A homogeneous algebraic definition of Euclidean space

Vaughan Pratt

Computer Science Department
Stanford University

BLAST 2013

Analogy: Boolean algebra

A Boolean algebra can be defined equivalently as either

- a Boolean ring $(B, +, \times, 0, 1)$, a ring satisfying $x^2 = x$; or
- a complemented distributive lattice $(B, \wedge, \vee, \neg, 0)$.

The former is arithmetical in character, the latter logical.

Halmos preferred the former definition for his *Lectures on Boolean algebras*.

Courses in logic tend to start with the latter.

At previous BLASTs

The variety $\mathrm{Aff}_{\mathbb{Q}}$ of affine spaces over the rationals can be defined equivalently with either an arithmetic or geometric signature:

Arithmetic: rational linear combinations whose coefficients sum to unity. Sublanguage of $\mathrm{Vct}_{\mathbb{Q}}$ (*all* linear combinations).

Geometric: (i) Extension $C = e(A, B)$ of segment AB s.t. $|AC| = 2|AB|$.
(ii) Centroid $c_n(A_1, \ldots, A_n)$ for $2 \leq n < \omega$.
No mention of $\mathbb{Q}$*!*

This talk: From affine to Euclidean geometry.
(Title of Wanda Szmielew's posthumously published monograph.)
Both Tarski and Szmielew axiomatized Euclidean geometry in first order logic, based on relations of congruence and betweenness.

Our goal: (i) operations in place of relations;
(ii) equations in place of logical wff's.

Main obstacle

Affine spaces over $\mathbb{Q}$ form a variety. Very nice.

Claim: Euclidean spaces don't form a variety. Not so nice.

Intuitive argument:

The Euclidean line has the p-norm for all positive p.
The Euclidean plane has only the 2-norm.
How could the direct square of $\mathbb{R}^2$ pick out the 2-norm?

(In fact, as an associative algebra, $\mathbb{R}^2$ is the hyperbolic plane, the other 2D Clifford algebra besides $\mathbb{C}$.)

Approaches

Arithmetical solution: introduce a quadratic form. Inner product spaces.
- Algebraic but based on a field, e.g. $\mathbb{R}$, so not homogeneous.

Logical solution: introduce congruence and linear order as 4-ary and 3-ary relations. Tarski's axioms.
- Homogeneous (points, no numbers) but not algebraic.

Geometrical solution: introduce circles [Euclid 350 BC] or spheres [Pieri 1908].
- Neither homogeneous nor algebraic. But can be made both.

Additional motivations

- Introduce (?) and demonstrate subjunction
- Pedagogical: Illustrate algebrization in a familiar setting, ruler-and-compass constructions, traditionally considered immune to algebrization.

A homogeneous algebraic approach

Logical framework: equations, but with an additional domain-independent operation $A \# B$ of *subjunction* (also $\Sigma_A X(A)$) serving to combine partial operations.

A **Euclidean space** is

- a partial algebra $(\mathbf{E}, c, a, b, t)$
- with signature 3-3-3-3 (four ternary partial operations),
- satisfying the equations that hold of $\mathbb{R}^n$ for all finite n
- between terms built from c, a, b, t and subjunction.

Variables $A, B, C, \ldots$ range over the points of the space $\mathbf{E}$.

Operation c: Line-line intersection via circumcenter

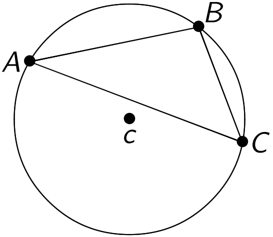


Figure 1. Circumcenter $c(A, B, C)$

Parameters: vertices of a triangle ΔABC.
Circumcenter is the intersection of the perpendicular bisectors of any two sides *in the plane of the triangle*.
Domain of c: nondegenerate triangles. (A, B, C may not be collinear.)
Exception: $c(A, A, A) = A$ (subjunction can't express $c(A, A, A) = \perp$).
Uses: multiplication, division, flatness (Euclid's 5th Postulate).

