) Chu spaces.

1. Yes. Associate points and states with respectively individuals (concrete) and predicates (conceptual).
2. Yes. The extension of a sort (property) in a universe is the set of its individuals of that sort (states of that property).
3. Yes. Lewis conceived of qualia such as red or heavy as both concrete and perceptual. Typed Chu spaces on $\mathcal{K} : \mathcal{L} \not\rightarrow \mathcal{J}$ make a quale *calico* : *cat* $\rightarrow$ *color* both the unique calico cat in the universe of colors and a "perceptual" color-state of the primitive cat.