Logic Without Syntax

(Extended abstract)

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Abstract

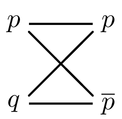
This paper presents an abstract, mathematical formulation of classical propositional logic. It proceeds layer by layer: (1) abstract, syntax-free propositions; (2) abstract, syntax-free contraction-weakening proofs; (3) distribution; (4) axioms $p \vee \overline{p}$.

Abstract propositions correspond to objects of the category $\mathbf{G}(\mathsf{Rel}^L)$ where $\mathbf{G}$ is the Hyland-Tan double glueing construction, Rel *is the standard category of sets and relations, and L is a set of literals. Abstract proofs are morphisms of a tight orthogonality subcategory of $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$, where we define $\mathbf{G}_{\leq}$ as a lax variant of $\mathbf{G}$. We prove that the free product-sum category (contraction-weakening logic) over L is a full subcategory of $\mathbf{G}(\mathsf{Rel}^L)$, and the free distributive lattice category (contraction-weakening-distribution logic) is a full subcategory of $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$. We explore general constructions for adding axioms, which are not* Rel*-specific or $(p \vee \overline{p})$-specific.*

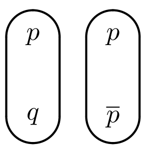
1 Introduction

Abstract propositions. Typically logicians define a proposition or formula as a labelled tree. Using de Morgan duality ($\neg(A \wedge B) = \neg A \vee \neg B$ *etc.*) one needs only trees labelled by literals (variables and their duals) and constants on leaves and $\vee$ and $\wedge$ on internal nodes. To quotient by associativity and commutivity, one may use graphs (cographs or P_4-free graphs [CLSB81]), for example,

$(p \vee q) \wedge (p \vee \overline{p}) \quad \mapsto$



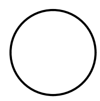
drawing an edge between leaves iff they meet in the parse tree at a $\wedge$. In this paper we go a step further, and define an *abstract proposition* as a set of *leaves* together with a set of subsets, called *resolutions*. For example,



has the four leaves of the formula/graph depicted earlier, and two resolutions, the maximal independent sets (maximal co-cliques) of the graph. Any syntactic proposition can be reconstructed from its abstract leaf/resolution presentation. (The terminology 'resolution' here comes from the definition of MALL proof net [HG03].)

A key advantage of this abstraction is a crisp mathematical treatment of the logical units/constants false 0 and true 1. In the traditional syntactic world, 0 and 1 have the same stature as literals, taking up actual ink on the page as labelled leaves. They are artificially dual to each other, by *fiat*, and artificially act as units for syntactic $\vee$ and $\wedge$. The graphical representation gets closer to a nice treatment of units: the empty graph ϵ is a unit for the operations $\vee$ and $\wedge$ (union and join) on graphs; however then one has degeneracy $\epsilon = 0 = 1$, so one must resort once again to artificially promoting 0 and 1 to actual labelled vertices, and the units remain ad hoc.

Abstract propositions succeed in having both units empty (no leaves), hence mathematically crisp as units for the operations $\vee$ and $\wedge$, *without* identifying them (as in the graph case $\epsilon = 0 = 1$):



1 (true) 0 (false)

The unit 1 has no leaves and no resolution, and 0 has no leaves with one (empty) resolution.

Abstract proofs. Abstract propositions correspond to certain objects of the category $\mathbf{G}(\mathsf{Rel}^L)$ studied in [Hug04a], where

- $\mathbf{G}$ is the Hyland-Tan double glueing construction [Tan97],
- Rel is the standard category of sets and relations, and
- $L = \{p, \overline{p}, q, \overline{q}, \ldots\}$ is a set of literals.

Thus a $\mathbf{G}(\mathsf{Rel}^L)$ morphism $A \to B$ provides an off-the-shelf notion of an abstract proof of B from A. By definition, a $\mathbf{G}(\mathsf{Rel}^l)$ morphism is a certain kind of binary relation between the leaves of A and the leaves of B. Figure 1 shows