Operations a, b, t: Line-sphere intersection

Sphere B_A [Pieri 1908]:

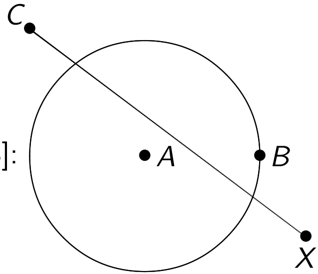


Figure 2. Line-sphere intersection $f(A, B, C, X)$

We reduce the four parameters to three by taking X to be one of A, B, or a tangent point T.

This specializes f to one of a, b, or t respectively.

Benefits:

- Eliminates the case of line CX completely missing sphere B_A.
- Parameters coplanar: ABC plane cuts sphere in great circle.
- Each special case has special properties.

Operation a: Normalization

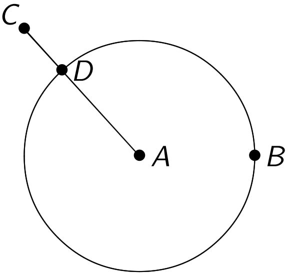


Figure 3. Normalization $D = a(A, B, C)$

D is the point on sphere B_A closest to C. Or, $D =$ ray $AC \cap B_A$.
Domain: Defined everywhere except $C = A \neq B$. ($t(A, A, C) = A$.)

AD is AC normalized to the same length as AB.
Any motion of C induces a rigid motion of AD about A.

Uses: metric, order (ray AC).

Operation b: Generic chord

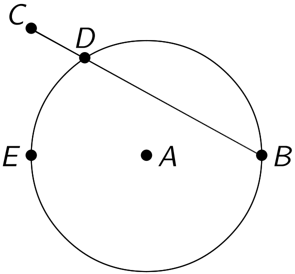


Figure 4. Generic chord $D = b(A, B, C)$

D is the intersection point that makes BD a chord.

Generic in the sense that any chord length from 0 to the diameter is possible depending on C.

Domain: Defined everywhere except $C = B \neq A$.

Important special case: $b(A, B, A) = e(B, A) = E$

(extend radius BA to a diameter BE). E total, affine. $ED \perp BC$.

Uses: projection (E onto BC, extension (BA to BE).

Operation t: Tangent point

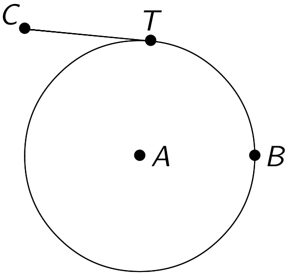


Figure 5. Tangent point $T = t(A, B, C)$

T is the tangent point from C nearest B.
Domain: Defined except when C is strictly inside B_A (no tangent point), or strictly outside and on line AB (ambiguous). (No ambiguity when C is on B_A, i.e. $|CA| = |BA|$.)

$$CT \perp AT$$

Main use: square root.

Subjunction: a domain-independent operation

Binary case: **subjunction** $A\#B$ is

- idempotent
- commutative
- unit is $\perp$ (undefined): $A\#\perp = A$
- $A \neq B \;\rightarrow\; A\#B = \perp$

Convention: Variables $A, B, \ldots$ range over points of the space and are therefore always defined. $X, Y, \ldots$ are metavariables denoting terms that may be undefined.

Higher arities: If a term $X(A)$ has a nonempty domain
on which it is constant,
then $\Sigma_A X(A)$ is that constant.
Otherwise $\Sigma_A X(A)$ is undefined.

Applications of subjunction

- Strengthens equational logic without logical connectives
- Complete a partial operation to a total one.
- More generally, combine compatible partial operations.
- Specify and compare domains.
- Formalize notion of *general position*

Subjunction: Completion to a total operation

- $a(A,B,C)\#(A\#C)$ extends a with $a(A,B,A)=A$.
- $b(A,B,C)\#(B\#C)$ extends b with $a(A,B,B)=B$.

More on subjunction later.

Euclidean spaces

Recall: a **Euclidean space** is

- a partial algebra (E, c, a, b, t)
- with signature 3-3-3-3 (four ternary partial operations),
- satisfying the equations that hold of $\mathbb{R}^n$ for all finite n
- between terms built from c, a, b, t and subjunction.

Dimension defined as for vector and affine spaces, with no restriction on cardinality of dimension.

Now what?

We've set up the machinery.

What would Euclid do?