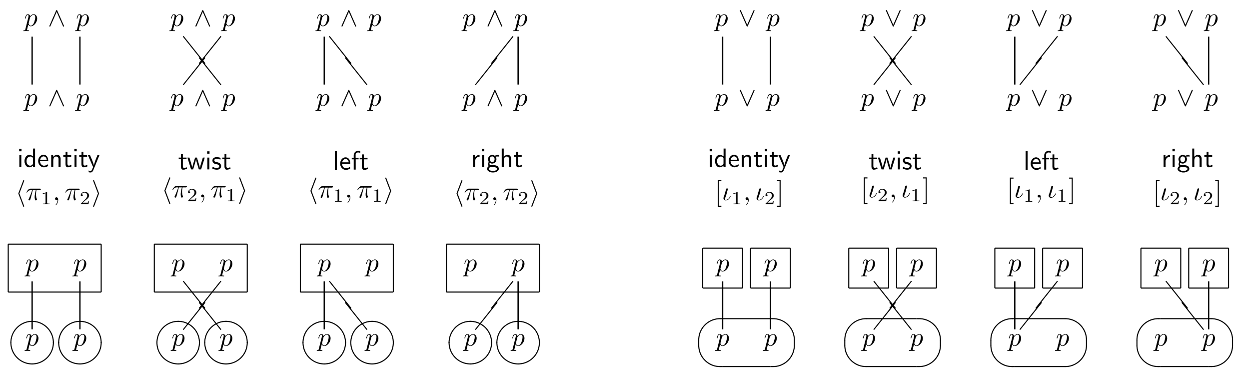


Figure 1. The four abstract proofs $p \wedge p \to p \wedge p$, and dually, $p \vee p \to p \vee p$. Each proof is a $\mathbf{G}(\mathsf{Rel}^L)$ morphism, a binary relation between leaves which satisfies the resolution condition. The top row shows the eight morphisms in syntactic form. The bottom row shows the same morphisms between the corresponding abstract propositions, where the target propositions are specified by their resolutions (curved regions), and the source propositions by their coresolutions (rectangular regions). Note that the resolution condition is satisfied: there is a unique edge between any output resolution (curved region) and input coresolution (square region). The resolution condition characterises free product-sum categories.

the four morphisms $p \wedge p \to p \wedge p$, and dually, the four morphisms $p \vee p \to p \vee p$. By definition of double glueing, a morphism R must satisfy:

($\mathcal{R}$) *Resolution condition.* R pulls resolutions backwards and pushes coresolutions forwards.

More precisely, $R : A \to B$ maps resolutions of B to resolutions of A, and coresolutions of A to coresolutions of B, where a coresolution of X is a resolution of its dual $\overline{X}$. In the special case that A and B are abstract propositions, this coincides with the usual resolution condition on MALL proof nets [HG03]:

($\mathcal{R}'$) *Resolution condition.* R has a unique edge between any output resolution and input coresolution.[1]

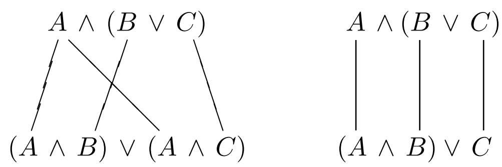
We prove:

> *Theorem.* $\mathbf{G}(\mathsf{Rel}^K)$ contains as a full subcategory the free product-sum category generated by the set K.

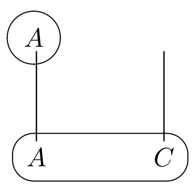
Thus we obtain an abstract, syntax-free formulation of pure contraction-weakening logic over a set of atoms K: every morphism (abstract proof) $A \to B$ is a composite of the units of the product/sum adjunctions, the natural transformations (inferences)

$$\begin{array}{llcccl} \pi_i & : & A_1 \wedge A_2 & \to & A_i & (\text{projection}) \\ \iota_i & : & A_i & \to & A_1 \vee A_2 & (\text{injection}) \\ \delta & : & A & \to & A \wedge A & (\text{diagonal}) \\ \epsilon & : & A \vee A & \to & A & (\text{codiagonal}) \end{array}$$

Adding distribution. The obvious candidates for a distribution



fail the resolution condition. They are not $\mathbf{G}(\mathsf{Rel}^L)$ morphisms. Condition ($\mathcal{R}$) fails because the image of an output resolution is strictly larger than an input resolution:



(1)

and uniqueness fails in the MALL resolution condition ($\mathcal{R}'$) since there are two edges between an output resolution and

[1] Recall from [HG03] the resolution condition on a set R of linkings on a sequent or proposition Γ: *R has a unique linking on any resolution of Γ.* In the current pure additive (*i.e.* product-sum) setting every linking has just one edge, *i.e.*, R is simply a set of edges. The main text quotes this condition with $\Gamma = A \to B = \overline{A} \vee B$. A resolution of $\overline{A} \vee B$ is a union of a coresulution of A and a resolution of B.