Proposition 1': Gram-Schmidt orthogonalization

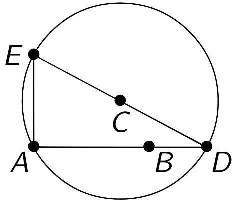


Figure 6. Gram-Schmidt: $E = e(b(C, A, B), C)$

Proposition 1'. *Given a segment AB and a point C not on the line AB, to erect a perpendicular AE to AB at A in the plane of* ΔABC.

Let $D = b(C, A, B)$, $E = e(D, C)$ $(= b(C, D, C))$.
Then $AE \perp AB$.

For *orthogonalization* the operation b sufficed.

Proposition 1: Gram-Schmidt orthonormalization

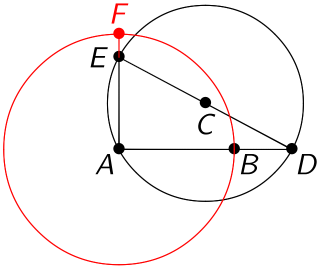


Figure 7. Gram-Schmidt: $F = a(A, B, E)$

Proposition 1. *Given a segment AB and a point C not on the line AB, to erect a perpendicular AF to AB at A s.t.* $|AF| = |AB|$.

Let $D = b(C, A, B)$, $E = e(D, C)$ $(= b(C, D, C))$, and $F = a(A, B, E)$. Then $AF \perp AB$, and $|AF| = |AB|$.

For *orthonormalization* we supplemented b with operation a.

The unit circle B_A

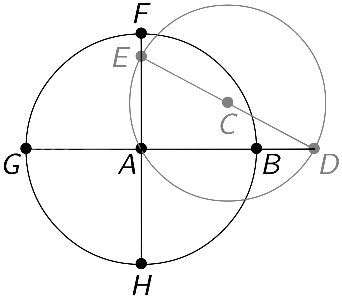


Figure 8. Unit circle B, F, G, H (taking $|AB| = 1$)

Complete radii BA and FA to respective diameters BG and FH.

$$G = e(B, A)$$

$$H = e(F, A)$$

Reciprocation $1/x$

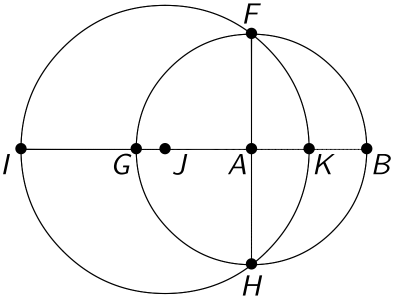


Figure 9. If $|AB| = 1$ then $|AK| = 1/|AI|$

Let I be any point on AB to the left of A.
Let $J = c(F, I, H)$ and $K = e(I, J)$ (the diameter). (Uses operation c.)
Chords intersecting at A: $|IA| * |AK| = |FA| * |AH|$.
If $|AB| = 1$ then $|FA| * |AH| = 1$, so $|AK| = 1/|IA|$.
If $I = G$ then $K = B$.
If $I = e(A, G)$ (as shown) then $K = m(A, B)$, the **midpoint** of AB.

Euclid's Proposition 1

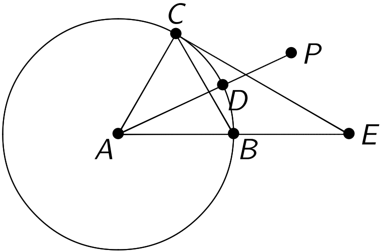


Figure 10. Find C making ABC equilateral.

Euclid P1. *Given points* A, B, *to construct an equilateral triangle* ΔABC.

Vague: what half-plane to draw C in?
Determine this with an additional point P lying properly within the desired half-plane. This time we use operation t.

Take $C = t(A, \underbrace{a(A, B, P)}_{D}, \underbrace{e(A, B)}_{E})$.

Euclid's Proposition 2

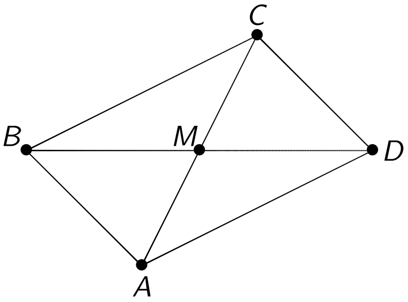


Figure 11. Translate BC to A: $D = p2(A, B, C)$

Euclid P2. To place a straight line equal to the given straight line BC with one end at the point A.