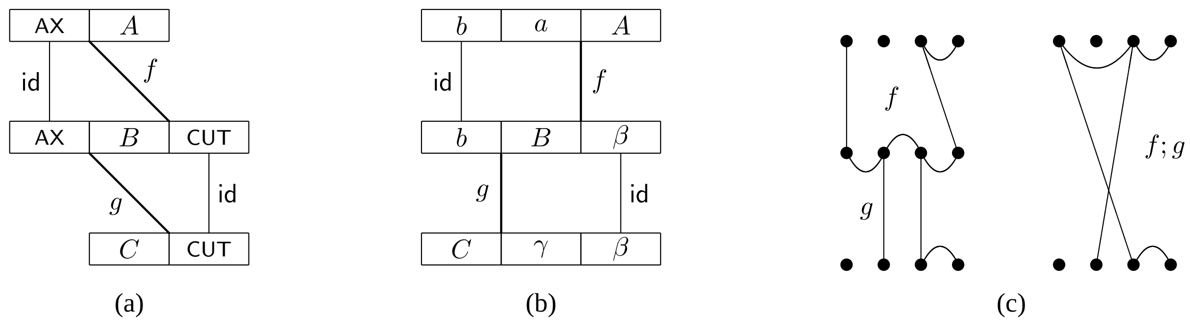
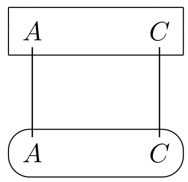


Figure 2. Three approaches to incorporating axioms $p \vee \overline{p}$. The diagrams above illustrate composition schematically. (a) Define an abstract classical proof $f : A \to B$ as a $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ morphism $f : \mathsf{AX} \wedge A \to B \vee \mathsf{CUT}$, where the (potentially infinite) *universal axiom* AX is the product of all axioms $p \vee \overline{p}$ and the *universal cut* is its dual $\mathsf{CUT} = \neg\mathsf{AX}$, the sum of all cuts $\overline{p} \wedge p$. (b) Define an abstract classical proof $A \to B$ as a $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ morphism $a \wedge A \to B \vee \beta$, where a is any product of axioms $p \vee \overline{p}$ and β is any sum of cuts $\overline{p} \wedge p$. In (a) and (b), linear distributivity is hidden at the interface layer. (c) Follow the usual linear logic pattern and relax the Rel morphisms of $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ to Link morphisms, where Link is the category of sets and linkings. Composition is the usual alternating path composition (*i.e.*, the 'smooth' paths in the example above).

an input coresolution:



(2)

(Both failures depicted above apply to both distributions.) These failures suggest naively relaxing the resolution conditions, to admit distribution:

($\mathcal{R}_{\leq}$) *Lax resolution condition.* R pulls resolutions back to super-resolutions and pushes coresolutions forwards to super-coresolutions.

Here a super-(co)resolution is a superset of a (co)resolution. Thus we have relaxed ($\mathcal{R}$) in the obvious way, admitting the first failure (1) depicted above, by allowing the images of (co)resolutions to spill beyond (co)resolutions. Similarly, we relax ($\mathcal{R}'$) in the obvious way to admit the second failure (2) depicted above, by simply dropping uniqueness:

($\mathcal{R}'_{\leq}$) *Lax resolution condition.* R has an edge between every output resolution and input coresolution.

Just as ($\mathcal{R}'$) coincided with ($\mathcal{R}$) on abstract propositions, so ($\mathcal{R}'_{\leq}$) coincides with ($\mathcal{R}_{\leq}$).

We define the *lax double glued category* $\mathbf{G}_{\leq}(\mathsf{Rel}^K)$ using ($\mathcal{R}_{\leq}$) in place of ($\mathcal{R}$). Surprisingly, this completely naive relaxation of the resolution conditions, stimulated by the distribution failures, works:

> *Theorem.* $\mathbf{G}_{\leq}(\mathsf{Rel}^K)$ contains as a full subcategory the free distributive lattice category generated by the set K.

Došen and Petrić [DP04] define a distributive lattice category as a product-sum category with a distribution, satisfying certain coherence laws. Thus we obtain an abstract, syntax-free formulation of contraction-weakening-distribution logic over a set of atoms K.

Axioms. The final step to an abstract, syntax-free formulation of classical propositional logic is to add axioms $1 \to p \vee \overline{p}$ (hence by duality, also cuts $\overline{p} \wedge p \to 0$). We explore three natural but distinct ways of achieving this. The first two constructions are quite general, and are not specific to our Rel-based abstract proposition approach, nor to the specific axioms $p \vee \overline{p}, q \vee \overline{q}, \ldots$. The third approach is more ad hoc and limited, being Rel-specific and ($p \vee \overline{p}$)-specific, but the style is more conventional in the literature. The three approaches are portrayed schematically in Figure 2.

(a) *Universal axiom construction.* Let the (potentially infinite) abstract proposition AX, the *universal axiom*, be the product of all $p \vee \overline{p}$ for complementary literals in L. Its dual $\mathsf{CUT} = \neg\mathsf{AX}$ is the *universal cut*. The *universal boolean category* B^u_L has the objects of $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ and a morphism $f : A \to B$ is a $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ morphism

$$\begin{array}{c} \mathsf{AX} \wedge A \\ \big\downarrow{\scriptstyle f} \\ B \vee \mathsf{CUT} \end{array}$$

Composition $f; g$ is defined in the obvious way, via linear distribution l at the interface. See Figure 2(a).

(b) *Local axiom construction.* The *local boolean category* B^a_L has objects as above, but a morphism $A \to B$ is a $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ morphism $a \wedge A \to B \vee \beta$ for a a product of axioms and β a sum of cuts. Composition is again defined in the obvious way, via linear distribution. See Figure 2(b).

(c) *Linkings.* We follow the standard recipe in linear logic and geometry of interaction [Gir87], imitating the step from pure linearly distributive categories (two-sided proof nets) to those with negation (one-sided nets) [BCST96]. The *linking boolean category* B^l_L is obtained from $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ by extending the homsets from Rel to the category Link of sets and linkings. A linking $X \to Y$ is a simple graph on $X + Y$, with composition along alternating paths, like Kelly-Mac Lane graphs [KM71] (see Figure 2(c)).

Related work. The category $\mathbf{G}(\mathsf{Rel}^K)$ was studied extensively in [Hug04a], where it was shown to fully embed the category of biextensional Chu spaces over K [Bar79, Bar98]. The definition of abstract proposition goes via a tight orthogonality [HS03] in $\mathbf{G}(\mathsf{Rel}^K)$, related to totality spaces [Loa94]. The observation that the MALL resolution condition [HG03] characterises the free product-sum category was in [Hug02][2] in a more syntactic guise (via the deductive system in [CS01]); prior to that, Hongde Hu had already characterised the free product-sum category in a similar manner, using P_4-free graphs (contractible coherence spaces) [Hu99]. In a syntactic setting, [LS05] also observes that relaxing uniqueness in the MALL resolution condition yields a classical proof net. The classical proof nets sketched in [Gir91] are fleshed out in [Rob03]. An abstract notion of classical proof net is presented in [Hug04b]. Categorical generalisations of boolean algebras are presented in [FP04] and [DP04].

2 Abstract propositions

Let (X, S) be a set system, *i.e.*, a set X and a set S of subsets of X. Subsets $s, t \subseteq X$ are ***orthogonal***, denoted $s \perp t$, if they intersect in a single point. The ***orthogonal of*** S is

$$S^{\perp} \;=\; \{t \subseteq X \;:\; t \perp s \text{ for all } s \in S\}$$

Fix a set of literals $L = \{p, \overline{p}, q, \overline{q}, \ldots\}$.

[2] Read this technical report with a pinch of salt, as the proof is far longer than it needs to be. At the time I wrote it, I was unaware of Hu's related work [Hu99]; thus [Hug02] was never published. Thanks to Robin Cockett and Robert Seely for pointing out the relationship with Hu's work.

DEFINITION 1 *An* ***abstract proposition*** (X, S) *is a set* X *of* ***leaves****, each labelled by a literal, and a set* S *of subsets of* X*, called* ***resolutions****, satisfying:*

- Double orthogonal: $S^{\perp\perp} = S$.