Approach: translate BC to AD.

Equivalently, add vectors BA and BC (B a local origin) to give BD.

Construction: $M = m(A, C)$, $D = e(B, M)$.

So $p2(A, B, C) = e(B, m(A, C))$.

Besides $1/x$ we now have $x + y$.

Metric

Any choice of distinct points 0,1 determines a distance metric $|AB|_{01}$ on the space in terms of points on the ray 01, as follows.

$$|AB|_{01} = a(0, p2(0, A, B), 1)$$

$p2(0, A, B) = D$ translates AB to $0D$.
$a(0, D, 1) = E$ rotates $0D$ to $0E$ in the ray 01.

Congruence of line segments AB, CD is then definable in the obvious way as $|AB| = |CD|$.

Congruence can be shown to satisfy all the axioms for Tarski's quaternary congruence relation.

Order

Euclidean space is ordered by the relation $Between(A, B, C)$, definable in our language as

$$a(A, B, C) = a(C, B, A) = B$$

The second equation is redundant except when $A = C$, where it forces B to equal A.

When B is between A and C the condition is obviously met, even when $A = C$ (since $a(A, A, A) = A$). To see the converse, assume $A \neq C$ and let D be the common value of $a(A, B, C)$ and $a(C, B, A)$. D must lie on the rays AC and CA and therefore on the segment AC, showing that D is between A and C. Now D must also lie on the spheres B_A and B_C, whence the spheres must share a tangent plane at D. But since A and C are on opposite sides of D the spheres cannot intersect elsewhere. But B lies on both spheres and must therefore be their unique intersection, Hence $B = D$ whence B is between A and C.

Operations p, r, i

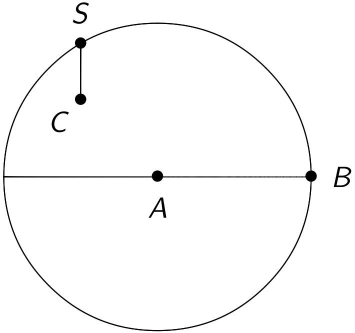
$p(A, B, C)$ denotes the foot of the *perpendicular* to AB from C.

$r(A, B, C)$ denotes the result of sliding C parallel to AB in order to *rectify* ΔABC at A, that is, to make $AC \perp AB$.

$i(A, B, C)$ *inverts* (reflects) C in AB, thinking of AB as a mirror.

$$
\begin{array}{ll}
p(A,B,C) = b(m(B,C),B,A) & b(A,B,C) = p(B,C,e(B,A)) \\
r(A,B,C) = p2(A,p(A,B,C),C) & p(A,B,C) = p2(A,r(A,B,C),C) \\
i(A,B,C) = e(C,p(A,B,C)) & p(A,B,C) = m(C,i(A,B,C))
\end{array}
$$

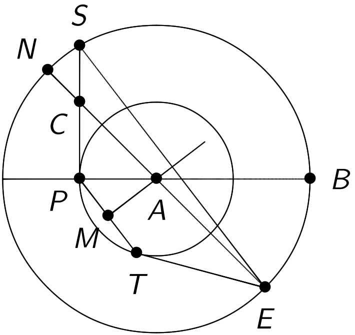
Operation $s(A, B, C)$



$S = s(A, B, C)$ is the point on sphere B_A in the plane of ΔABC, such that $CS \perp AB$.

s is derivable from t as follows.

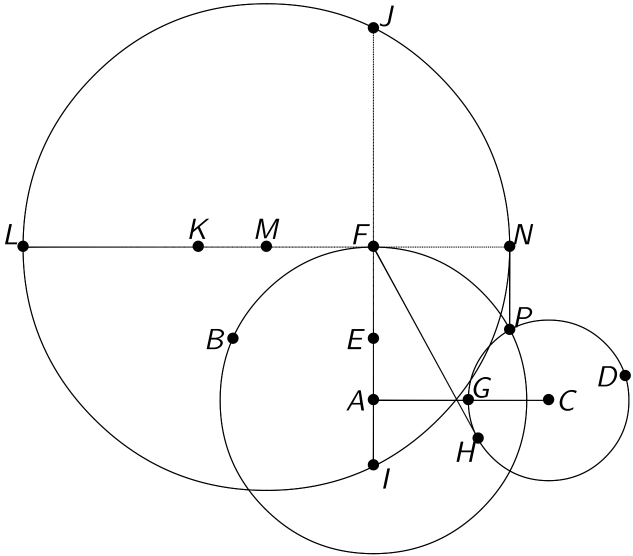
Derivation of s from t



$$P = p(A, B, C) \quad M = m(P, T)$$
$$N = a(A, B, C) \quad S' = i(A, M, E)$$
$$E = e(N, A) \quad S'' = e(E, A\#P)$$
$$T = t(A, P, E) \quad S = S'\#S''$$