Every syntactic $\wedge\vee$-formula ϕ over the set of literals L (*e.g.* $(p \vee q) \wedge (p \vee \overline{p})$) defines an abstract formula (X, S) with X the leaves of ϕ: let (X, E) be the simple graph with an edge $xy \in E$ iff the leaves x and y meet at a $\wedge$ in the parse tree of ϕ, and let S be the set of maximal stable sets of (X, E). (A stable set of a graph is a maximal set of vertices which contains no edge.) This represents ϕ modulo associativity and commutativity of $\wedge$ and $\vee$. See page 1 for an example. Any abstract proposition so obtained is ***syntactic***.

We define the following constants and operations.

- *True*. $1 = (\emptyset, \emptyset)$, no leaves and no resolutions.
- *False*. $0 = (\emptyset, \{\emptyset\})$, no leaves and the empty resolution.
- *Negation/not*. $\neg(X, S) = (\overline{X}, S^{\perp})$, where $\overline{X}$ relabels positive literals p to negative literals $\overline{p}$, and vice versa.
- *Sum/union/or*.

  $$(X, S) \vee (Y, T) \;=\; (X + Y,\, \{s + t : s \in S, t \in T\}).$$

  Thus the sum $A \vee B$ takes disjoint union on leaves, and a resolution of $A \vee B$ is the union of a resolution in A and a resolution of B.
- *Product/join/and*. $A \wedge B = \neg(\neg A \vee \neg B)$.

Note that $\neg\neg A = A$, $A \wedge 1 = A$ and $A \vee 0 = A$.

Abstract truth. An abstract proposition is ***true*** if every resolution contains a complementary pair of leaves (*e.g.* p and $\overline{p}$). Thus the constant 1 defined above (depicted on page 1) is true (since it has no resolution) but 0 is not (since it has an empty resolution). This definition of truth simply extends a well-known characterisation of truth for syntactic formulas: a syntactic formula A is true iff every component of its conjunctive normal form[3] $\textsc{cnf}(A)$ contains a complementary pair, and resolutions of A are in bijection with components of $\textsc{cnf}(A)$. The abstract leaf/resolution representation of propositions can be seen as "CNF + superposition information", where the latter indicates which literal occurrences in different components of $\textsc{cnf}(A)$ came from the same occurrence in A.

[3] The result of exhaustively applying (co)distribution $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$.

3 Abstract proofs

Abstract propositions correspond to objects of the category $\mathbf{G}(\mathsf{Rel}^L)$ where $\mathbf{G}$ is the Hyland-Tan double glueing construction and Rel is the standard category of sets and relations: (X, S) corresponds to the $\mathbf{G}(\mathsf{Rel}^L)$ object $(X, S^\perp, S)$. This category fully embeds the category of biextensional Chu spaces over L [Hug04a]. Abstract propostions correspond precisely to the objects of the tight orthogonality subcategory, in the sense of [HS03], for $s \perp t$ defined above. The definitions of 0, 1, $\wedge$ and $\vee$ above correspond to the standard initial object, terminal object, product and sum in a tight orthogonality category.

The morphisms of $\mathbf{G}(\mathsf{Rel}^L)$ provide a ready-made notion of abstract proof. A ***coresolution*** of an abstract proposition (X, S) is an element of $S^\perp$. An ***abstract proof*** $(X, S) \to (Y, T)$ between abstract propositions is a binary relation R between X and Y which respects labelling, *i.e.*, xRy only if x and y are labelled with the same literal, and satisfies:

($\mathcal{R}$) *Resolution condition.* R pulls resolutions backwards and pushes coresolutions forwards.[4]

More precisely, the inverse image of R is a function $T \to S$ and the direct image of R is a function $S^\perp \to T^\perp$. As discussed in the Introduction, this coincides with the usual resolution condition on MALL proof nets [HG03]:

PROPOSITION 1 *Between abstract propositions, the resolution condition ($\mathcal{R}$) on $R : A \to B$ coincides with:*

($\mathcal{R}'$) Resolution condition. *R has a unique edge between any output resolution and input coresolution.*

Proof. Suppose R satisfies ($\mathcal{R}$), let b be a resolution of B, and let α be a coresolution of A. Let a be the resolution of A which is the R-image of b. By double orthogonality, a and α intersect at a single leaf x. Since R maps b onto a, we have xRy for some leaf y in b. This provides the unique edge between α and b, for if xRy' for some other $y \in b$, the coresolution $R(\alpha)$ of B would intersect b in two leaves, a contradiction.

Conversely, suppose R satisfies ($\mathcal{R}'$), and let b be resolution of B. Given a coresolution α of A write $\widehat{\alpha}$ for the leaf of $A = (X, S)$ which is in the unique edge of R between α and b. Thus the R-image of b is $a = \{\widehat{\alpha} : \alpha \in S^\perp\}$. Since a intersects each α in exactly one leaf, namely $\widehat{\alpha}$, by the double orthogonality condition a is resolution. $\square$

Let CW_K be the full subcategory of $\mathbf{G}(\mathsf{Rel}^K)$ whose objects are syntactic propositions.

[4] For logical reaons, we have taken resolutions as the contravariant part, so the "co" is opposite to usual $\mathbf{G}(\mathcal{C})$. In the general $\mathbf{G}(\mathcal{C})$ case, the sets of resolutions and coresolutions are independent, rather than the one determining the other by orthogonality, as we have with abstract propositions.

THEOREM 1 *CW_K is the free product-sum category generated by the set K.*

Proof. By the Whitman-style theorem for product-sum categories in [CS01], it essentially suffices to show that CW_K is soft in the sense of [Joy95]: every morphism $A \wedge B \to C \vee D$ factors through a projection on the left or an injection on the right. If softness failed, there would be edges A–C and B–D (or A–D and B–C), breaking the resolution condition. $\square$

4 Distribution forces lax resolution

The Introduction discussed how the obvious candidates for distribution fail the resolution condition. We proceed completely naively, and relax the resolution condition on a binary relation R in the obvious way to accomodate distribution.

($\mathcal{R}_\leq$) *Lax resolution condition.* R pulls resolutions back to super-resolutions and pushes coresolutions forwards to super-coresolutions.

Here a super-(co)resolution is a superset of a (co)resolution.

($\mathcal{R}'_\leq$) *Lax resolution condition.* R has an edge between every output resolution and input coresolution.

PROPOSITION 2 *Between abstract propositions, the conditions ($\mathcal{R}_\leq$) and ($\mathcal{R}'_\leq$)coincide.*

Proof. Similar to the proof of Proposition 1. $\square$

Define the ***lax double glued category*** $\mathbf{G}_\leq(\mathsf{Rel}^K)$ using ($\mathcal{R}_\leq$) in place of ($\mathcal{R}$). (It is relatively easy to see that composition of binary relations preserves the lax resolution condition.) This $\mathbf{G}_\leq$ is a general lax double glueing construction $\mathbf{G}_\leq(\mathcal{C})$ only when the homsets of $\mathcal{C}$ are equipped with a suitable $\leq$ relation. In the case $\mathcal{C} = \mathsf{Rel}$ (or Rel^K), $\leq$ is the inclusion order.

The abstract propositions 0 and 1 remain initial and terminal in $\mathbf{G}_\leq(\mathsf{Rel}^K)$, since $\mathbf{G}_\leq(\mathsf{Rel}^K)$ remains structured over Rel.

PROPOSITION 3 *In the lax setting, $\wedge$ and $\vee$ continue to be product and sum on abstract propositions, i.e., in the tight orthogonality subcategory of $\mathbf{G}_\leq(\mathsf{Rel}^K)$.*

Proof. Straightforward from the definition of the lax resolution condition. $\square$

Let D_K be the full subcategory of $\mathbf{G}_\leq(\mathsf{Rel}^K)$ whose objects are syntactic propositions. A *distributive lattice category* is a category with products, sums and a distribution, satisfying certain coherence conditions [DP04].

THEOREM 2 *D_K is the free distributive lattice category generated by the set K.*

The proof uses two key factorisation lemmas.

LEMMA 1 (DISTRIBUTION FACTORISATION) *Any D_K morphism $R : A \wedge (B \vee C) \rightarrow D$ factorises through distribution d, i.e., there exists R' such that R equals*

$$A \wedge (B \vee C) \xrightarrow{d} (A \wedge B) \vee (A \wedge C) \xrightarrow{R'} D.$$

Proof. Coresolutions of $A \wedge (B \vee C)$ are in bijection with coresolutions of $(A \wedge B) \vee (A \wedge C)$. Thus the R' induced by R in the obvious way (duplicating every edge from a leaf of A) is well-defined, and factorises through distribution. □

Let ***mix*** $m_{A,B} : A \wedge B \rightarrow A \vee B$ be the composite

$$A \wedge B \rightarrow A \wedge (Z \vee B) \rightarrow (A \wedge Z) \vee (A \wedge B) \rightarrow A \vee B$$

of injection, distribution and a two projections, for some Z. Thus the binary relation of $m_{A,B}$ is simply the identity between leaves.