Figure 12. s from t

Sphere-sphere intersection: Mohr-Mascheroni$^\perp$



$$E_B = r(A, C, B)$$
$$F_B = a(A, B, E_B)$$
$$E_D = r(A, C, D)$$
$$F_D = a(A, B, E_D)$$
$$F = (F_B \# F_D) \# F_B$$
$$G = a(C, D, A)$$
$$H = t(C, G, F)$$
$$I = a(F, H, A)$$
$$J = e(I, F)$$
$$K = p2(F, C, A)$$
$$L = e(F, K)$$
$$M = c(I, J, L)$$
$$N = e(L, M)$$
$$P = s(C, G, N)$$

Arithmetic operations

Arithmetic is performed on the line 01 for arbitrary choice of $0 \neq 1$.

Addition $z = x + y$: $z = p2(x, 0, y)$.
Negation $z = -x$: $z = e(x, 0)$. (Invert x in 0.)

Multiplication and square root:
Using s, it is straightforward to construct intersecting orthogonal chords of respective lengths u and v. In general neither are diameters.
For product $z = x * y$, intersect the chords so as to decompose $u = x + y$, $v = 1 + z$.

For square root $z = \sqrt{x}$, intersect the chords as $u = 1 + x$, $v = z + z$.

Reciprocal $z - 1/x$: already done.

Domains of a, b, t, c

Definition: $X \restriction \mathrm{dom}(Y) = e(e(X, Y), Y)$.

a: $B \restriction \mathrm{dom}(a(A, B, C)) = B\#(A\#C)$

b: $A \restriction \mathrm{dom}(b(A, B, C)) = A\#(B\#C)$

t: $C \restriction \mathrm{dom}(t(A, B, C)) = X\#(Y\#Z)$

where $W = a(A, B, C)\#(B \restriction \mathrm{dom}(A\#C))$

$X = a(W, C, e(A, W))\#C$

$Y = b(A, B, C)\#e(A, B)$

$Z = a(B, C, e(B, A))\#a(e(B, A), C, B)$

$c(A, a(A, B, C)\#(A\#C), C) = A\#C$ (so $c(A, A, A) = A$)

Axioms

(Note: Substitution of terms for variables $A, B, C, \ldots$ is permitted only for total terms.)
The foregoing domain axioms.
Circumcenter $D = c(A, B, C)$.

$$c(A, B, C) = c(B, A, C) = c(A, C, B)$$

$$|DA| = |DB| = |DC|$$

$$\mathrm{coplanar}(A, B, C, D)$$

Normalize $D = a(A, B, C)$

$$\mathrm{collinear}(A, C, D)$$
$$|AD| = |AB|$$
$$a(A, A, C) = A$$

Axioms (cont.)

Generic chord: $D = b(A, B, C)$

$$\text{collinear}(B, C, D)$$

$$|AD| = |AB|$$

$$t(e(B, A), D, C) = D$$

$$b(A, A, C) = A$$

$$b(A, B, D) = D$$

$$e(e(A, B), B) = A$$

$$e(e(A, B), e(C, D)) = e(e(A, C), e(B, D))$$

Axioms (cont.)

Tangent: $T = t(A, B, C)$

$$|AT| = |AB|$$
$$b(m(A, C), A, T) = T$$
$$t(A, A, C) = A$$
$$t(A, B, T) = T$$

Subjunction:

$$X\#X = X$$
$$X\#Y = Y\#X$$
$$(A\#B)\#A = A$$
$$(((X\#Y)\#X)\#Y)\#X = (X\#Y)\#X$$

(Convention: $A, B, \ldots$ range over points, hence always defined, X over terms and may therefore be undefined.)