LEMMA 2 (MIX-SOFTNESS FACTORISATION) *Any D_K morphism $R : A \rightarrow B$ with A a pure product*[5] *and B a pure sum is **mix-soft**: unless R the identity on a single leaf, it factorises through an injection, a projection, or mix.*

Proof. If any leaf is not covered, R factorises through an injection or an injection. Otherwise, assuming R is not the identity on a single leaf, it factorises through mix at whichever of A or B contains more than one leaf. □

Proof (of Theorem 2). Coherence of distributive lattice categories with respect to a faithful functor to Rel was proved in [DP04], and Rel is the underlying morphism category of $\mathbf{G}_{\leq}(\mathsf{Rel}^K)$. Thus we need only show that every morphism of D_K is canonical, *i.e.*, generated from the canonical maps defining a distributive lattice category.

Given $R : A \rightarrow B$, using the distrubution factorisation lemma we may assume A is in disjunctive normal form (a sum of products of leaves), and since D_K has sums, we may further assume A is a pure product of leaves. Dually, we may assume B is a sum of leaves. Apply the mix-softness factorisation lemma. □

Note that the above proof, in terms of the two factorisation lemmas, amounts to a Whitman-style characterisation theorem for free distributive lattice categories.[6] (See [CS01] for the pure product-sum case.)

[5] *I.e.*, A is a product of one or more leaves, *i.e.*, $A = (X, S)$ with S comprising every singleton $\{x\}$ for $x \in X$.

[6] Hence the generality of defining mix as a distribution composite, rather than directly as the identity binary relation between leaves.

5 Axioms

So far, with the lax double glued category $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$, we have an abstract setting for contraction-weakening-distribution logic. The setting is canonical in the sense that it fully embeds the free distributive lattice category. The category $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ is also equipped with a duality $\neg$, a contravariant full and faithful functor (a de Morgan duality). The final step to an abstract, syntax-free formulation of classical propositional logic is to add axioms $1 \rightarrow p \vee \overline{p}$ (hence by duality, also cuts $\overline{p} \wedge p \rightarrow 0$).

We explore three natural but distinct ways of achieving this. Each was discussed and motivated in the Introduction, and the idea behind composition was sketched in Figure 2.

The first two constructions are quite general, and are not specific to our Rel-based abstract proposition approach, nor to the specific axioms $p \vee \overline{p}, q \vee \overline{q}, \ldots$. The third approach is more ad hoc and limited, being Rel-specific and $(p \vee \overline{p})$-specific, but the style is more conventional in the literature, *e.g.* [KM71, Gir87, BCST96].

5.1 Universal axiom construction

Let the (potentially infinite) abstract proposition AX, the ***universal axiom***, be the product of all $p \vee \overline{p}$ for complementary literals in L. Its dual $\mathsf{CUT} = \neg\mathsf{AX}$ is the ***universal cut***. The ***universal boolean category*** B_L^u has the objects of $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ and a morphism $f : A \rightarrow B$ is a $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ morphism

$$\begin{tikzcd} \mathsf{AX} \wedge A \arrow[d, "f"] \\ B \vee \mathsf{CUT} \end{tikzcd}$$

Composition $f; g$ is defined in the obvious way, via linear distribution l at the interface:

$$\begin{tikzcd} \mathsf{AX} \wedge A \arrow[d, "\langle \pi_1, f \rangle"'] \\ \mathsf{AX} \wedge (B \vee \mathsf{CUT}) \arrow[d, "l"'] \\ (\mathsf{AX} \wedge B) \vee \mathsf{CUT} \arrow[d, "{[g, \iota_2]}"'] \\ C \vee \mathsf{CUT} \end{tikzcd}$$

On syntactic propositions, there is a morphism $A \rightarrow B$ iff $A \Rightarrow B = \neg A \vee B$ is true. An ***abstract classical proof*** of A in B_L^u is a morphism $1 \rightarrow A$. Thus A has an abstract classical proof in B_L^u iff it is true.

The universal axiom construction is quite general. Writing the product pairing $\langle h, k \rangle : U \rightarrow V \wedge W$ of $h : U \rightarrow V$

and $k : U \to W$ and sum pairing $[h, k] : U \vee V \to W$ of $h : U \to W$ and $k : V \to W$ with explicit adjunction units,

$$
\begin{array}{c}
\mathsf{AX} \wedge A \\
\downarrow \delta \\
(\mathsf{AX} \wedge A) \wedge (\mathsf{AX} \wedge A) \\
\downarrow \pi_1 \wedge f \\
\mathsf{AX} \wedge (B \vee \mathsf{CUT}) \\
\downarrow l \\
(\mathsf{AX} \wedge B) \vee \mathsf{CUT} \\
\downarrow g \vee \iota_2 \\
(C \vee \mathsf{CUT}) \vee (C \vee \mathsf{CUT}) \\
\downarrow \epsilon \\
C \vee \mathsf{CUT}
\end{array}
$$

we see that we can apply the universal axiom construction to a category equipped with

- a tensor, with a linear distribution over its dual;
- a choice of axioms to be tensored together in forming AX (possibly an infinite tensor);
- indexed families of morphisms for contraction (typed like ϵ), weakening (typed like inclusion ι_i), copying (typed like δ) and deletion (typed like projection π_j);
- sufficient coherence laws to ensure associativity of the above composition by diagram pasting.

5.2 Local axiom construction

This construction is similar to the one above. The ***local boolean category*** B_L^a has objects as above, but a morphism $A \to B$ is a $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ morphism $a \wedge A \to B \vee \beta$ for a a product of (zero or more) axioms $p \wedge \overline{p}$ and β a sum of (zero or more) cuts $\overline{p} \wedge p$. Composition is again defined in the obvious way, via associativity and linear distribution at the interface:

$$
\begin{array}{c}
(b \wedge a) \wedge A \\
\downarrow \mathsf{assoc} \\
b \wedge (a \wedge A) \\
\downarrow \mathsf{id} \wedge f \\
b \wedge (B \vee \beta) \\
\downarrow l \\
(b \wedge B) \vee \beta \\
\downarrow g \vee \mathsf{id} \\
(C \vee \gamma) \vee \beta \\
\downarrow \mathsf{assoc} \\
C \vee (\gamma \vee \beta)
\end{array}
$$

See Figure 2(b) for a schematic supressing canonical maps. In a manner similar to the universal axiom construction, the local axiom construction generalises to a category equipped with

- a tensor, with a linear distribution over its dual;
- a choice of axioms to be tensored together;
- indexed families of morphisms for contraction (typed like ϵ), weakening (typed like inclusion ι_i), copying (typed like δ) and deletion (typed like projection π_j);
- sufficient coherence laws to ensure associativity of the above composition by diagram pasting.

In contrast to the universal construction, the local construction does not require the existence of an object corresponding to a possibly infinite tensor of axioms $p \vee \overline{p}$.

5.3 Linkings

Although the following approach is more ad hoc, in that it does not generalise as the two constructions above, it is a standard idea in linear logic [Gir87], traceable as far back as Kelly-Mac Lane graphs for closed categories [KM71].

Let Link denote the category of sets and linkings, where a linking $X \to Y$ is a simple graph on the disjoint union $X + Y$. Composition is the usual alternating path composition (see Figure 2(c)). Let the category Link_L be the hom-set extension of $\mathbf{G}_{\leq}(\mathsf{Rel}^L)$ obtained by permitting arbitrary linkings between leaves which respect labelling (with edges of $A \to B$ within A or B going between dual leaves p—$\overline{p}$). Since Link is compact closed under disjoint union, Link_L is star-autonomous under $\wedge$. Let B_L^l, the ***linking boolean category***, be the restriction of Link_L to syntactic propositions while retaining the lax MALL resolution condition ($\mathcal{R}'_{\leq}$) from $\mathbf{G}_{\leq}(\mathsf{Rel}^K)$, *i.e.*, there is an edge in every resolution of $A \Rightarrow B = \neg A \vee B$. Since the ambient category Link_L is star-autonomous, and objects are syntactic, by a routine structural induction ($\mathcal{R}'_{\leq}$) is preserved by composition. It is immediate from ($\mathcal{R}'_{\leq}$) that $A \Rightarrow B$ is true iff there is a morphism $A \to B$. The lax MALL resolution condition is also studied in [LS05], in a more syntactic setting.

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